Random polynomials

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Complex analytic polynomials

Consider a holomorphic polynomial of degree N

$$p_N(z) = \sum_{j=0}^{N} c_j z^j = a_N \prod_{j=1}^{N} (z - \zeta_j).$$

We are interested in the zeros

$$Z_{p_N} = \{\zeta_1, \ldots, \zeta_N\}$$

of p_N. We are interested in properties of zeros as the degree $N \to \infty$.

Coefficients and zeros

The Newton-Vieta formula,

$$\Pi_{j=1}^{N}(z-\zeta_j) = \sum_{k=0}^{N}(-1)^k e_{N-k}(\zeta_1,\ldots,\zeta_N)z^k$$

gives a formula for the coefficients c_j in terms of the zeros. Here, the elementary symmetric functions are defined by

$$e_j = \sum_{1\le p_1<\cdots<p_j\le N} z_{p_1}\cdots z_{p_j}.$$

Conversely, the formula for the zeros in terms of the coefficients is by comparison extremely complicated.

Why study 'random' polynomials?

Rather than study individual polynomials, we study ensembles of polynomials and ask how the zeros are distributed for typical (in a measure sense) polynomials. Motivation:

- It is very difficult to find the zeros from the coefficients. Zeros are very 'unstable' as the coefficients are changed. See notes at the end of the slide.
- It is not so difficult to find out where most zeros are for 'most' polynomials in a probability space;

Random holomorphic polynomials of one complex variable

The space of polynomials of degree N is a complex vector space $\mathcal{P}_N$ of dimension $N+1$. We put a probability measure on this vector space by viewing the coefficients

$$p_N(z) = \sum_{j=0}^{N} c_j z^j$$

as random variables. I.e. we put a probability measure on $\mathcal{P}_N$.

Complex Kac-Hammersley polynomials

One of the first random polynomials

$$p_N(z) = \sum_{j=0}^{N} c_j z^j$$

was defined by stipulating that the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. Complex Gaussian:

$$\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k), \quad \mathbf{E}(c_j \bar{c}_k) = \delta_{jk}.$$

Here, **E** denotes the expectation.

Probability measure

We identify

$$\mathcal{P}_N \simeq \mathbb{C}^{N+1}, \;\; p_N \to (c_0, \ldots, c_N),$$

The complex Gaussian measure above is the γ_{KAC} on $\mathcal{P}_N$:

$$d\gamma_{KAC}(p_N) = e^{-|c|^2/2} dc.$$

For any random variable (= function) on $\mathcal{P}_N$,

$$\mathbf{E}(X) := \int_{\mathcal{P}_N} X(p_N) e^{-|c|^2/2} dc.$$

Expected distribution of zeros

The empirical measure of zeros of a polynomial of degree N is the probability measure on $\mathbb{C}$ defined by

$$Z_{p_N} = \mu_\zeta = \frac{1}{N} \sum_{z:p_N(z)=0} \delta_z,$$

where δ_z is the Dirac delta-function at z.

Definition: The expected distribution of zeros of random polynomials of degree N with measure P is the probability measure $\mathbf{E}_P Z_f$ on $\mathbb{C}$ defined by

$$\langle \mathbf{E}_P Z_{p_N}, \varphi \rangle = \int_{\mathcal{P}_N} \{ \frac{1}{N} \sum_{z:p_N(z)=0} \varphi(z) \} dP(p_N),$$

for $\varphi \in C_c(\mathbb{C})$.

How are zeros of complex Kac polynomials distributed?

Complex zeros concentrate in small annuli around the unit circle S^1. In the limit as the degree $N \to \infty$, the zeros asymptotically concentrate exactly on S^1:

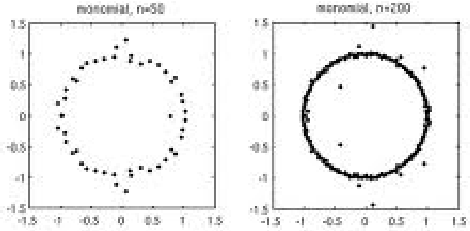
THEOREM (Kac-Hammersley-Shepp-Vanderbei)

The expected distribution of zeros of polynomials of degree N in the Kac ensemble has the asymptotics:

$$\mathbf{E}^N_{KAC}(Z_{p_N}) \to \delta_{S^1} \quad \textit{as} \quad N \to \infty\ ,$$

$$\textit{where} \qquad (\delta_{S^1}, \varphi) := \tfrac{1}{2\pi} \int_{S^1} \varphi(e^{i\theta})\, d\theta.$$

Complex zeros of the Hammersley-Kac polynomial



Why do the zeros concentrate on the unit circle?

