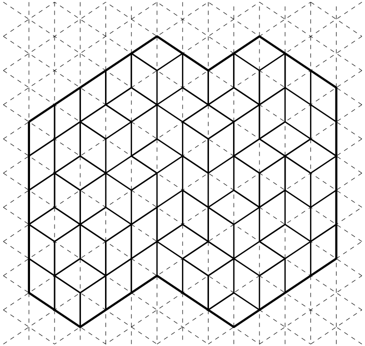
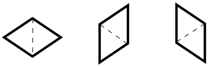
Lozenge tilings: universal bulk limits, global fluctuations.

Vadim Gorin
MIT (Cambridge) and IITP (Moscow)

May 2016

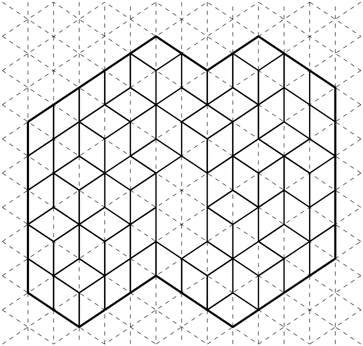
Random lozenge tilings

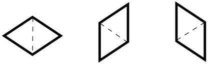




Random tilings of finite and infinite planar domains with **uniform Gibbs property**.

Random lozenge tilings





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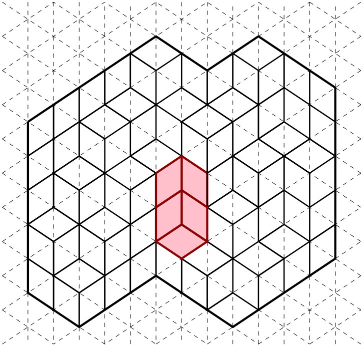

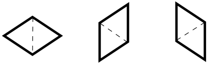

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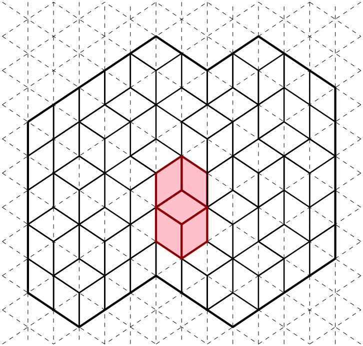

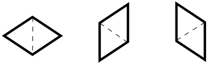

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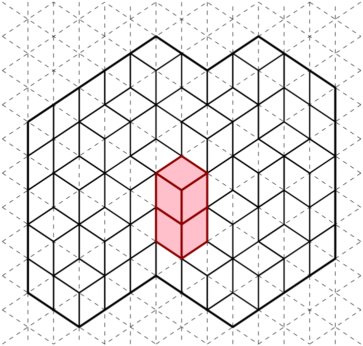
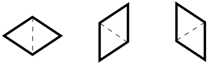
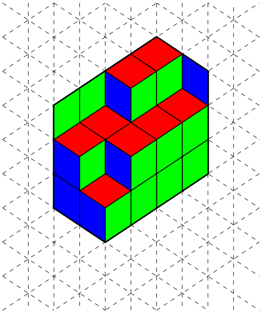


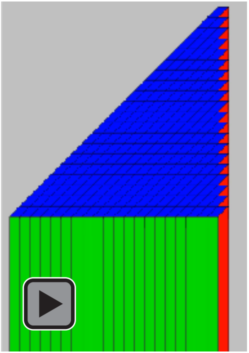


Random tilings of finite and infinite planar domains with **uniform Gibbs property**.

Random lozenge tilings: examples



1) Uniformly random tilings of a finite domain

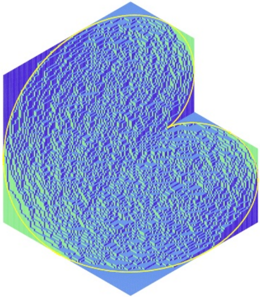


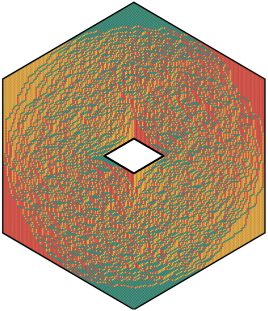
2) Surface growth (simulation of Patrik Ferrari)

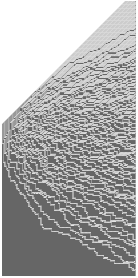
3) Path–measures in **Gelfand–Tsetlin graph** of asymptotic representation theory.

Random lozenge tilings: questions

(Kenyon–Okounkov) (Petrov) (Borodin-Ferrari)



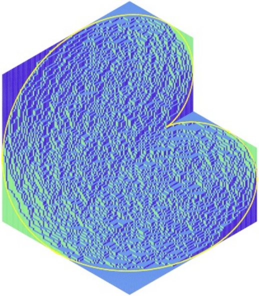




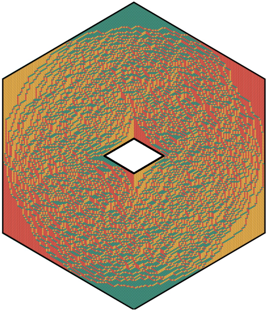
Asymptotics as mesh size $\to 0$ or size of the system $\to \infty$?

Random lozenge tilings: questions

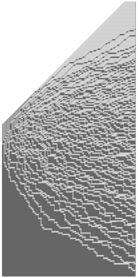
(Kenyon–Okounkov)



(Petrov)



(Borodin-Ferrari)



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Universality belief:

main features do not depend on exact specifications.

Random lozenge tilings: questions

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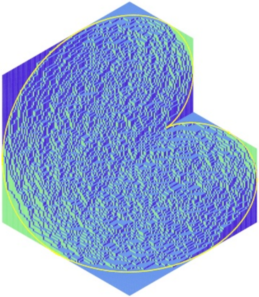
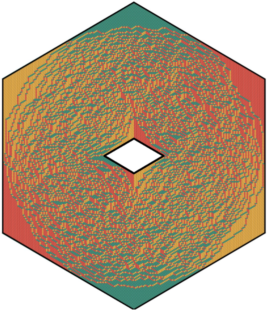


(Petrov)



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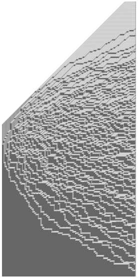


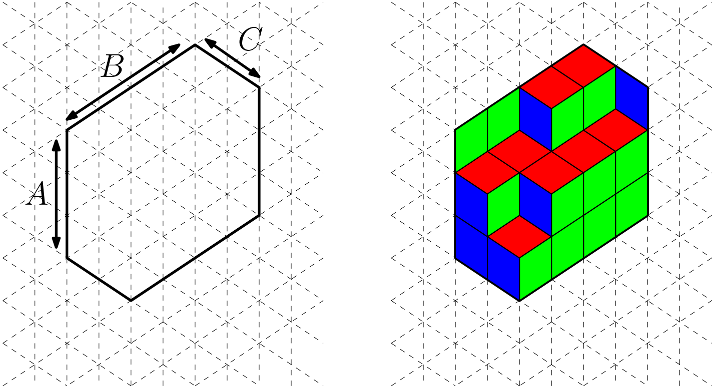
Asymptotics as mesh size $\to 0$ or size of the system $\to \infty$?

