Almost Everywhere Convergence and the Work of Alexandra Bellow

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Let (X, Σ, m) be a complete non-atomic probability space.

$\tau : X \to X$ invertible, measure preserving, ergodic ($\tau^{-1}(A) = A$ implies $A = X$ or $A = \phi$).

Pointwise Ergodic Theorem, (Birkoff, 1931):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\tau^k x) = \int_X f dm$$

a.e. for all $f \in L^1(X)$.

Alexandra Bellow: There is an $f \in L^1(X)$ such that the averages

$$\frac{1}{n}\sum_{k=1}^{n} f(\tau^{2^k} x)$$

fail to converge a.e.

More generally, the sequence $\{2^k\}$ can be replaced with any lacunary sequence, i.e. any sequence such that there is a constant $c > 1$ and

$$\frac{n_{k+1}}{n_k} \geq c$$

for all k.

A. P. Calderón, *Ergodic theory and translation invariant operators*, Proc. Nat. Acad. Sci. **59** (1968)

Are there any sequences of density zero that are good averaging sequences in the sense that the averages converge a.e. for all $f \in L^1$.

Alexandra Bellow and Victor Losert introduced a class of sequences called block sequences. The idea was to fix an increasing sequence of positive integers, $\{n_k\}$, and a second sequence $\{\ell_k\}$ with $n_{k+1} > n_k + \ell_k$. Let B_k denote the interval $[n_k, n_k + \ell_k - 1]$, and let $\Omega = \bigcup_{k=1}^{\infty} B_k$.

Define

$$A_N f(x) = \frac{1}{\#\{j : j \leq N, j \in \Omega\}} \sum_{j \in \Omega, j \leq N} f(\tau^j x)$$

Bellow and Losert were able to show that if the $\{n_k\}$ and $\{\ell_k\}$ grow very rapidly, then convergence occurs for all $f \in L^1(X)$.

With $n_k = 2^{2^k}$ and $\ell_k = \sqrt{n_k}$, almost everywhere convergence occurs.

(Alexandra Bellow, Two Problems, Proc. of the Oberwolfach Conference on Measure Theory, June 1981)

Does

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\tau^{k^2} x)$$

exist a.e. for all $f \in L^1$?

She also asked the same question for averages along the sequence of primes.

At the Calderón-Zygmund seminear, Rubio de Francia talked about joint work with Duoandikoetxea. He discussed a technique of proving maximal functions inequalities using Fourier Transforms.

For practice we decided to try the technique on the block sequences, where we already knew convergence.

We looked at the case $n_k = 2^{2^k}$ and $\ell_k = \sqrt{n_k}$ and considered the averages

$$M_k f(x) = \frac{1}{\ell_k} \sum_{j=n_k}^{n_k+\ell_k-1} f(\tau^j x).$$

Using Fourier transforms techniques, we were able to show that the averages $\{M_k f\}$ converge a.e. for all $f \in L^p$, $p > 1$. Because of the methods used we were not able to obtain results for $f \in L^1$.

Earlier Akcoglu and Del Junco had shown that if $n_k = k$ and $\ell_k = \sqrt{k}$ then the averages $M_k f(x)$ can diverge even for $f \in L^\infty$.

Alexandra constructed examples of sequences such that given $p_0 > 1$, averages along that sequence converged a.e. for all $f \in L^p$, $p > p_0$ and diverged for some $f \in L^p$ for any $p \leq p_0$.

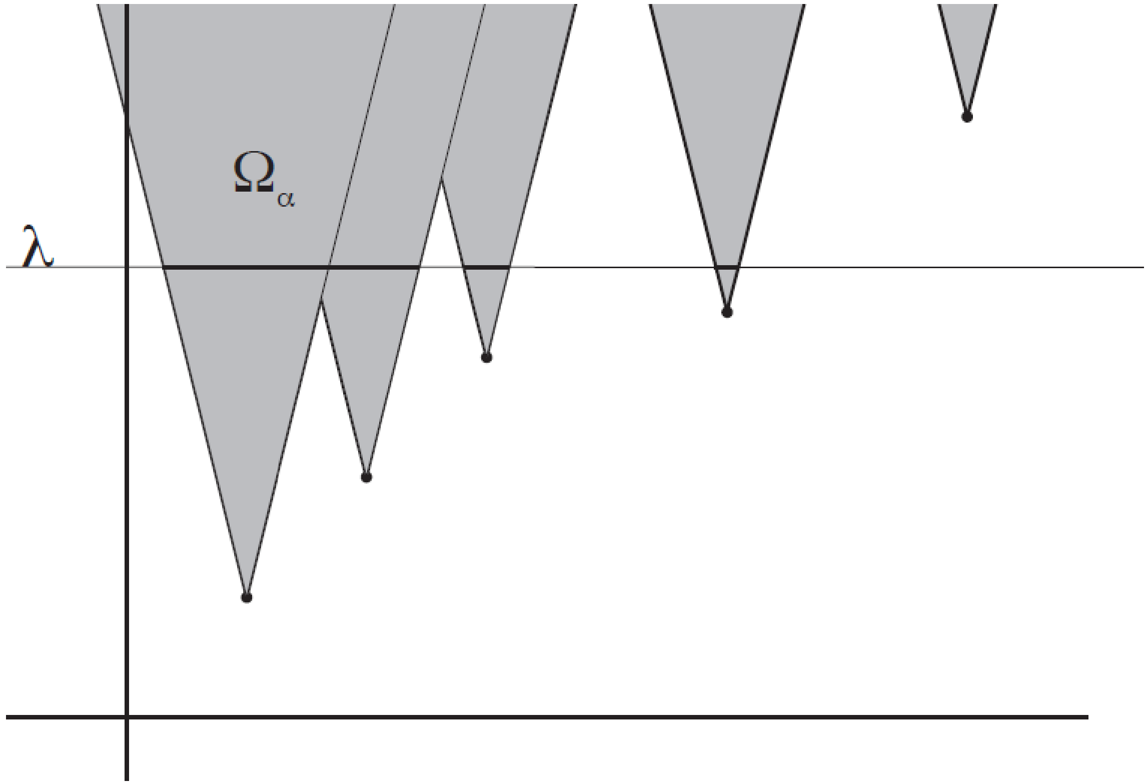
Later Karin Rhinehold extended this to include other cases.