This is obviously not true of general polynomials. It was a consequence of our choice of probability measure, which weighted polynomials most strongly which had all zeros near S^1. How did this happen?

It was the (implicit) choice of inner product that produced this concentration of zeros on S^1.

Gaussian measure and inner product

An inner product on $\mathcal{P}_N$ induces an orthonormal basis $\{S_j\}$ and associated associated Gaussian measure dγ:

$$S = \sum_{j=1}^{d} c_j S_j,$$

where $\{c_j\}$ are independent complex normal random variables. Thus the measure on coefficient space is

$$e^{-|c|^2/2} dc.$$

Implicit inner product for the Kac-Hammersley ensemble

The inner product underlying the Kac Gaussian measure on $\mathcal{P}_N$ is defined by the basis $\{z^j\}$ being orthonormal. The inner product which makes $\{z^j\}$ orthonormal is δ_{S^1} (Fourier series).

Orthonormalizing on S^1 made zeros concentrate on S^1 uniformly wrt Lebesgue measure $d\theta$.

What is $d\theta$? It is the equilibrium measure of the unit disc (or circle). To see that this is the right viewpoint, we consider general domains and weights.

Equilibrium measure

In classical potential theory, the equilibrium measure of a compact set K is the unique probability measure $d\mu_K$ supported on K which minimizes the logarithmic energy

$$E(\mu) = -\Sigma(\mu) = -\int_K \int_K \log|z-w|\, d\mu(z)\, d\mu(w).$$

In weighted potential theory with weight $e^{-\varphi}$ one modifies the logarithmic energy as follows

$$E_{\varphi,K}(\mu) = -\int_K \int_K \log\left(e^{-\frac{1}{2}\varphi(z)} e^{-\frac{1}{2}\varphi(w)} |z-w|\right)\, d\mu(z)\, d\mu(w).$$

Theorem: There exists a unique minimizing probability measure μ. See notes at the end of the slides.

Gaussian random polynomials adapted to domains and weights

We now orthonormalize polynomials on the interior Ω or boundary $\partial\Omega$ of any simply connected, bounded domain $\Omega \subset \mathbb{C}$. Introduce a weight $e^{-N\varphi}$ and a probability measure $d\nu$ on Ω and define

$$\langle f, \bar{g} \rangle_{\Omega,\varphi} := \int_\Omega f(z)\overline{g(z)}\, e^{-N\varphi(z)} d\nu \quad .$$

Let $\gamma^N_{\Omega,\varphi}$ = the Gaussian measure induced by $\langle f, \bar{g} \rangle_{\Omega,\varphi}$ on $\mathcal{P}^{(1)}_N$. How do zeros of random polynomials adapted to Ω concentrate?

Equilibrium distribution of zeros

The basic phenomenon is that the expected distribution of zeros of random polynomials (or any random holomorphic sections) tends to the equilibrium measure defined by (Ω, φ, ν) with $K = \text{supp}\ \nu$. This fact was proved in increasing generality:

- For positive line bundles over Kähler manifolds (Nonnenmacher dim $M = 1$, Shiffman-Zelditch dim $M = m$ (1998-9))
- For real analytic plane domains and flat line bundles (Shiffman-Zelditch, 2003);
- For general plane domains with Bernstein-Markov measures (Bloom, 2005).
- For general big line bundles, smooth metrics and B-M measures (Berman 2007...).

Equilibrium distribution of zeros: unweighted case

Denote the expectation relative to the ensemble $(\mathcal{P}_N, \gamma_\Omega^N)$ by $\mathbf{E}_\Omega^N$.

THEOREM

(Shiffman-Z, 2003)

$$\mathbf{E}_\Omega^N(Z_{p_N}) = \mu_\Omega + O(1/N) \ ,$$

where μ_Ω is the equilibrium measure of $\bar{\Omega}$.