Universality belief:

main features do not depend on exact specifications.

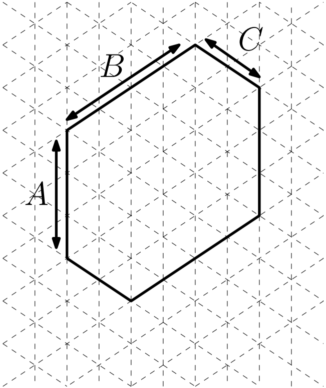
What are these features?

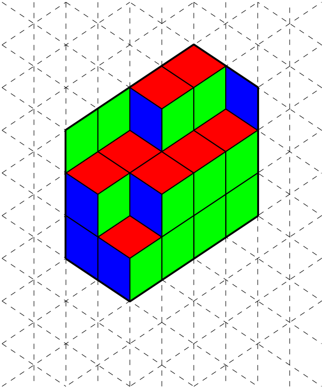
Random lozenge tilings: hexagon



Representative example: uniformly random lozenge tiling of $A \times B \times C$ hexagon.

Random lozenge tilings: hexagon





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Equivalently: decomposition of irreducible representation of $U(B+C)$ with signature $(A^B, 0^C)$.

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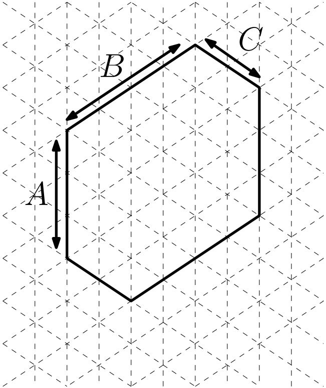
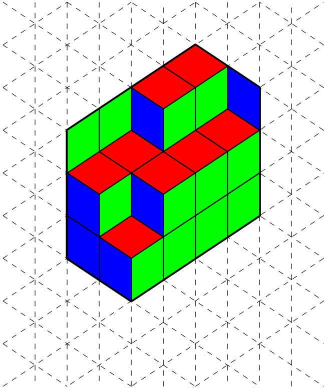


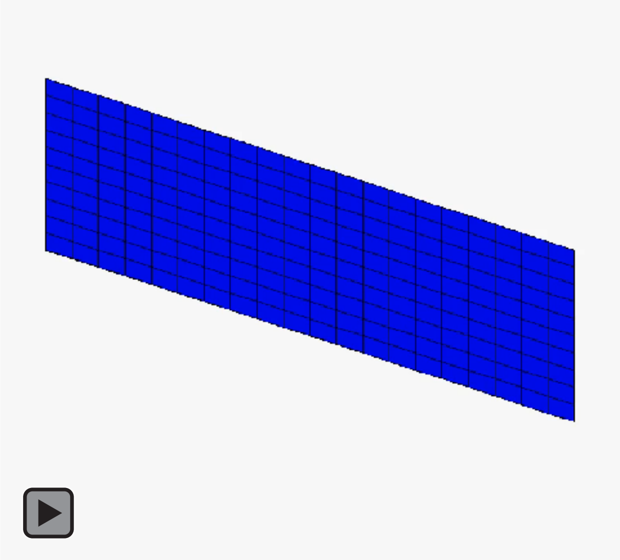


Representative example: uniformly random lozenge tiling of $A \times B \times C$ hexagon.

Equivalently: decomposition of irreducible representation of $U(B + C)$ with signature $(A^B, 0^C)$.

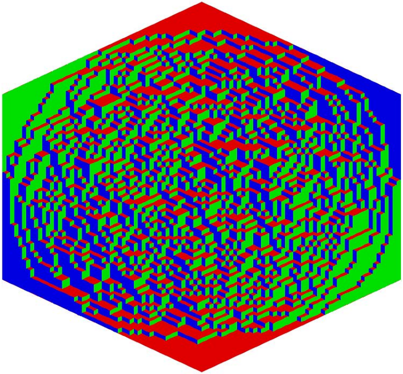
Equivalently: fixed time distribution of a $2d$-particle system.

Random lozenge tilings: hexagon



Shuffling algorithm (Borodin–Gorin)

Random lozenge tilings: features



Law of Large Numbers
(Cohn–Larsen–Propp)

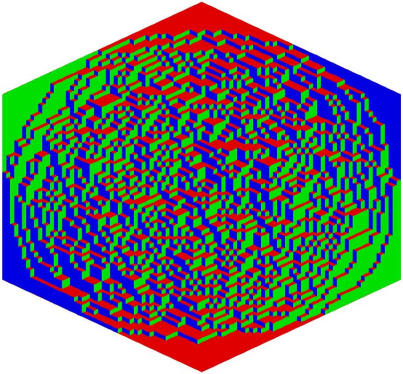
And for general domains
(Cohn–Kenyon–Propp)
(Kenyon–Okounkov)
(Bufetov–Gorin)

$$A = aL, B = bL, c = cL$$

$$L \to \infty$$

Theorem. Average proportions of three types of lozenges converge in probability to explicit deterministic functions of a point inside the hexagon. Equivalently, the rescaled height function $\frac{1}{L}H(Lx, Ly)$ converges to a deterministic limit shape.

Random lozenge tilings: features



Central Limit Theorem
(Kenyon), (Borodin-Ferrari), (Petrov), (Duits), (Bufetov–Gorin)

Liquid region: all types of lozenges are present

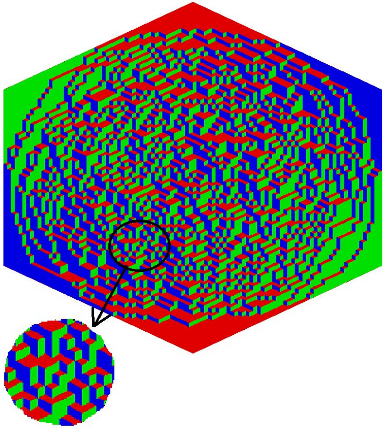
Frozen region: only one type

$$A = aL, B = bL, c = cL$$

$$L \to \infty$$

Theorem. The centered height function $H(Lx, Ly) - \mathbb{E}H(Lx, Ly)$ converges in the liquid region to a generalized Gaussian field, which can be identified with a pullback of the 2d **Gaussian Free Field**.

Random lozenge tilings: features



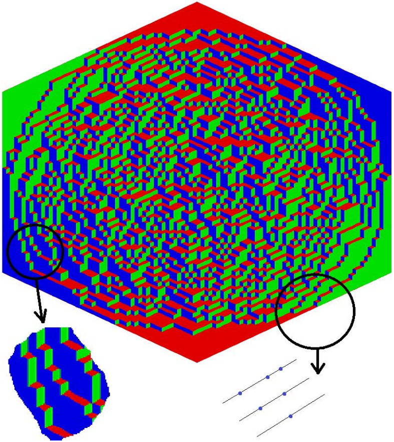
Bulk local limit
(Okounkov–Reshetikhin),
(Baik-Kriecherbauer-McLaughlin-Miller),
(Gorin), (Petrov)

$$A = aL, B = bL, c = cL$$

$$L \to \infty$$

Theorem. Near each point (xL, yL) the point process of lozenges converges to a (unique) **translation invariant ergodic Gibbs measure** on tilings of plane of the slope given by the limit shape.

