Best and random approximation of convex bodies by polytopes

a convex body K in $\mathbb{R}^n$ is compact convex set with non-empty interior

a polytope P in $\mathbb{R}^n$ is the convex hull of finitely many points $x_1, \ldots, x_N$

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How well can a convex body be approximated by a polytope?

1. Approximation by a polytope P with

(i) a fixed number of vertices

(ii) a fixed number of facets = $(n-1)$-dimensional faces

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When $K \subset L$,

$$\Delta_v(K,L) = |L| - |K|$$

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BEST APPROXIMATION

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$$= \frac{1}{2}\mathrm{del}_{n-1} \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x) \right)^{\frac{n+1}{n-1}}$$

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μ is the surface area measure on ∂K
del_{n-1} is a constant that depends only on n

Best Approximation for N large

$$
\begin{aligned}
\Delta_v(K, P_{\text{best}}) &= |K| - |P_{\text{best}}| \\
&\sim \frac{1}{2}\mathrm{del}_{n-1}\left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, d\mu(x)\right)^{\frac{n+1}{n-1}} \frac{1}{N^{\frac{2}{n-1}}}
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Theorem (Mankiewicz&Schütt)

There is a numerical constant $c > 0$ such that

$$\frac{n-1}{n+1}\left(\frac{1}{|B_2^{n-1}|}\right)^{\frac{2}{n-1}} \leq \text{del}_{n-1} \leq \left(1+\frac{c \log n}{n}\right) \frac{n-1}{n+1}\left(\frac{1}{|B_2^{n-1}|}\right)^{\frac{2}{n-1}}$$

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Affine surface area appears

$$as(K)=\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\,d\mu(x)$$

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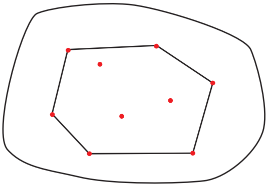
when chosen in K, not ALL points become vertices



More generally:

$f : \partial K \to \mathbb{R}^+$ strictly positive a.e. on ∂K, $\int_{\partial K} f d\mu = 1$

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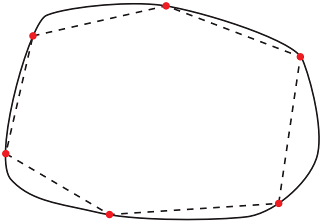
Choose N points $x_1, \dots x_N$ w.r. to $\mathbb{P}_f$ on ∂K

As before we call their convex hull $[x_1, \dots, x_N]$ a **random polytope**. **Every** point chosen becomes a vertex



The **expected volume** of such a random polytope is

$$\mathbb{E}_N(\partial K, \mathbb{P}_f) = \int_{\partial K} \cdots \int_{\partial K} \Big| [x_1, \dots, x_N] \Big| d\mathbb{P}_f(x_1) \dots d\mathbb{P}_f(x_N)$$

Theorem (Schütt&Werner)

Let K be a convex body in $\mathbb{R}^n$ with sufficiently regular boundary.

$$\lim_{N\to\infty} \frac{|K| - \mathbb{E}_N(\partial K, \mathbb{P}_f)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} =$$

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- $$c_n = \frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}$$

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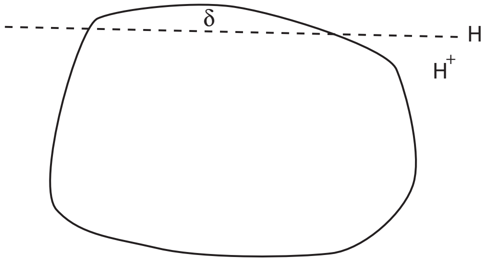

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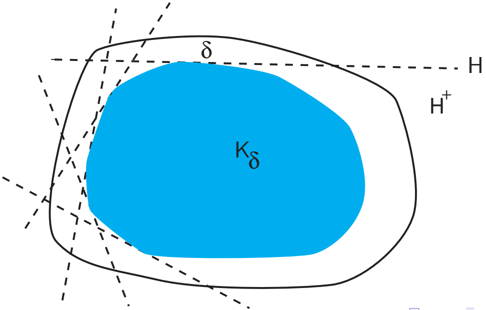

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How do best and random approximation compare?

- $|K| - |P_{\text{best}}| \sim \left(\frac{1}{N}\right)^{\frac{2}{n-1}} \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x)\right)^{\frac{n+1}{n-1}} \frac{1}{2}\mathrm{del}_{n-1}$
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With an absolute constant c

$$\frac{1}{2}\mathrm{del}_{n-1} \le c_n \le \left(1 + \frac{c \log n}{n}\right) \frac{1}{2}\mathrm{del}_{n-1}$$

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These restrictions need to be dropped

Again: we concentrate on the vertex case

Drop requirement that $P_N \subset K$ resp. $K \subset P_M$

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Theorem (Ludwig, Schütt, Werner)

Let K be C^2_+. There is a constant $c > 0$ s.th. for all $n \in \mathbb{N}$ there is $N_n \in \mathbb{N}$ s.th. for all $N \geq N_n$ there is a polytope P in $\mathbb{R}^n$ with at most N vertices s.th.