The equilibrium measure of a compact set K is the unique probability measure $d\mu_K$ supported on K which minimizes the energy

$$E(\mu) = -\int\int \log|z-w|\, d\mu(z)\, d\mu(w).$$

Thus, zeros behave like electric charges repelling with the Coulomb force $\log|z-w|$.

First weighted case: $SU(2)$ polynomials

Can we construct an inner product $\int |p_N(z)|^2 e^{-N\varphi} d\nu$ which spreads out the zeros of random polynomials uniformly on the Riemann sphere $\mathbb{CP}^1$? Yes:

We define an inner product on $\mathcal{P}_N^{(1)}$ which depends on N:

$$\langle z^j, z^k \rangle_N = \frac{1}{\binom{N}{j}} \delta_{jk}.$$

Thus, a random $SU(2)$ polynomial has the form

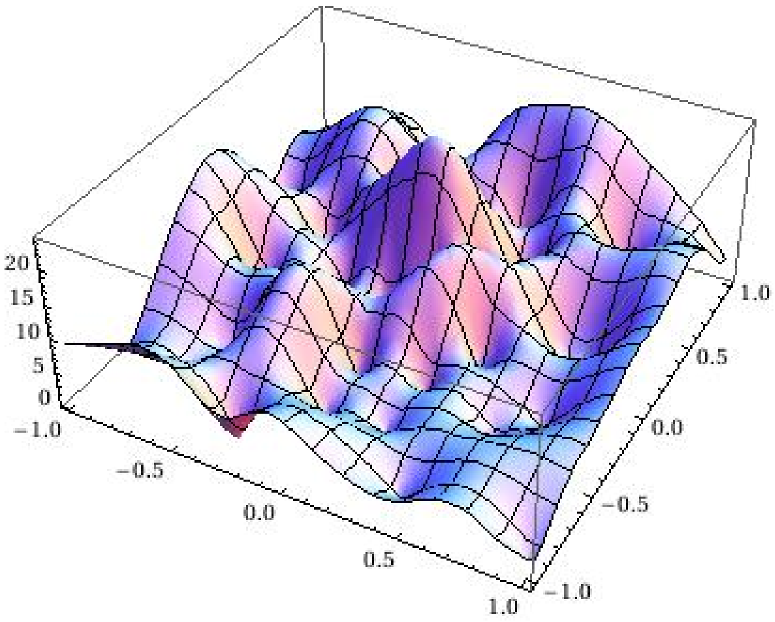
$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}} \, z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \overline{\lambda}_\beta) = \delta_{\alpha\beta}.$$

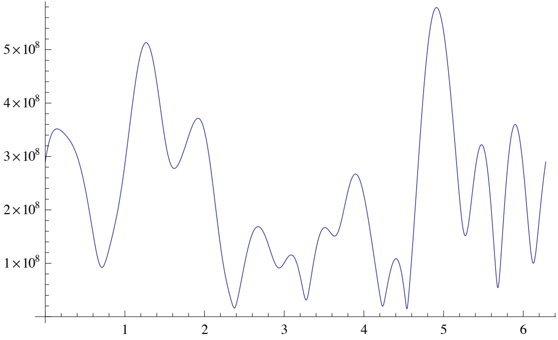
PROPOSITION

In the $SU(2)$ ensemble, $\mathbf{E}(Z_f) = \omega_{FS}$, the Fubini-Study area form on $\mathbb{CP}^1$.

Degree 50 SU(2) polynomial: graph of $|p(z)|^2 e^{-N\varphi}$



Degree 50 SU(2) polynomial : graph of $|p(z)|^2$ on $[0, 2\pi]$



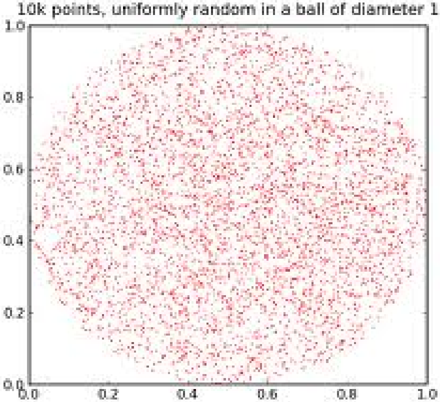
$SU(2)$ and holomorphic line bundles

The $SU(2)$ inner products may be written in the form

$$\int_{\mathbb{C}} f(z)\overline{g(z)} e^{-N\log(1+|z|^2)} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

The factor $e^{-N\log(1+|z|^2)}$ defines a Hermitian metric on the line bundle $\mathcal{O}(N)$, and its curvature form is $\omega = \frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$.
This has a simple geometric interpretation, without which it is hard to understand. Namely, we view polynomials of degree N as holomorphic sections of a line bundle $\mathcal{O}(N) \to \mathbb{CP}^1$. Then, $e^{-N\log(1+|z|^2)}$ is a Hermitian metric on $\mathcal{O}(N)$ and $\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$ is the usual area form on $\mathbb{CP}^1$.