Random lozenge tilings: features

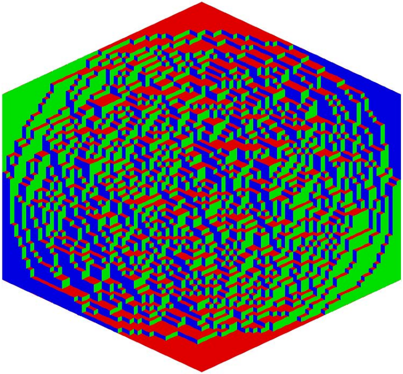


Edge local limit at a generic point
(Ferrari–Spohn),
(Baik-Kriecherbauer-McLaughlin-Miller),
(Petrov)

Edge local limit at a tangency point
(Johansson–Nordenstam),
(Okounkov–Reshetikhin),
(Gorin–Panova), (Novak)
$A = aL, B = bL, c = cL, L \to \infty$

Theorem. Near a generic (or tangency) point of the frozen boundary its fluctuations are governed by the **Airy line ensemble** (or **GUE–corners process**, respectfully)

Random lozenge tilings: features



1. Law of Large Numbers
2. Central Limit Theorem
3. Bulk local limits
4. Edge local limits at generic and tangency points

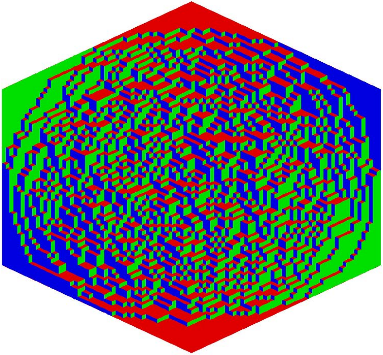
$$A = aL, B = bL, c = cL,$$

$$L \to \infty$$

Universality predicts that the same features should be present in generic random tilings models.

This is rigorously established only for the Law of Large Numbers.

Random lozenge tilings: what's new?



1. Law of Large Numbers
2. Central Limit Theorem
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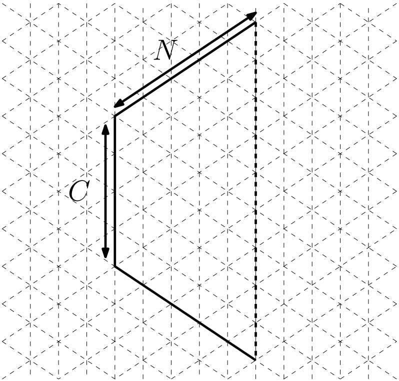
4. Edge local limits at generic and tangency points

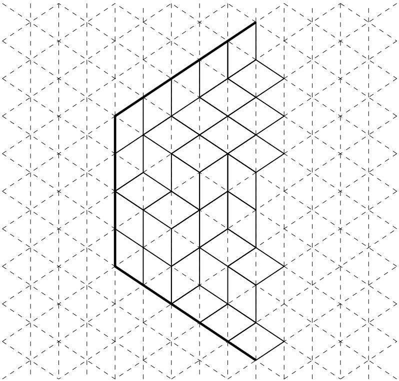
Conjecturally, should hold for generic random tilings

Today:

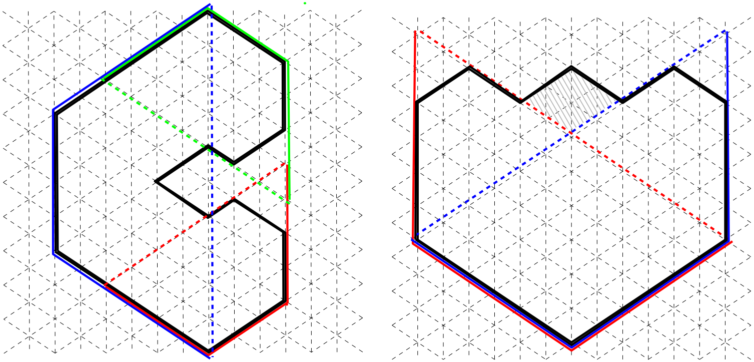
- Partial universality result for bulk local limits
- Description of limit shapes (in LLN) via quantized Voiculescu R–transform
- Universal Central Limit Theorem for "trapezoid domains".

Trapezoids





Bulk local limits: universality



Theorem. (G.-16) Let $\Omega(L)$ be a regularly growing sequence of domains. For any part of $\Omega(L)$ covered by a trapezoid, near any point in the liquid region in this part, the uniformly random lozenge tilings of $\Omega(L)$ converge locally as $L \to \infty$ to the **ergodic translation–invariant Gibbs measure** of the slope given by the limit shape.

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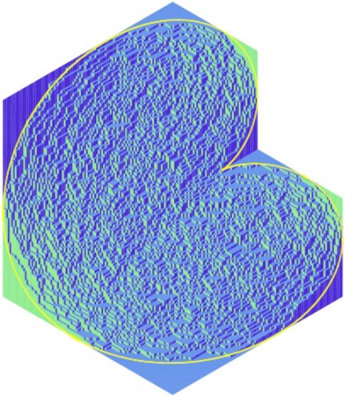
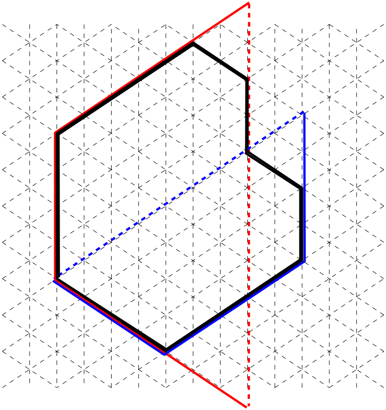


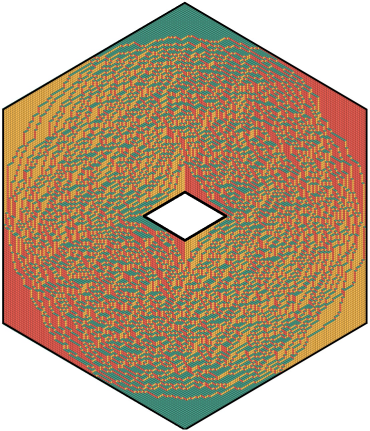
Picture from (Kenyon–Okounkov)



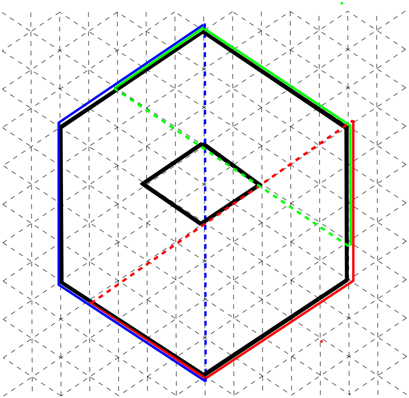
Bulk limits were not known for this domain before

Many domains are **completely covered** by trapezoids and therefore the conjectural bulk universality is now a **theorem** for them.

