Random Walks on Infinite Discrete Groups
Problem Set 1

A. Subadditivity

Exercise 1. Show that for any subadditive sequence a_n,

$$\alpha := \lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n} \geq -\infty.$$

Exercise 2. It is easily checked that the exponential growth rate of $\mathbb{Z}^d$ is 0, while the exponential growth rate of the tree $\mathbb{T}_d$ is $\log(d-1)$. What is the exponential growth rate of the lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2^{\mathbb{Z}}$? NOTE: There are two natural generating sets for the lamplighter group. The exponential growth rate will depend on which generating set is used.

Exercise 3. (A) Show that for a *symmetric* random walk, if the spectral radius is strictly less than 1 then the Avez entropy is strictly positive. (B) Show that for the asymmetric p, q random walk on the integers $\mathbb{Z}$ (i.e., the random walk that at each step jumps $+1$ with probability p and -1 with probability q) the spectral radius is less than 1 but the Avez entropy is 0. HINT: For (B) you will want to use the *Hoeffding inequality*. See the Appendix of the Lecture Notes for the statement.

Exercise 4. (A) Show that a random walk has positive entropy h if and only if its lazy version also has positive entropy. (B) Show that a random walk has positive speed if and only if its lazy version also has positive speed.

Exercise 5. Show that the speed and Avez entropy of a random walk and the exponential growth rate of the ambient group satisfy the basic inequality

$$h \leq \beta \ell.$$

One of the interesting open problems in the subject is to characterize those groups for which equality can hold in this relation. HINT: Show that the *Shannon entropy* of a probability measure on a finite set $\mathcal{Y}$ is maximal for the uniform distribution. Note: The Shannon entropy of a probability measure ν on $\mathcal{Y}$ is defined (cf. section **??** below) by

$$H(\mu) := -\sum_{y\in\mathcal{Y}} \nu(y) \log \nu(y).$$

Exercise 6. Assume that $P\{X_1 = 1\} > 0$. This assumption ensures that if $\mu^{*n}(y) > 0$ then $\mu^{*(n+1)}(y) > 0$.

(A) Show that for each $\alpha \geq 1$ the limit

$$\psi(\alpha) := -\lim_{n\to\infty} (\alpha n)^{-1} E \log \mu^{*[\alpha n]}(X_n) \quad \text{exists.}$$

(B) Show that the function $\psi(\alpha)$ (called the *entropy profile*) is convex.
(C) Show that $\lim_{\alpha\to\infty} \psi(\alpha) = -\log \varrho$ where $\varrho =$ spectral radius.
(D) What is the entropy profile of the simple random walk on $\mathbb{Z}$?

B. The Direct Sum Group $\Gamma = \oplus_{\mathbb{N}}\mathbb{Z}_2$

The *direct sum* $\Gamma = \oplus_{\mathbb{N}}\mathbb{Z}_2$ of infinitely many copies of the two-element group $\mathbb{Z}_2$ consists of all infinite sequences $\mathbf{x} = (x_1, x_2, \cdots)$ of 0s and 1s such that $\sum_{i=1}^{\infty} x_i < \infty$. The group operation + is coordinatewise addition modulo 2; thus, every element is its own inverse. No infinite group in which every element is of finite order is finitely generated (why?), so the group Γ does not satisfy the standing hypothesis of the course. It does, however, have a countable generating set $\{e_m\}_{m\geq 1}$, where e_m is the sequence with a 1 in the mth coordinate and 0s in all other coordinates.

The following exercises concern an interesting class of random walks $S_n = \sum_{i=1}^{n} \xi_i$ whose step distribution is

$$P\{\xi_n = e_m\} = p_m,$$

where $p_1 \geq p_2 \geq p_3 \geq \cdots > 0$ is a given probability distribution on $\mathbb{N}$. As usual, the increments $\xi_1, \xi_2, \cdots$ are independent and identically distributed. The random walk therefore evolves as follows: at each step (i) choose $m \in \mathbb{N}$ at random according to $(p_i)_{i\in\mathbb{N}}$; then (ii) flip the mth coordinate.

Exercise 7. *Fourier Analysis on* Γ. The *dual group* $\hat{\Gamma}$ is the multiplicative group of *characters* of Γ. This group can be realized as the closed interval $[-1, 1]$ with the uniform distribution $Q(= \frac{1}{2}\text{Lebesgue})$. Each element $\alpha \in [-1, 1]$ corresponds to a group homomorphism $\chi_\alpha : \Gamma \to \{-1, 1\}$; this homomorphism is defined by

$$\chi_\alpha(\mathbf{x}) = \prod_{i=1}^{\infty} a_i^{x_i}$$

where the numbers $a_i \in \{-1, 1\}$ are the coefficients in the binary representation $\alpha = \sum_{k=1}^{\infty} a_k/2^k$.

For any finite measure μ on Γ, define the *Fourier transform* $\hat{\mu} : \hat{\Gamma} = [-1, 1] \to \mathbb{C}$ by

$$\hat{\mu}(\alpha) := \sum_{\mathbf{x}\in\Gamma} \mu(x)\chi_\alpha(\mathbf{x}).$$

(A) Prove the *orthogonality relations*

$$\int_{[-1,1]} \chi_\alpha(\mathbf{x})\chi_\alpha(\mathbf{x}')\, dQ(\alpha) = \delta(\mathbf{x}, \mathbf{x}').$$

(B) Prove the *Fourier inversion formula*

$$\mu(x) = \int_{[-1,1]} \hat{\mu}(\alpha)\chi_\alpha(\mathbf{x})\, dQ(\alpha).$$

(C) Prove the convolution formula

$$E\chi_\alpha(S_n) = \hat{\mu}(\alpha)^n \quad \text{where} \quad \mu = \sum_{m=1}^{\infty} p_m \delta_{e_m}.$$

(D) Deduce that

$$P\{S_n = \mathbf{x}\} = \int_{[-1,1]} \chi_\alpha(\mathbf{x})\hat{\mu}(\alpha)^n\, dQ(\alpha) = \int_{[-1,1]} \chi_\alpha(\mathbf{x}) \left(\sum_{m=1}^{\infty} p_m a_m\right)^n dQ(\alpha).$$

Exercise 8. Let $f_k = \sum_{i=k}^{\infty} p_i$.

(A) Show that the random walk S_n is recurrent if and only if

$$\int_{[-1,1]} (1 - \hat{\mu}(\alpha))^{-1}\, dQ(\alpha) = \infty.$$

(B) Conclude that S_n is recurrent if

$$\sum_{k=1}^{\infty} \frac{1}{2^k f_k} = \infty.$$

(C) Show that S_n is transient if

$$\sum_{k=1}^{\infty} \frac{1}{2^k p_k} < \infty.$$

Note: DARLING & ERDÖS showed that the condition $\sum_{k=1}^{\infty} \frac{1}{2^k f_k} = \infty$ is actually *necessary and sufficient* for recurrence.