J. Sueiro, "A note on maximal operators of Hardy-Littlewood type"

Let $\{n_k\}$ be an increasing sequence, and $\{\ell_k\}$ be a second sequence of positive integers. Let

$$M_k f(x) = \frac{1}{\ell_k} \sum_{j=n_k}^{n_k+\ell_k-1} f(\tau^j x).$$

Let $\Omega = \{(n_k, \ell_k) : k = 1, 2, \cdots\}$ Fix an angle α and let Ω_α be the union of all cones with base at a point of the form (n_k, ℓ_k) and angle α. Let $\Omega_\alpha(s)$ denote the cross section of Ω_α at height s.



Theorem 0.1. *If there is a constant* $A > 0$ *such that* $|\Omega_\alpha(\lambda)| \le A\lambda$ *then the averages* $M_k f(x)$ *converge a.e. for all* $f \in L^1$. *Further, if no such constant exists then the averages fail to converge even for some function* $f \in L^\infty$.

Strong sweeping out was a concept formalized by Alexandra Bellow. A sequence of operators $\{T_k\}$ has the strong sweeping out property if given $\epsilon > 0$ there is a set A with $m(A) < \epsilon$ but such that $\liminf T_k \chi_A(x) = 0$ a.e. and $\limsup T_k \chi_A(x) = 1$ a.e.

Theorem 0.2. *Let*

$$B_k f(x) = \frac{1}{2^{2\ell_k}} \sum_{j=-\ell_k}^{\ell_k} \binom{2\ell_k}{\ell_k + j} f(\tau^{n_k+j} x.)$$

The averages $B_k f(x)$ converge a.e. if the cone condition is satisfied with the sequence $\{(n_k, \sqrt{\ell_k})\}_{k=1}^{\infty}$ and diverge for some $f \in L^\infty$ if the cone condition for this sequence fails.

As an example, let $n_k = k$ and $\ell_k = k$. Then the averages

$$\frac{1}{2^{2k}} \sum_{j=-k}^{k} \binom{2k}{k+j} f(\tau^{k+j} x)$$

will diverge for some $f \in L^\infty$.

If $n_k = 2^{2^k}$ and $\ell_k = 2^{2^k}$ then the averages converge a.e. for all $f \in L^1(X)$.

More generally, let μ be a measure on the integers, and let

$$\mu^n(k) = \sum_{j=-\infty}^{\infty} \mu(j)\mu^{n-1}(k-j).$$

That is μ^n denotes the measure μ convolved with itself n times.

If

$$\sum_{k=-\infty}^{\infty} k\mu(k) \neq 0$$

and

$$\sum_{k=-\infty}^{\infty} k^2\mu(k) < \infty$$

then the averages

$$\mu^n f(x) = \sum_{k=-\infty}^{\infty} \mu^n(k) f(\tau^k x)$$

have the strong sweeping out property.

On the other hand, if the support of μ is not contained in a proper subgroup of Z, $\sum_{k=-\infty}^{\infty} k\mu(k) = 0$ and $\sum_{k=-\infty}^{\infty} k^2\mu(k) < \infty$ then the averages $\mu^n f(x)$ converge a.e. for all $f \in L^p$, $p > 1$.

The case $p = 1$ was answered under the stronger assumption $\sum_{k=-\infty}^{\infty} k^{2+\delta}\mu(k) < \infty$, $\delta \geq \frac{\sqrt{17}-3}{2}$, by Karin Reinhold-Larsson in 1993.

Bellow and Calderón established a.e. convergence in the $p = 1$ case under the original assumptions of finite second moment.

Congratulations Alexandra on a great career !!

Thanks for being
a great friend!!