Bulk local limits: universality



Simulation by L. Petrov



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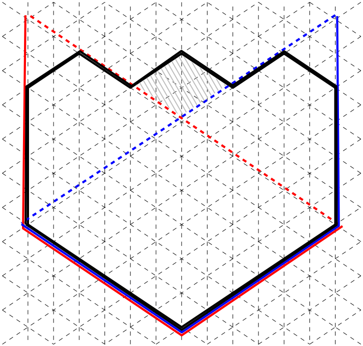

Some domains are only **partially** covered by trapezoids.

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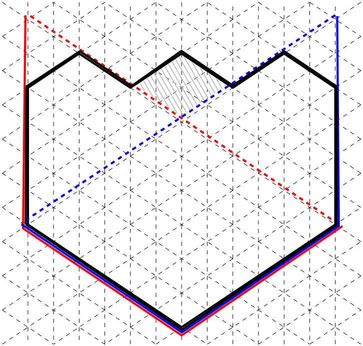

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The theorem also holds for more general **Gibbs measures** on tilings covered by trapezoids ($2+1$–dimensional interacting particle systems, asymptotic representation theory).

Bulk local limits: universality

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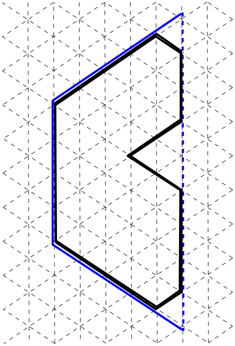
Previous results:



(Petrov-12)
Local bulk limits for **polygons** covered by **single** trapezoid.

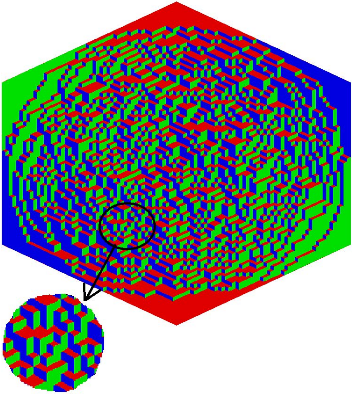
(Kenyon–04)
Local bulk limits for a class of domains with **no** straight boundaries.

(Borodin–Kuan–07)
Local bulk limits for Gibbs measures arising from **characters of $U(\infty)$**

(Okounkov–Reshetikhin–01)
Local bulk limits for Schur processes

Ergodic translation–invariant Gibbs measures

Theorem. ... near any point in the liquid region as $L \to \infty$ we observe an **ergodic translation–invariant Gibbs measure**.



Theorem. (Sheffield). For each slope, i.e. average proportions of lozenges $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ there is a unique e.t.-i.G. measure.

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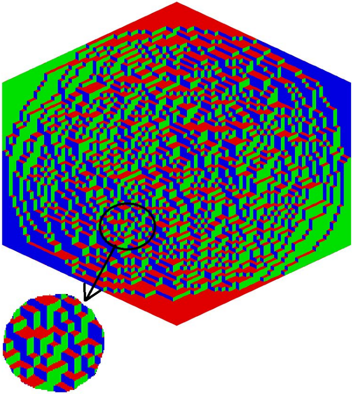

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Description. (Cohn–Kenyon–Propp, Okounkov–Reshetikhin) Red lozenges in e.t.-i.G. measure form a **determinantal point process** with incomplete Beta kernel.

$$\rho_k((x_1, n_1), \ldots, (x_k, n_k)) = \det_{i,j=1}^{n} \left[\frac{1}{2\pi\mathbf{i}} \int_{\bar{\xi}}^{\xi} w^{x_j - x_i - 1} (1 - w)^{n_j - n_i} dw \right]$$

contour intersects $(0, 1)$ when $n_j \geq n_i$ and $(-\infty, 0)$ otherwise.

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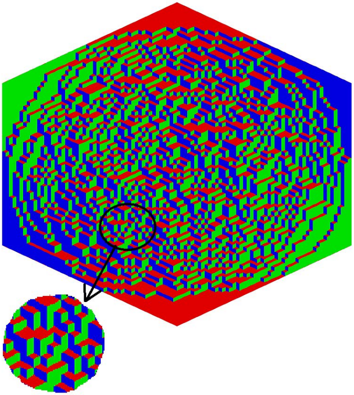

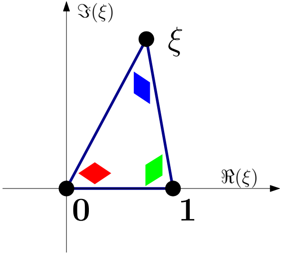

Description. (Cohn–Kenyon–Propp, Okounkov–Reshetikhin) Red lozenges in e.t.-i.G. measure form a **determinantal point process**

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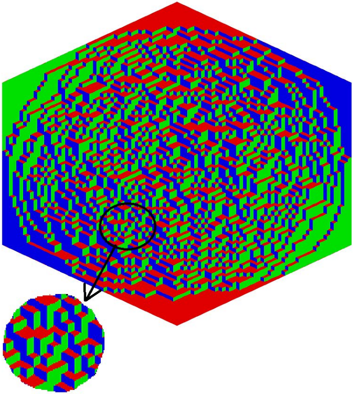
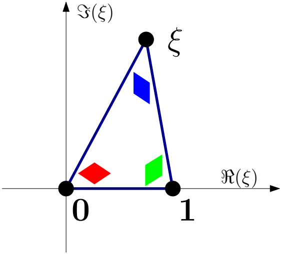
Local vs global meanings of slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$





Meaning 1: It describes the **e.t.-i.G. measure** in the bulk

$$\frac{1}{2\pi \mathbf{i}} \int_{\bar{\xi}}^{\xi} w^{x_j - x_i - 1}(1 - w)^{n_j - n_i} dw$$

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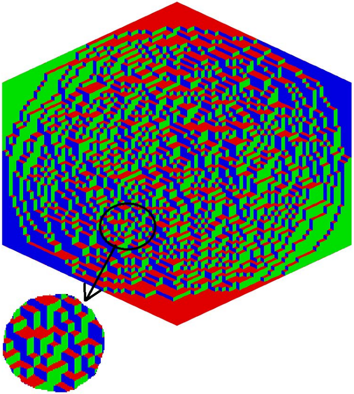
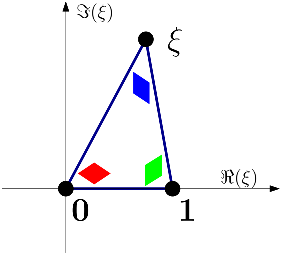