Uniform zeros wrt $\mathbb{CP}^1, \omega_{FS}$



Why do the zeros spread out?

Proposition

In the $SU(2)$ ensemble, $\mathbf{E}(Z_f) = \omega_{FS}$, the Fubini-Study area form on $\mathbb{CP}^1$.

The inner product

$$\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-N\log(1+|z|^2)}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

is $SU(2)$ invariant. Hence, the expected distribution of zeros is $SU(2)$ invariant.

Gaussian random holomorphic sections of line bundles

We may consider more general Hermitian metrics $h = e^{-\varphi}$ on $\mathcal{O}(1) \to \mathbb{CP}^1$ and probability measures on $\mathbb{CP}^1$. Everything we do generalizes to any Riemann surface M of any genus.
The Hermitian metric h on $\mathcal{O}(1)$ induces Hermitian metrics $h^N = e^{-N\varphi}$ on the powers $\mathcal{O}(N)$, a probability measure $d\nu$, and an inner product

$$\langle s_1, s_2 \rangle_N = \int_M s_1(z)\overline{s_2(z)} e^{-N\varphi} d\nu(z).$$

We let $\{S_j\}$ denote an orthonormal basis of the space $H^0(M, L^N)$ of holomorphic sections of L^N.

Inner products and Gaussian measures

The inner product induces the complex Gaussian probability measure

$$d\gamma(s) = \frac{1}{\pi^m} e^{-|c|^2} dc\,, \qquad s = \sum_{j=1}^{n} c_j S_j\,, \tag{1}$$

on $\mathcal{S}$, where $\{S_j\}$ is an orthonormal basis for $\mathcal{S}$ and dc is $2n$-dimensional Lebesgue measure. This Gaussian is characterized by the property that the $2N$ real variables $\Re c_j, \Im c_j$ $(j = 1, \ldots, n)$ are independent Gaussian random variables with mean 0 and variance $\frac{1}{2}$; i.e.,

$$\mathbf{E} c_j = 0, \quad \mathbf{E} c_j c_k = 0, \quad \mathbf{E} c_j \bar{c}_k = \delta_{jk}\,.$$

When $\nu = \omega_h$ we call the induced Gaussian measure the Hermitian Gaussian measure.

Expected distribution of zeros

For $s \in H^0(C, L^N)$ over a Riemann surface, we let Z_s denote empirical measure of zeros,

$$(Z_s, \varphi) = \frac{1}{N} \sum_{z : s(z) = 0} \delta_z, .$$

This is a random probability measure on C. Its expectation is a measure called the expected distribution of zeros: Paired with a continuous test function f,

$$(\mathbf{E} Z_{s_N}, f) := \int_{H^0(C, L^N)} \left(\frac{1}{N} \sum_{z : s(z) = 0} f(z) \right) d\gamma_{h^N, \nu}(s_N).$$

Limit distribution of zeros: positive line bundles

THEOREM

Let $(L, h) \to C$ *be a positive line bundle, and consider the Hermitian Gaussian measure induced by* (h^N, ω_h)*. Then,*

$$\mathbf{E} Z_{s_N} \to \omega$$

weakly in the sense of measures; in other words,

$$\lim_{N \to \infty} (\mathbf{E} Z_{s_N}, \varphi) = \int_C \omega \wedge \varphi$$

for all continuous functions φ*. In particular,*

$$\lim_{N \to \infty} \frac{1}{N} \mathbf{E} \# \{ z \in U : s_N(z) = 0 \} = m \,\mathrm{vol}_2\, U \ ,$$

for U *open in* C*.*

Equilibrium distribution of zeros

In the opposite extreme when $\varphi = 0$, we have:
Suppose that $\varphi = 0$ and that ν is a 'Bernstein-Markov measure'.

