1. Problem Session 2

Exercise 1. *Prove the inequality (3) from lecture notes: For fixed ε, denote by $\mathcal{A}_\varepsilon$ the set of all sequences $\overrightarrow{x} = (x_1, \ldots, x_n)$ such that*

$$\mathbb{P}\left(\overrightarrow{X} = \overrightarrow{x}\right) \in \left(2^{-n(H+\varepsilon)}, 2^{-n(H-\varepsilon)}\right).$$

Prove that there exists n_0 such that for every $n \geq n_0$ the following holds

$$\mathbb{P}\left(\overrightarrow{X} \in \mathcal{A}_\varepsilon\right) > 1 - \varepsilon.$$

Exercise 2. *Prove that*

$$|\mathcal{A}_\varepsilon| > (1-\varepsilon) \cdot 2^{n(H-\varepsilon)}.$$

Exercise 3. *If $\mathcal{P}_n$ is the set of all types on $\mathcal{A}^n$, prove that*

$$|\mathcal{P}_n| = \binom{n + |\mathcal{A}| - 1}{n} < (n+1)^{|\mathcal{A}|}.$$

Exercise 4. *The random variable X has values in the set $\{-2, 0, 2\}$ and the expectation 1. What is the smallest and the largest entropy that the distribution of X can have?*

Exercise 5. *Prove that $H(\mu|\nu) \geq 0$. Prove that $H(\mu|\nu) = 0$ if and only if $\mu = \nu$.*

Exercise 6. *Prove that if $\mathbb{P}, \mathbb{Q} \in \mathcal{P}_n$ are two types then*

$$\mathbb{P}\left(\overrightarrow{X} \in T(\mathbb{P})\right) \quad \geq \quad \mathbb{P}\left(\overrightarrow{X} \in T(\mathbb{Q})\right). \tag{1}$$

Exercise 7. *Let us define $E = \left\{\mathbb{Q} \in \mathcal{P}_n : \int_{\mathcal{A}} a \, d\mathbb{Q}(a) \geq \alpha\right\}$.*

(a) *Assume that $X_1, \ldots, X_n$ are iid with distribution $\mathbb{P}$ on $\mathcal{A}$. Prove that*

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n}{n} \geq \alpha\right) = \mathbb{P}\left(\mathbb{P}_{\overrightarrow{X}} \in E\right).$$

(b) *Use the method of Lagrange multipliers to prove that*

$$\inf_{\mathbb{Q} \in E} H(\mathbb{Q}|\,\mathbb{P}) = \alpha\theta - \log_2\left(\sum_{i=1}^{k} p_i 2^{\theta a_i}\right),$$

where θ is the unique solution of $\dfrac{\sum_{i=1}^{k} a_i p_i 2^{\theta a_i}}{\sum_{i=1}^{k} p_i 2^{\theta a_i}} = \alpha.$

(c) *Prove that Sanov's theorem implies Cramér's theorem in the case that the probability space is discrete.*

Exercise 8. *Prove that*

$$\sum_{i=1}^{n} \frac{1}{2^{m_i}} = 1, \tag{2}$$

where m_i is the depth of the ith vertex in a binary tree.

Exercise 9. *For every sequence of n positive integers $m_1, m_2, \ldots, m_n$ that satisfy (2), there exists a tree with n nodes such that the height of the i-th node is exactly m_i.*

Hint: Prove that there are two of the numbers m_i and m_j that are equal. Then consider the problem in which these two numbers are replaced with a single number equal to $m_i - 1$.

Exercise 10. *For $x \in (0,1)$ prove that*

$$1 + x \log_2 x + (1-x)\log_2(1-x) \geq 0.$$

Exercise 11. *If $\hat{m}_1, \ldots, \hat{m}_n$ are positive integers such that $\displaystyle\sum_{i=1}^{n} \frac{1}{2^{\hat{m}_i}} \leq 1$ there are positive integers $m_1, \ldots, m_n$ such that $m_i \leq \hat{m}_i$ for all $i \in \{1, \ldots, n\}$ and*

$$\sum_{i=1}^{n} \frac{1}{2^{m_i}} = 1.$$