Meaning 1: It describes the **e.t.-i.G. measure** in the bulk

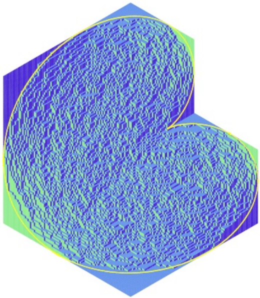
$$\frac{1}{2\pi \mathbf{i}} \int_{\bar{\xi}}^{\xi} w^{x_j - x_i - 1} (1 - w)^{n_j - n_i} dw$$

Meaning 2: Law of Large Numbers. Normalized lozenge counts inside a subdomain $\mathcal{D}$ converge to **deterministic vector**

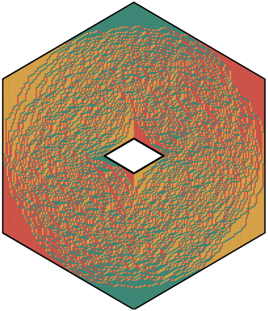
$$\left(\int_{\mathcal{D}} p^{\lozenge}(\mathbf{x}, \eta) d\mathbf{x} d\eta, \int_{\mathcal{D}} p^{\lozenge}(\mathbf{x}, \eta) d\mathbf{x} d\eta, \int_{\mathcal{D}} p^{\lozenge}(\mathbf{x}, \eta) d\mathbf{x} d\eta \right)$$

How to find slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$?

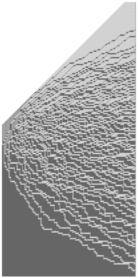
(Kenyon–Okounkov)



(Petrov)

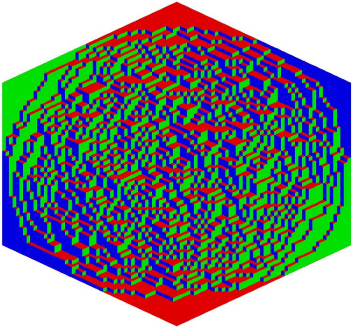


(Borodin-Ferrari)



Both local bulk limits and global law of large numbers are parameterized by the **same** position–dependent slope which one needs to find.

How to find slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$?

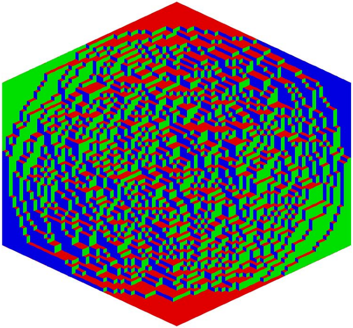


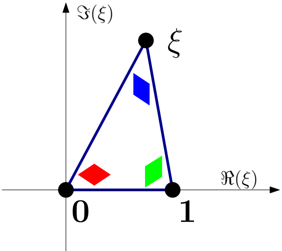
Method 1. (Cohn–Kenyon–Propp)
Solve **variational problem** for tilings of a generic domain Ω.

$$\int_{\Omega} \sigma \left(p^{\lozenge}(\mathbf{x}, \eta), p^{\lozenge}(\mathbf{x}, \eta), p^{\lozenge}(\mathbf{x}, \eta) \right) d\mathbf{x} d\eta \longrightarrow \max$$

$\sigma(\cdot, \cdot, \cdot)$ is an explicitly known **entropy** (or surface tension)

How to find slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$?





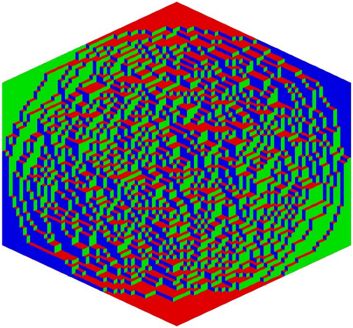
Method 1. (Cohn–Kenyon–Propp)
Solve **variational problem** for tilings of a generic domain.

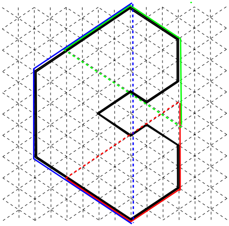
Method 2. (Kenyon–Okounkov)
For **simply–connected polygons** the solution is found through an algebraic procedure.

$$Q(\xi, 1-\xi) = \mathbf{x}\xi + \boldsymbol{\eta}(1-\xi)$$

Q is a **polynomial** uniquely defined by a set of algebraic conditions such as degree and tangency to polygon's sides.

How to find slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$?



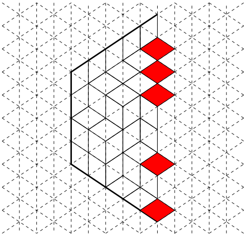


Method 1. (Cohn–Kenyon–Propp)
Solve **variational problem** for tilings of a generic domain.

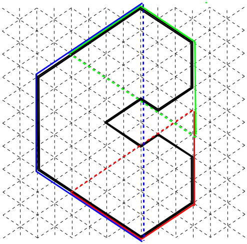
Method 2. (Kenyon–Okounkov)
For **simply–connected polygons** the solution is found through an algebraic procedure.

Method 3. (Bufetov–Gorin-13)
For **trapezoids** the solution is found through a quantization of the Voiculescu R–transform from free probability.

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ for trapezoids.



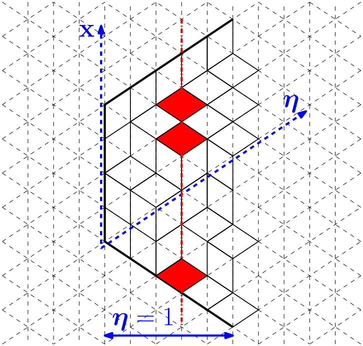
Various origins for the measure on tilings of trapezoid, e.g.:



Setup. We know the asymptotic profile of $p^{\lozenge}$ along **the right boundary** of a trapezoid. The distribution of tilings of trapezoid is conditionally uniform given the right boundary (which might be random).

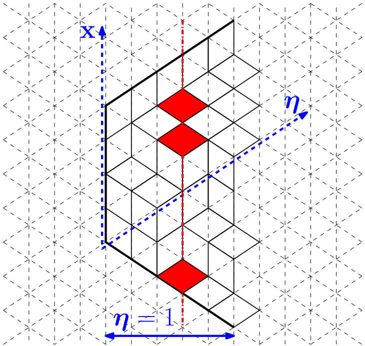
Question. How to find $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ inside the trapezoid?

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ for trapezoids.



$\mu[\eta]$, $0 < \eta \leq 1$ is a probability measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\diamond}(\eta\mathbf{x} - \eta, \eta)$

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ for trapezoids.



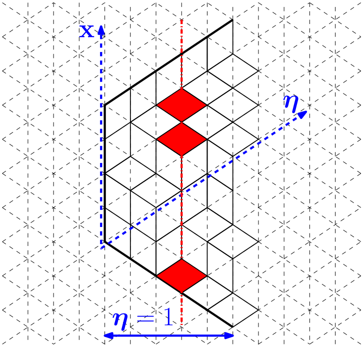
$\mu[\eta]$, $0 < \eta \le 1$ is a **probability** measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\lozenge}(\eta\mathbf{x} - \eta, \eta)$

$$E_\mu(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z-x}\mu(dx)\right).$$

$$R_\mu(z) = E_\mu^{(-1)}(z) - \frac{z}{z-1},$$

Deformation (quantization) of the **Voiculescu R transform** from the free probability theory

Slope $(p^{\lozenge_1}, p^{\lozenge_2}, p^{\lozenge})$ for trapezoids.