THEOREM
(Shiffman-Z, 2003; Bloom, 2005)

$$\mathbf{E}_N(Z_f^N) = \mu_K + O\left(1/N\right) \ ,$$

where μ_K is the weighted equilibrium measure of $K = supp\nu$.
I.e. μ_K minimizes the logarithmic energy $\mathcal{E}(\mu) =$

$$-\int_K \int_K \log\left(|z - w|\right) \, d\mu(z) \, d\mu(w).$$

General equilibrium measures

Compare:

- Kähler case: the limit distribution of zeros was the Kähler form $\omega_\varphi = i\partial\bar{\partial}\varphi$
- unweighted case: the limit is μ_K.

Unifying theme (Shiffman-Z; Bloom; R. Berman): both are equilibrium measures. In all dimensions, for smooth weights and B-M measures, the limit distribution of zeros should be the equilibrium measure $\mu_{\underline{K},\varphi}$
In general: $\mu_{K,\varphi} = i\partial\bar{\partial}V^*_{K,\varphi}$ (a certain pluri-complex Green's function).

Equilibrium measure

Given a weight φ, the weighted equilibrium measure of a compact set K is the unique probability measure $d\mu_{\varphi,K}$ which minimizes the weighted logarithmic energy on the space $\mathcal{M}(K)$ of probability measures on K:

$$\mathcal{E}_\varphi(\mu) = -\int_K \int_K \log\left(|z-w|e^{-\varphi(z)/2}e^{-\varphi(w)/2}\right)\, d\mu(z)\, d\mu(w).$$

Examples:

- $\varphi = 0, K = S^1 : d\mu_{\varphi,K} = \delta_{S^1}$;
- $\varphi = \log(1+|z|^2),\ K = \mathbb{CP}^1; d\mu_{\varphi,K} = \frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$.

Behavior of almost all sequences: positive line bundles

We form the probability space $\prod_{N=1}^{\infty} H^0(C, L^N)$ with the product measure μ. Its elements are sequences (s_N) of sections (chosen independently).

Theorem

Let $(L, h) \to C$ be a positive line bundle, and consider the Hermitian Gaussian measure induced by (h^N, ω_h). Then, for μ-almost all $\mathbf{s} = \{s_N\} \in \mathcal{S}$, $\frac{1}{N} Z_{s_N} \to \omega$ weakly in the sense of measures; in other words,

$$\lim_{N\to\infty} \left(\frac{1}{N} Z_{s_N}, \varphi \right) = \int_M \omega \wedge \varphi$$

for all continuous functions φ. In particular,

$$\lim_{N\to\infty} \frac{1}{N} \#\{z \in U : s_N(z) = 0\} = m \operatorname{vol}_2 U \ ,$$

for U open in C.

Comparison to plane domain result

Recap: In the case of plane domains with 'flat' Hermitian metric $dd^c\varphi = 0$ and a rather general measure $d\nu$ we got:

$$\mathbf{E}_\Omega^N(Z_f^N) = \nu_\Omega + O\left(1/N\right) \ ,$$

where ν_Ω is the equilibrium measure of $\bar{\Omega}$.
In the case of line bundles where $dd^c\varphi >> 0$ we got

$$\lim_{N\to\infty} (\mathbf{E}Z_{s_N}, \varphi) = \int_C \omega \wedge \varphi.$$

There is a generalization of potential theory to Kähler manifolds, and ω is the equilibrium measure for $(C, L, e^{-\varphi})$.

Sketch of Proof: Step 1: Individual distribution of zeros

For $s \in H^0(C, L^N)$ over a Riemann surface, we let Z_s denote empirical measure of zeros,

$$(Z_s, \varphi) = \frac{1}{N} \sum_{z:s(z)=0} \delta_z, .$$

When $s = fe_L^{\otimes N}$, we have by the Poincare-Lelong formula,

$$Z_s = \frac{i}{N\pi} \partial\bar{\partial} \log |f| = \frac{i}{N\pi} \partial\bar{\partial} \log \|s\|_{h^N} + \omega_h \ . \tag{2}$$

Two point function = Szegö kernel

The two point function (on the diagonal) is defined by

$$\Pi_{h^N,\nu}(z,z) = \mathbf{E}_\gamma \left(\|s(z)\|_h^2 \right) = \sum_{j=1}^{n} \|S_j(z)\|_h^2 \,, \qquad z \in C \,.$$

It is the (contracted) value on the diagonal of the orthogonal projection on $H^0(C, L^N)$ with respect to the inner product $G(h^N, \nu)$.