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- When $P \subset K$ (Bronsteyn&Ivanov)

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- If we drop the restriction, we gain by a factor of dimension: n

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Theorem (Böröczky)

For every polytope P_N with at most N vertices

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- Δ_s is not a metric: it fails the triangle inequality

- There are many other metrics (Hausdorff, ...).
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$$1 \leq \lim_{N \to \infty} \frac{\mathbb{E}_N\left(\Delta_v(K, P_{\mathbb{P}_f})\right)}{\Delta_v(K, P_{\text{best}})} \leq 1 + c\,\frac{\log n}{n}$$

Surface deviation

Theorem (Hoehner, Schütt & Werner)

There exists an absolute constant $c > 0$ such that for every integer $n \geq 3$, there is an N_n such that for every $N \geq N_n$ there is a polytope P_N in $\mathbb{R}^n$ with N vertices such that

$$\Delta_s(B_2^n, P_N) \leq c\, \frac{\mathrm{vol}_{n-1}\left(\partial B_2^n\right)}{N^{\frac{2}{n-1}}}$$

Surface deviation

Theorem (Hoehner, Schütt & Werner)

There exists an absolute constant $c > 0$ such that for every integer $n \geq 3$, there is an N_n such that for every $N \geq N_n$ there is a polytope P_N in $\mathbb{R}^n$ with N vertices such that

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Theorem (J. Müller)

Let P_N be the convex hull of N points chosen i.i.d. from ∂B_2^n with respect to the normalized surface measure. Then for N large

$$\mathrm{vol}_{n-1}(\partial B_2^n) - \mathbb{E}\mathrm{vol}_{n-1}(\partial P_N) \sim n\, \frac{\mathrm{vol}_{n-1}\left(\partial B_2^n\right)}{N^{\frac{2}{n-1}}}$$

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Proposition (Hoehner, Schütt & Werner)

There is a constant $c > 0$ and $N_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq 2$, all $N \in \mathbb{N}$ with $N \geq N_0$ and all **simple** polytopes P_N in $\mathbb{R}^n$ with no more than N vertices

$$\Delta_s(B_2^n, P_N) \geq c \frac{\mathrm{vol_{n-1}}(\partial \mathrm{B_2^n})}{N^{\frac{2}{n-1}}}.$$

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We choose at random points $x_1, \ldots, x_N$ with respect to $\mathbb{P} = \frac{\mu_{\partial B_2^n}}{|\partial B_2^n|}$ on ∂B_2^n. Get random polytope

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We choose the points from ∂B_2^n and we approximate

$$(1-r)B_2^n, \quad r \text{ chosen appropriately}$$

We want to estimate

$$\mathbb{E}[\Delta_s((1-r)B_2^n, P_N)] = \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \left| \partial((1-r)B_2^n \cup [x_1, \ldots, x_N]) \right| - \left| \partial((1-r)B_2^n \cap [x_1, \ldots, x_N]) \right| \, d\mathbb{P}(x_1) \ldots d\mathbb{P}(x_N)$$

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$$= \mathbb{E}\left(\left| \partial((1-r)B_2^n) \cap P_N^C \right| \right) - \mathbb{E}\left(\left| (1-r)B_2^n \cap \partial P_N \right| \right) + \mathbb{E}\left(\left| \, ((1-r)B_2^n)^C \cap \partial P_N \right| \right) - \mathbb{E}\left(\left| \partial\left((1-r)B_2^n\right) \cap P_N \right| \right)$$

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$$\left| \partial((1-r)B_2^n \cap [x_1, \ldots, x_N]) \right| \, d\mathbb{P}(x_1) \ldots d\mathbb{P}(x_N)$$

$$= \mathbb{E}\left(\left| \partial((1-r)B_2^n) \cap P_N^C \right| \right) - \mathbb{E}\left(\left| (1-r)B_2^n \cap \partial P_N \right| \right)$$

$$+ \mathbb{E}\left(\left| \left((1-r)B_2^n\right)^C \cap \partial P_N \right| \right) - \mathbb{E}\left(\left| \partial\left((1-r)B_2^n\right) \cap P_N \right| \right)$$

we choose r such that

$$= 2\left(\mathbb{E}\left(\left| \partial\left((1-r)B_2^n\right) \cap P_N^C \right| \right) - \mathbb{E}\left(\left| (1-r)B_2^n \cap \partial P_N \right| \right) \right)$$

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We choose $r = r(n, N)$ such that

$$\mathbb{E}\left(\left|\partial((1-r)B_2^n) \cap P_N^C\right|\right) - \mathbb{E}\left(\left|(1-r)B_2^n \cap \partial P_N\right|\right) =$$