$\mu[\eta]$, $0 < \eta \le 1$ is a **probability** measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\lozenge}(\eta \mathbf{x} - \eta, \eta)$

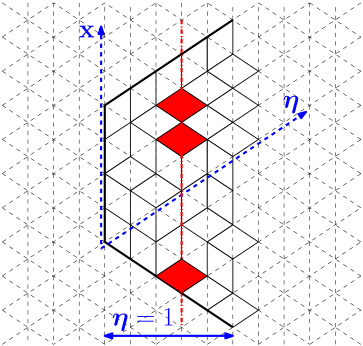
$$E_\mu(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z-x} \mu(dx) \right).$$

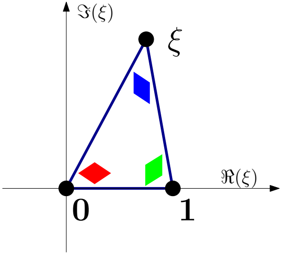
$$R_\mu(z) = E_\mu^{(-1)}(z) - \frac{z}{z-1},$$

Theorem. (Bufetov–Gorin-13) If $(p^{\lozenge_1}, p^{\lozenge_2}, p^{\lozenge})$ describes the Law of Large Numbers for Gibbs measures on tilings of trapezoids, then

$$R_{\mu[\eta]}(z) = \frac{1}{\eta} R_{\mu[1]}(z).$$

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\diamond})$ for trapezoids.





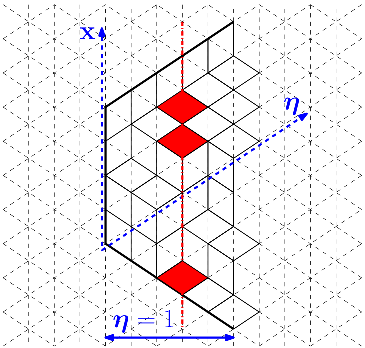
$\mu[\boldsymbol{\eta}]$, $0 < \boldsymbol{\eta} \leq 1$ is a probability measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\diamond}(\boldsymbol{\eta}\mathbf{x} - \boldsymbol{\eta}, \boldsymbol{\eta})$

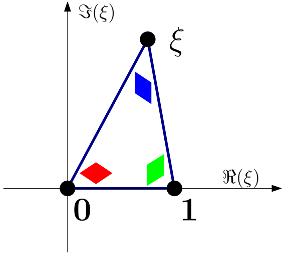
$$E_{\mu}(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z - x}\mu(dx)\right).$$

Corollary. (Bufetov–Gorin-13) For tilings of trapezoids also

$$\xi(\boldsymbol{\eta}\mathbf{x} - \boldsymbol{\eta}, \boldsymbol{\eta}) = E_{\mu[\eta]}(\mathbf{x} - 0\mathbf{i})$$

Slope $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ for trapezoids.





$\mu[\eta]$, $0 < \eta \leq 1$ is a probability measure on $\mathbb{R}$ with density at a point $\mathbf{x}$ equal to $p^{\lozenge}(\eta\mathbf{x} - \eta, \eta)$

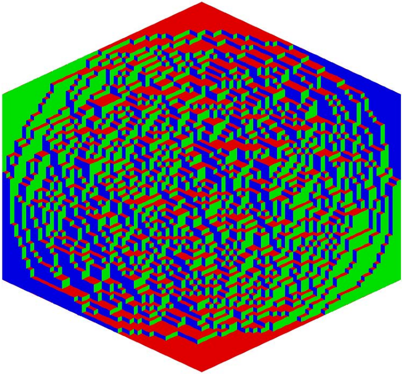
$$E_\mu(z) = \exp\left(\int_{\mathbb{R}} \frac{1}{z - x}\mu(dx)\right).$$

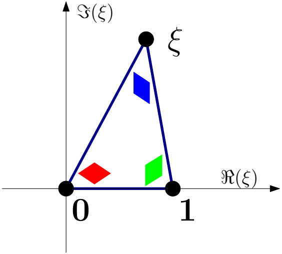
Corollary. (Bufetov–Gorin-13)
For tilings of trapezoids also

$$\xi(\eta\mathbf{x} - \eta, \eta) = E_{\mu[\eta]}\left(\mathbf{x} - 0\mathbf{i}\right)$$

Angle of red lozenge is clear.
Others are very mysterious.

Complex structure and CLT

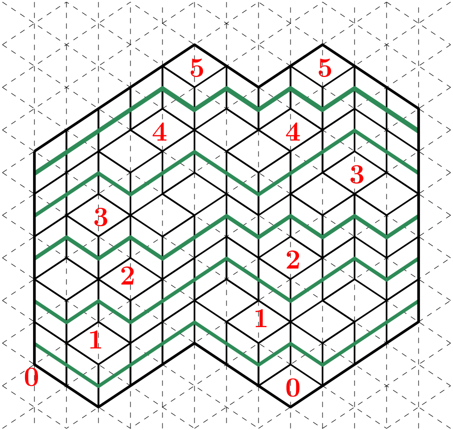




The proportions $(p^{\lozenge}, p^{\lozenge}, p^{\lozenge})$ define a **complex structure** $\xi(\mathbf{x}, \eta)$ inside the liquid region.

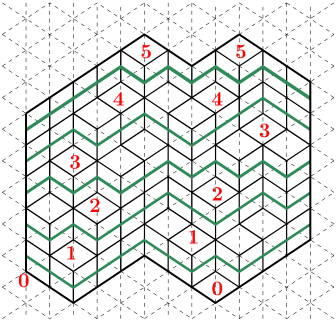
Which arises in the Central Limit Theorem for fluctuations of the **height function**.

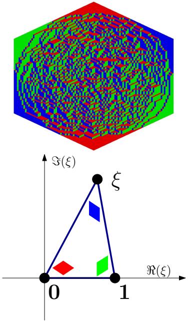
Height function



Tiling defines stepped surface, parameterized by height function.
LLN: rescaled height function converges to a limit shape.
CLT: what are the fluctuations?

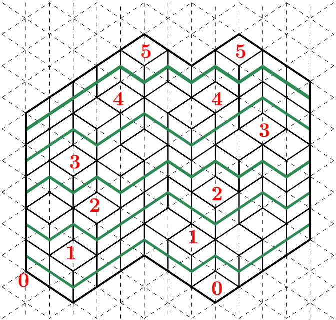
Height function

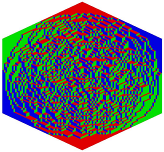


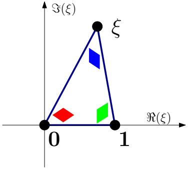


Conjecture. (Kenyon–Okounkov) For any regularly growing simply-connected domains $\Omega(L)$, the centered height functions of uniformly random tilings $H_L(L\mathbf{x}, L\eta) - \mathbb{E}H_L(L\mathbf{x}, L\eta)$ converge in the liquid region to the **Gaussian Free Field** with respect to the complex structure ξ and with Dirichlet boundary conditions.