Asymptotics of Szego kernels on positive line bundles

We are interested in $\Pi_{h^N}(z,z) = \sum_j ||S_j^N z)||^2_{h^N}$. In the case of the Hermitian inner product of a positive line bundle, we have the following asymptotics

Theorem

(TYZC) Let M be a compact complex manifold of dimension m (over $\mathbb{C}$) and let $(L,h) \to M$ be a positive Hermitian holomorphic line bundle. Let $\{S_1^N, \ldots, S_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^N)$ (with respect to the inner product defined above). Then there exists a complete asymptotic expansion

$$\sum_{j=1}^{d_N} \|S_j^N(z)\|^2_{h_N} = a_0 N^m + a_1(z) N^{m-1} + a_2(z) N^{m-2} + \ldots$$

Expected distribution

We then take expected values:

LEMMA

For N sufficiently large,

$$E(\widetilde{Z_s^N}) = \frac{\sqrt{-1}}{2\pi N}\partial\bar{\partial}\log\sum_{j=1}^{d_N}|f_j^N|^2$$

$$= \frac{\sqrt{-1}}{2\pi N}\partial\bar{\partial}\log\Pi_{h^N}(z,z) + \omega = \omega + O(N^{-1}),$$

completing the proof.

Sketch of proof

Let $\varphi \in C(C)$. We must show

$$\frac{\sqrt{-1}}{\pi N} \int_{\mathbb{C}^{d_N}} \int_C \partial\bar{\partial} \log |\langle a, f\rangle| \wedge \varphi d\gamma_N(a) = (\omega_N, \varphi) \; . \qquad (3)$$

To compute the integral, we write $f = |f|u$ where $|u| \equiv 1$. Evidently, $\log |\langle a, f\rangle| = \log |f| + \log |\langle a, u\rangle|$. The first term gives

$$\frac{\sqrt{-1}}{\pi N} \int_C \partial\bar{\partial} \log |f| \wedge \varphi = \int_C \omega_C \wedge \varphi. \qquad (4)$$

Other terms

We now look at the second term. We have

$$\frac{\sqrt{-1}}{\pi}\int_{H^0(C,L^N)}\int_C \partial\bar\partial \log|\langle a,u\rangle| \wedge \varphi d\gamma_N(a)$$

$$= \frac{\sqrt{-1}}{\pi}\int_C \partial\bar\partial\left[\int_{H^0(C,L^N)} \log|\langle a,u\rangle| d\mu_N(a)\right]\wedge\varphi = 0,$$

since the average $\int \log|\langle a,\omega\rangle| d\mu_N(a)$ is a constant independent of u for $|u| = 1$, and thus the operator $\partial\bar\partial$ kills it.
Bergman kernel asymptotics then give:

$$E(\widetilde{Z_s^N}) = \omega + O(\frac{1}{N})$$

Sketch of proof of equilibrium distribution of zeros

The main point of the proof is to gain control over asymptotics of the partial Szegö and Bergman kernels. Let $\{P_N\}$ be an ONB of $\mathcal{P}_N$ (polynomials of degree N) with respect to the inner product. The Szegö kernel is:

$$S(z,w) := \sum_{k=0}^{\infty} P_k(z)\overline{P_k(w)}, \quad (z,w) \in \overline{\Omega} \times \overline{\Omega} \tag{5}$$

By the regularity theorem, one has that $S(z,z) < \infty$ for $z \in \Omega$, and thus $P_N(z) \to 0$ for $z \in \mathrm{int}\Omega$. Hence,

$$S_N(z,z) \to S(z,z), \quad \text{uniformly on compact subsets of } \Omega,$$

where $S_N(z,w) := \sum_{k=0}^{N} P_k(z)\overline{P_k(w)}$ is the partial Szegö kernel.

Kac ensemble: Inside S^1

We do the simplest case: show that the expected distribution of zeros in the Kac ensemble tends to $\frac{1}{2\pi} d\theta$.
We have:

$$\frac{1}{N}\partial\bar{\partial} \log S_N(z,z) \sim \frac{1}{N}\partial\bar{\partial} \log(1-|z|^{2N}).$$

Clearly, in any annulus $|z| \le r < 1$, $(1-|z|^{2N}) \to 1$ rapidly with its derivatives, and the limit equals zero. So the limit distribution of zeros vanishes there.