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$$.$$

$$\mathbb{E}\left(\left|\left((1-r)B_2^n\right)^C \cap \partial P_N\right|\right) - \mathbb{E}\left(\left|\partial\left((1-r)B_2^n\right) \cap P_N\right|\right)$$

which is equivalent to

$$\mathbb{E}\left(\left|\partial((1-r)B_2^n) \cap P_N^C\right|\right) + \mathbb{E}\left(\left|\partial\left((1-r)B_2^n\right) \cap P_N\right|\right) =$$

$$\mathbb{E}\left(\left|\left((1-r)B_2^n\right)^C \cap \partial P_N\right|\right) + \mathbb{E}\left(\left|(1-r)B_2^n \cap \partial P_N\right|\right)$$

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.

$$\mathbb{E}\left(\left|\,((1-r)B_2^n)^C \cap \partial P_N\right|\right) - \mathbb{E}\left(\left|\partial\,((1-r)B_2^n) \cap P_N\right|\right)$$

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or

$$\left|\partial\,((1-r)B_2^n)\,\right| = \mathbb{E}\left(\left|\partial P_N\right|\right)$$

$$\left|\partial\left((1-r)B_2^n\right)\right| = \mathbb{E}\left(\left|\partial P_N\right|\right)$$

or

$$\left|\partial\left(B_2^n\right)\right| - \left|\partial\left((1-r)B_2^n\right)\right| \quad \sim$$

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$$\begin{aligned} \left|\partial\left(B_2^n\right)\right| - \left|\partial\left((1-r)B_2^n\right)\right| &\sim n\, r\left|\partial\left(B_2^n\right)\right| \\ &= \left|\partial\left(B_2^n\right)\right| - \mathbb{E}\left(\left|\partial P_N\right|\right) \end{aligned}$$

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J. Müller implies for N large enough

$$\left|\partial\left(B_2^n\right)\right| - \mathbb{E}\left(\left|\partial P_N\right|\right) \sim n\,\frac{\left|\partial\left(B_2^n\right)\right|}{N^{\frac{2}{n-1}}}$$

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$$r \sim \frac{1}{N^{\frac{2}{n-1}}}$$

$$\mathbb{E}[\Delta_s((1-r)B_2^n, P_N)] =$$

.

$$= 2\left(\mathbb{E}\left(\left|\partial\left((1-r)B_2^n\right)\cap P_N^C\right|\right) - \mathbb{E}\left(\left|\left(1-r\right)B_2^n \cap \partial P_N\right)\right|\right)\right)$$

.

$$= 2\left(I_1 - I_2\right)$$

First consider

$$
\begin{aligned}
I_1 &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_{n-1}\left[\partial\left((1-r)B_2^n\right) \cap P_N^c\right] \mathbb{1}_{\{0 \in \mathrm{int}(P_N)\}} d\mathbb{P}(x) \\
&+ \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_{n-1}\left[\partial\left((1-r)B_2^n\right) \cap P_N^c\right] \cdot \mathbb{1}_{\{0 \notin \mathrm{int}(P_N)\}} d\mathbb{P}(x)
\end{aligned}
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I_1 &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_{n-1}\left[\partial\left((1-r)B_2^n\right) \cap P_N^c\right] \mathbb{1}_{\{0 \in \mathrm{int}(P_N)\}} d\mathbb{P}(x) \\
&+ \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_{n-1}\left[\partial\left((1-r)B_2^n\right) \cap P_N^c\right] \cdot \mathbb{1}_{\{0 \notin \mathrm{int}(P_N)\}} d\mathbb{P}(x) \\
&\le \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_{n-1}\left[\partial\left((1-r)B_2^n\right) \cap P_N^c\right] \mathbb{1}_{\{0 \in \mathrm{int}(P_N)\}} d\mathbb{P}(x) \\
&+ \mathrm{vol}_{n-1}(\partial B_2^n)\, \mathbb{P}(0 \notin \mathrm{int}(P_N))
\end{aligned}
$$

where $d\mathbb{P}(x) = d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)$

By a result of Wendel, the second summand equals

$$\begin{aligned}\mathrm{vol}_{n-1}(\partial B_2^n)\,\mathbb{P}(0 \notin \mathsf{int}(P_N)) &= \mathrm{vol}_{n-1}\,(\partial B_2^n)\,2^{-N+1}\sum_{k=0}^{n-1}\binom{N-1}{k} \\ &\le \mathrm{vol}_{n-1}\,(\partial B_2^n)\,2^{-N+1} n N^n.\end{aligned}$$

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The second summand is essentially of the order 2^{-N}, and it turns out that the first summand is of the order of $N^{-\frac{2}{n-1}}$. Therefore, we can concentrate on the first summand

$$\int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_{n-1}\left[\partial\left((1-r)B_2^n\right) \cap P_N^c\right] \mathbb{1}_{\{0 \in \mathsf{int}(P_N)\}} d\mathbb{P}(x)$$