Height function







Conjecture. $H_L(L\mathbf{x}, L\eta) - \mathbb{E}H_L(L\mathbf{x}, L\eta)$ converge in the liquid region to the **Gaussian Free Field** with respect to the complex structure ξ and with Dirichlet boundary conditions.

- (Kenyon–04) proved for domains with no frozen regions
- (Petrov–12) proved for polygons covered by a single trapezoid
- (Bufetov–Gorin-16) extend by a different method to arbitrary trapezoids with **deterministic** boundary conditions.

Gaussian Free Field

Definition. The Gaussian Free Field (with Dirichlet boundary conditions) in the upper halfplane $\mathbb{U}$ — is a generalized centered Gaussian random field $\mathcal{F}$ on $\mathbb{U}$ with covariance

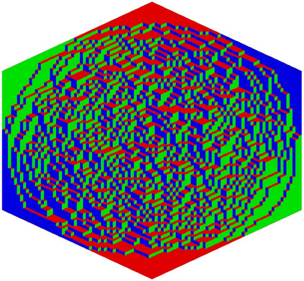
$$\mathbb{E}(\mathcal{F}(z)\mathcal{F}(w)) = -\frac{1}{2\pi} \ln \left| \frac{z-w}{z-\overline{w}} \right|, \quad z, w \in \mathbb{U}$$

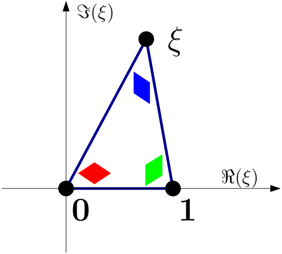
Equivalently, for any smooth compactly supported g_1, g_2 on $\mathbb{U}$,

$$\mathbb{E}\left[\Big(\int_{\mathbb{U}} g_1(u)\mathcal{F}(u)du \Big) \cdot \Big(\int_{\mathbb{U}} g_2(u)\mathcal{F}(u)du \Big) \right] = \int_{\mathbb{U}} g_1(u)\Delta^{-1} g_2(u)du.$$

GFF is a conformally invariant $2d$–analogue of Brownian motion.

Uniformization of the complex structure





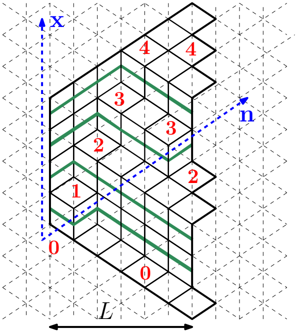
$\xi(\mathbf{x}, \eta)$ turns the liquid region into a simply connected complex Riemann surface.

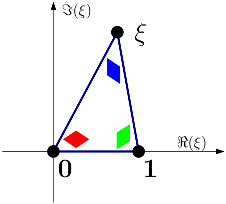
$(\mathbf{x}, \eta) \to z(\mathbf{x}, \eta)$ is a **conformal uniformization map** to the upper half–plane $\mathbb{H}$.

(unique up to 3 parameters, but GFF is invariant)

Definition. The Gaussian Free Field **in the liquid region** with Dirichlet boundary conditions is the pullback of GFF in $\mathbb{H}$ with respect to the map $(\mathbf{x}, \eta) \to z(\mathbf{x}, \eta)$.

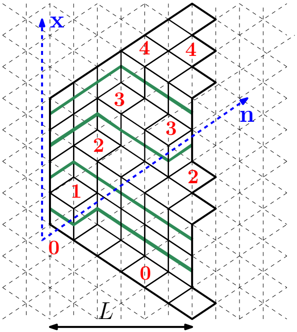
CLT for trapezoids

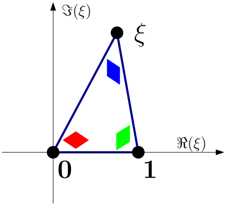




Theorem. (Bufetov-Gorin-16) Take a sequence of trapezoids with fixed deterministic right boundaries and such that the rescaled height functions along the boundary approach a limit profile. The centered height functions $H_L(L\mathbf{x}, L\eta) - \mathbb{E}H_L(L\mathbf{x}, L\eta)$ converge to the Gaussian Free Field in the liquid region with respect to the complex structure ξ.

CLT for trapezoids

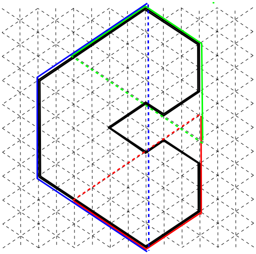


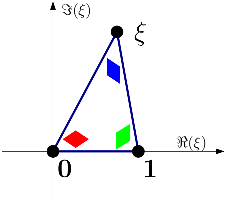


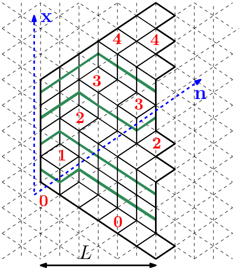
$$z(\mathbf{x}, \eta) = \frac{1-\eta}{1-\xi(\mathbf{x}, \eta)} + \mathbf{x} + \eta - 1$$

Theorem. (Bufetov-Gorin-16) Take a sequence of trapezoids with fixed deterministic right boundaries and such that the rescaled height functions along the boundary approach a limit profile. The centered height functions $H_L(L\mathbf{x}, L\eta) - \mathbb{E}H_L(L\mathbf{x}, L\eta)$ converge to the Gaussian Free Field in the liquid region with respect to the complex structure ξ. The uniformization map is explicit.

Ingredients of proofs



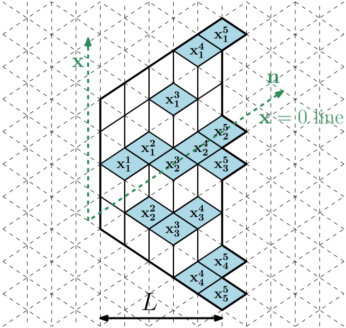




- Partial universality result for bulk local limits
- Description of limit shapes (in LLN) via quantized Voiculescu R–transform
- Universal Central Limit Theorem for "trapezoid domains".

We use two key approaches to random tilings of trapezoids.