Kac ensemble: outside S^1.

In any annulus $|z| \geq r > 1$ we may write $(1 - |z|^{2N}) = |z|^{2N}(|z|^{-2N} - 1)$ and separate the factors after taking log. The second again tends to zero rapidly, while the first factor, $\log |z|^{2N}$, is killed by $\partial\bar{\partial}$ (note that $z \neq 0$ in this part). It follows that the limit measure must be supported on S^1. Since it is SO(2)-invariant (radial), and since it is a probability measure, it must be $\frac{1}{2\pi} d\theta$.

Kac ensemble: Asymptotics

In fact, we have the following explicit formula and asymptotics for the circular case: Let $\nu = \frac{d\theta}{2\pi}$ denote Haar measure on S^1. Then

$$\mathbf{E}_\nu^N(Z_f) = \left[\frac{1}{(|z|^2-1)^2} - \frac{(N+1)^2|z|^{2N}}{(|z|^{2N+2}-1)^2}\right]\frac{\sqrt{-1}}{2\pi}\, dz \wedge d\bar{z}\ ,$$

Furthermore, $\mathbf{E}_\nu^N(Z_f) = N\nu + O(1)$; i.e., for all test forms $\varphi \in \mathcal{D}(\mathbb{C})$, we have

$$\mathbf{E}_\nu^N\left(\sum_{\{z:f(z)=0\}} \varphi(z)\right) = \frac{N}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})\, d\theta + O(1)\,.$$

In particular, $\mathbf{E}_\nu^N(\widetilde{Z}_f^N) \to \nu$ in $\mathcal{D}'(\mathbb{C})$.

Complex Green's function with pole at infinity

There is a proof of the equilibrium distribution of zeros which is based on the extremal function V_K^* ,

$$V_K(z) = \sup\{u(z) : \ u \in \mathcal{L}, \ u \leq 0 \text{ on } K\}.$$

Here, $\mathcal{L}$ is the Lelong class,

$$\mathcal{L} = \{u : u \in SH(\mathbb{C}), \ \ u(z) \leq \log^+ |z| + C_u\}.$$

Complex Green's function and equilibrium measure

Theorem

$\nu_K = \frac{1}{2\pi} dd^c V_K^*$.

A proof can be found in Saff-Totik.

Hence it suffices to show that

$$\frac{1}{N} \log \Pi_N(z, z) \to V_\Omega^*(z).$$

This is done by relating the log of the partial Szego kernel to the Siciak extremal function.

Siciak-Zaharyuta theorem

The Siciak functions are

$$\Phi_K^N(z) = \sup\{\frac{1}{N}\log|p(z)| :$$

$$p \text{ is a polynomial of degree } N, ||p||_K \leq 1\}.$$

He proved that $\frac{1}{N}\log\Phi_K^N \to V_K^*$.
The complex Green's function can be expressed entirely in terms of logarithms of polynomials:

THEOREM

$$V_K(z) = \sup\{\frac{1}{\text{degree } p}\log|p(z)| :$$

$$p \text{ is a polynomial of degree } \geq 1, ||p||_K \leq 1\}.$$

Here, $||p||_K = \sup_{z\in K}|p(z)|$.

Siciak extremal function and partial Szego kernel

One has

Proposition

$\frac{1}{N} \leq \frac{S_N(z,z)}{\Phi_\Omega^N(z)} \leq Ce^{\epsilon N} N$ *for all* $\epsilon > 0$.

Taking $\frac{1}{N}$ log shows that

$$\frac{1}{N} \log S_N(z,z) \sim \frac{1}{N} \log \Phi_\Omega^N(z) \to V_K^*$$

by Siciak's theorem.

More on weighted equilibrium measure

We review the definition of equilibirium measure with respect to the data (φ, ν). The data defines an inner product $\mathrm{Hilb_N}(\varphi, \nu)$ with weight $e^{-N\varphi} d\nu$.There are two characterizations of $\nu_{h,K}$:

(i) $\nu_{h,K}$ is the minimizer of the Green's energy functional among measures supported on K.

(ii) The potential of $\nu_{h,K}$ is the maximal ω_h-subharmonic function of K.