Ingredients of proofs



For L–tuple $(\mathbf{t}_1 > \mathbf{t}_2 > \dots \mathbf{t}_L)$, let $\{x_i^j\}$, $1 \le i \le j \le L$ be horizontal lozenges of uniformly random lozenge tiling with positions $\mathbf{t}$ on the right boundary

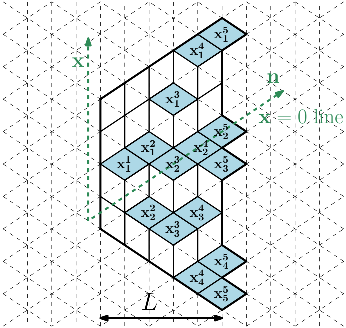
Theorem. (Petrov-2012) For any collection of distinct pairs $(x(1), n(1)), \dots, (x(k), n(k))$

$$\mathrm{P}\left[x(i) \in \{x_1^{n(i)}, x_2^{n(i)}, \dots, x_j^{n(i)}\},\ i = 1, \dots, k\right] = \det_{i,j=1}^{k} \left[K(x(i), n(i); x(j), n(j))\right]$$

$$K(x_1, n_1; x_2, n_2) = -\mathbf{1}_{n_2<n_1}\mathbf{1}_{x_2 \le x_1} \frac{(x_1 - x_2 + 1)_{n_1-n_2-1}}{(n_1 - n_2 - 1)!} + \frac{(L - n_1)!}{(L - n_2 - 1)!}$$

$$\times \frac{1}{(2\pi \mathbf{i})^2} \oint_{\mathcal{C}(x_2,\dots,\mathbf{t}_1-1)} dz \oint_{\mathcal{C}(\infty)} dw \frac{(z - x_2 + 1)_{L-n_2-1}}{(w - x_1)_{L-n_1+1}} \frac{1}{w - z} \prod_{r=1}^{L} \frac{w - \mathbf{t}_r}{z - \mathbf{t}_r},$$

Ingredients of proofs



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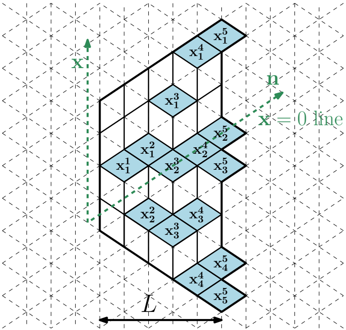
Observation. (G.-16) the bulk limit of $K(\cdot)$ depends only on the asymptotic limit shape of $\mathbf{t}$. This allows to pass from deterministic to random $\mathbf{t}$ and prove bulk universality.

Ingredients of proofs



For L–tuple $(\mathbf{t}_1 > \mathbf{t}_2 > \dots \mathbf{t}_L)$, let $\{x_i^j\}$, $1 \le i \le j \le L$ be horizontal lozenges of uniformly random lozenge tiling with positions $\mathbf{t}$ on the right boundary

Schur generating function of a vertical section k

$$\sum_{\lambda} \mathrm{P}\left[(x_1^k, x_2^k, \dots, x_k^k) = \lambda\right] \frac{s_\lambda(u_1, \dots, u_k)}{s_\lambda(1, \dots, 1)} = \frac{s_{\mathbf{t}}(u_1, \dots, u_k, 1^{L-k})}{s_{\mathbf{t}}(1^L)}$$

$$s_\lambda(u_1, \dots, u_k) = \frac{\det_{i,j=1}^{k}[u_i^{\lambda_j}]}{\prod_{i<j}(u_i - u_j)}, \quad \lambda = (\lambda_1, \dots, \lambda_k).$$

Ingredients of proofs

Schur generating function of a vertical section k

$$\sum_{\lambda} \mathrm{P}\left[(x_1^k, x_2^k, \dots, x_k^k) = \lambda\right] \frac{s_\lambda(u_1, \dots, u_k)}{s_\lambda(1, \dots, 1)} = \frac{s_{\mathbf{t}}(u_1, \dots, u_k, 1^{L-k})}{s_{\mathbf{t}}(1^L)}$$

$$s_\lambda(u_1, \dots, u_k) = \frac{\det_{i,j=1}^{k}[u_i^{\lambda_j}]}{\prod_{i<j}(u_i - u_j)}, \qquad \lambda = (\lambda_1, \dots, \lambda_k).$$

Apply $$\prod_{i<j}(u_i - u_j)^{-1} \left(\sum_{i=1}^{k} \left(u_i \frac{\partial}{\partial u_i} \right)^m \right)^r \prod_{i<j}(u_i - u_j)$$
and set $u_1 = \cdots = u_k = 1$ to get

$$\mathbb{E}\left[\left((x_1^k)^m + (x_2^k)^m + \cdots + (x_k^k)^m \right)^r \right].$$

Ingredients of proofs

Schur generating function of a vertical section k

$$\sum_{\lambda} \mathrm{P}\left[(x_1^k, x_2^k, \dots, x_k^k) = \lambda\right] \frac{s_\lambda(u_1, \dots, u_k)}{s_\lambda(1, \dots, 1)} = \frac{s_{\mathbf{t}}(u_1, \dots, u_k, 1^{L-k})}{s_{\mathbf{t}}(1^L)}$$

$$s_\lambda(u_1, \dots, u_k) = \frac{\det_{i,j=1}^{k}[u_i^{\lambda_j}]}{\prod_{i<j}(u_i - u_j)}, \quad \lambda = (\lambda_1, \dots, \lambda_k).$$

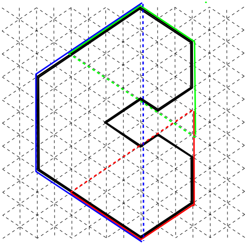
Apply $\displaystyle\prod_{i<j}(u_i - u_j)^{-1} \left(\sum_{i=1}^{k} \left(u_i \frac{\partial}{\partial u_i} \right)^m \right)^r \prod_{i<j}(u_i - u_j)$

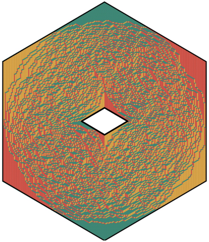
and set $u_1 = \dots = u_k = 1$ to get

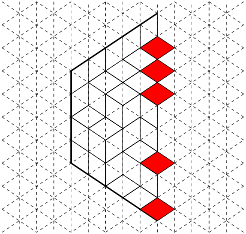
$$\mathbb{E}\left[\left((x_1^k)^m + (x_2^k)^m + \dots + (x_k^k)^m \right)^r \right].$$

Asymptotics of Schur functions (Gorin-Panova-12) + combinatorial analysis turns this observation into LLN and CLT for trapezoids.

Summary







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- Limit shapes (in LLN) via quantized Voiculescu R–transform
- Universal Central Limit Theorem for “trapezoid domains” leading to $2d$ Gaussian Free Field via uniformization map.

Key tools: double contour integral expression for the correlation kernel, asymptotic of Schur polynomials, differential operators acting on Schur generating functions.

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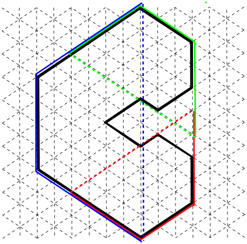

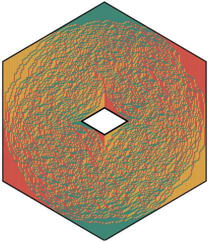

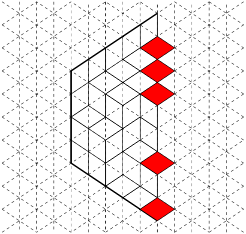

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How do we extend **universality** further?