Green's energy

$$\mathcal{E}_h(\mu) = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_h(z,w) d\mu(z) d\mu(w) \,. \tag{6}$$

where G_h is the (weighted) Green's function,

$$G_h(z,w) = 2\log|z-w| - \varphi(z) - \varphi(w). \tag{7}$$

Minimizing energy on a compact set

We fix a compact non-polar subset $K \subset \mathbb{CP}^1$ and consider the restriction of the energy functional $\mathcal{E}_h : \mathcal{M}(K) \to \mathbb{R}$ to probability measures supported on K.

Proposition

If $K \subset \mathbb{CP}^1$ is non-polar, then $\mathcal{E}_h$ is bounded above on $\mathcal{M}(K)$. It has a unique maximizer $\nu_{K,h} \in \mathcal{M}(K)$.

(When we use G_h, where the log is $-\infty$ on the diagonal, we look for a maximizer. When we use $-G_h$, where the log is $+\infty$ on the diagonal, we look for a minimizer. The choice of sign differs from author to author and from slide to slide).

Instability of zeros

- z^N has N zeros at $z = 0$. But for any $\epsilon > 0$, $z^N - \epsilon = 0$ has the N roots $\{\epsilon^{\frac{1}{N}} e^{\frac{2\pi ik}{N}}\}_{k=0}^{N}$ and $\epsilon^{\frac{1}{N}} \to 1$ as $N \to \infty$.
- Wilkinson's polynomial $\prod_{j=1}^{N}(z - j)$ has unstable roots even though they are well separated. E.g. if $N = 20$, the coefficien of z^{19} is -210. If it is decreased to -210.0000001192, the zero at $z = 20$ grows to $\simeq 20.8$.

What makes the roots unstable?

If you perturb the coefficients continuously in a 1 parameter family of polynomials $p_N(t) = p_N + tc_N$ of degree N, the roots $\alpha_j(t)$ move as

$$\frac{d\alpha_j}{dt} = \frac{c(\alpha_j)}{p'_N(\alpha)}.$$

When $p'_N(\alpha)$ is small, the roots move quickly. For the degree 20 Wilkinson polynomial, with $c_{20}(x) = x^{19}$,

$$\frac{d\alpha_j}{dt} = \frac{\alpha_j^{19}}{\prod_{k\neq j}(\alpha_j - \alpha_k)} = -\frac{\alpha_j}{\prod_{k\neq j}(\alpha_j - \alpha_k)}.$$

The right side is large when there are many roots α_k such that $|\alpha_j - \alpha_k| << |\alpha_j|$.

Ostrowski bound

Let

$p(z) = z^N + a_1 z^{N-1} + \cdots + a_N,\ q(z) = z^N + b_1 z^{N-1} + \cdots + b_N.$

Then the roots of p_N resp. q_N can be enumerated as $\alpha_1, \ldots, \alpha_N$ resp. $\beta_1, \ldots, \beta_N$ in such a way that

$$\max_j |\alpha_j - \beta_j| \leq (2N-1) \left(\sum_{k=1}^{N} |a_k - b_k| \gamma^{N-k} \right)^{1/N}.$$

Here, $\gamma = 2 \max_{1 \leq k \leq N} \{|a_k|^{1/k}, |b_k|^{1/k}\}$.Bhatia showed that the

factor $(2N-1)$ can be replaced by $4 \times 2^{-1/N}$.

Let

$$||p_N - q_N||_2 := \left(\sum_{j=1}^{N} |a_j - b_j|^2 \right)^{\frac{1}{2}}.$$

Then

$$||p_N - q_N||_2 < \epsilon \implies |\alpha_j - \beta_j| \leq C\ N \epsilon^{1/N}.$$

Bombieri norm

The Bombieri norm of $p_N(z) = \sum_{j=0}^{N} a_j z^{N-j}$ is defined by

$$[p_N]_B := \left(\sum_{j=0}^{N} \frac{|a_j|^2}{\binom{N}{j}} \right)^{\frac{1}{2}} .$$

Suppose that $[p_N - q_N] \le \epsilon$.
Then for any root α of p_N there exists a root β of q_N so that

$$|\alpha - \beta| \le \frac{N(1 + |\alpha|^2)^{N/2}}{|q_N'(\alpha)|} \epsilon.$$

Some references

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