Gaussian Random Fields: Geometric Properties and Extremes

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Outline

Lecture 1: Gaussian random fields and their regularity

Lecture 2: Hausdorff dimension results and hitting probabilities

Lecture 3: Strong local nondeterminism and fine properties, I

Lecture 4: Strong local nondeterminism and fine properties, II

Lecture 5: Extremes and excursion probabilities

Lecture 1 Gaussian random fields and their regularity

- Introduction
- Construction of Gaussian random fields
- Regularity of Gaussian random fields
- A review of Hausdorff measure and dimension

1.1 Introduction

A random field $X = \{X(t), t \in T\}$ is a family of random variables with values in state space S, where T is the parameter set.

If $T \subseteq \mathbb{R}^N$ and $S = \mathbb{R}^d$ $(d \geq 1)$, then X is called an (N, d) random field. They arise naturally in

- turbulence (e.g., A. N. Kolmogorov, 1941)
- oceanography (M.S. Longuet-Higgins, 1953, ...)
- spatial statistics, spatio-temporal geostatistics (G. Mathron, 1962)
- image and signal processing

Examples:

$X(t, \mathbf{x})$ = the height of an ocean surface above certain nominal plane at time $t \geq 0$ and location $\mathbf{x} \in \mathbb{R}^2$.

$X(t, \mathbf{x})$ = wind speed at time $t \geq 0$ and location $\mathbf{x} \in \mathbb{R}^3$.

$X(t, \mathbf{x})$ = the levels of d pollutants (e.g., ozone, $PM_{2.5}$, nitric oxide, carbon monoxide, etc) measured at location $\mathbf{x} \in \mathbb{R}^3$ and time $t \geq 0$.

Theory of random fields

1. How to construct random fields?
2. How to characterize and analyze random fields?
3. How to estimate parameters in random fields?
4. How to use random fields to make predictions?

In this short course, we provide a brief introduction to (1) and (2).

1.2 Construction and characterization of random fields

- Construct covariance functions
- For stationary Gaussian random fields, use spectral representation theorem
- For random fields with stationary increments or random intrinsic functions, use Yaglom (1957) and Matheron (1973)
- Stochastic partial differential equations
- Scaling limits of discrete systems

1.2.1 Stationary random fields and their spectral representations

A real-valued random field $\{X(t), t \in \mathbb{R}^N\}$ is called second-order stationary if $\mathbb{E}(X(t)) \equiv m$, where m is a constant, and the covariance function depends on $s - t$ only:

$$\mathbb{E}\big[(X(s) - m)(X(t) - m)\big] = C(s - t), \quad \forall s, t \in \mathbb{R}^N.$$

Note that C is positive definite: For all $n \geq 1$, $t^j \in \mathbb{R}^N$ and all complex numbers $a_j \in \mathbb{C}$ $(j = 1, \ldots, n)$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} C(t^i - t^j) \geq 0.$$

Bochner's theorem

Theorem (Bochner, 1932)

A bounded *continuous* function C is positive definite if and only if there is a finite Borel measure μ such that

$$C(t) = \int_{\mathbb{R}^N} e^{i\langle t, x\rangle}\, d\mu(x), \qquad \forall t \in \mathbb{R}^N.$$

Spectral representation theorem

In particular, if $X = \{X(t), t \in \mathbb{R}^N\}$ is a centered, stationary Gaussian random field with values in $\mathbb{R}$ whose covariance function is the Fourier transform of μ, then there is a complex-valued Gaussian random measure $\widetilde{W}$ on $\mathcal{A} = \{A \in \mathcal{B}(\mathbb{R}^N) : \mu(A) < \infty\}$ such that $\mathbb{E}\big(\widetilde{W}(A)\big) = 0$,

$$\mathbb{E}\big(\widetilde{W}(A)\overline{\widetilde{W}(B)}\big) = \mu(A \cap B) \quad \text{and} \quad \widetilde{W}(-A) = \overline{\widetilde{W}(A)}$$

and X has the following Wiener integral representation:

$$X(t) = \int_{\mathbb{R}^N} e^{i\langle t, x\rangle}\, d\widetilde{W}(x).$$

The finite measure μ is called the spectral measure of X.

The Matérn class

An important class of isotropic stationary random fields are those with the Matérn covariance function

$$C(t) = \frac{1}{\Gamma(\nu)2^{\nu-1}}\left(\sqrt{2\nu}\frac{|t|}{\rho}\right)^{\nu} K_\nu\left(\sqrt{2\nu}\frac{|t|}{\rho}\right),$$

where Γ is the gamma function, K_ν is the modified Bessel function of the second kind, and ρ and ν are non-negative parameters.
Since the covariance function $C(t)$ depends only on the Euclidean norm $|t|$, the corresponding Gaussian field X is called isotropic.

By the inverse Fourier transform, one can show that the spectral measure of X has the following density function:

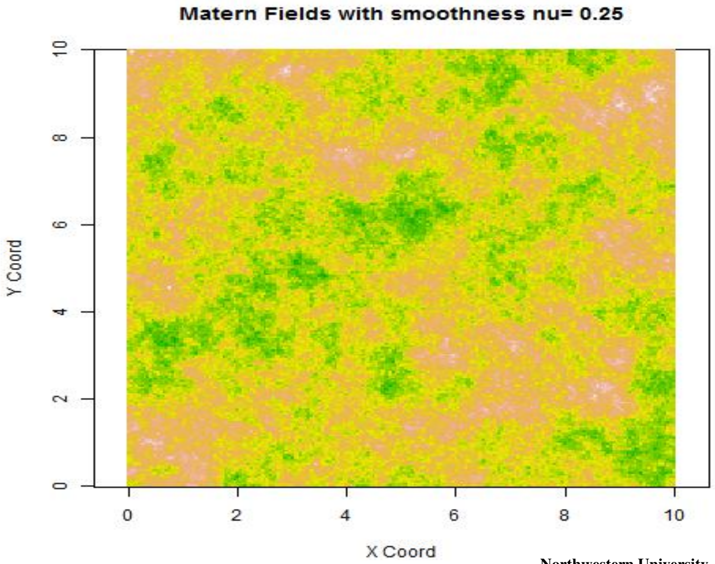
$$f(\lambda) = \frac{1}{(2\pi)^N} \frac{1}{(|\lambda|^2 + \frac{\rho^2}{2\nu})^{\nu + \frac{N}{2}}}, \quad \forall \lambda \in \mathbb{R}^N.$$

Whittle (1954) showed that the Gaussian random field X can be obtained as the solution to the following fractional SPDE

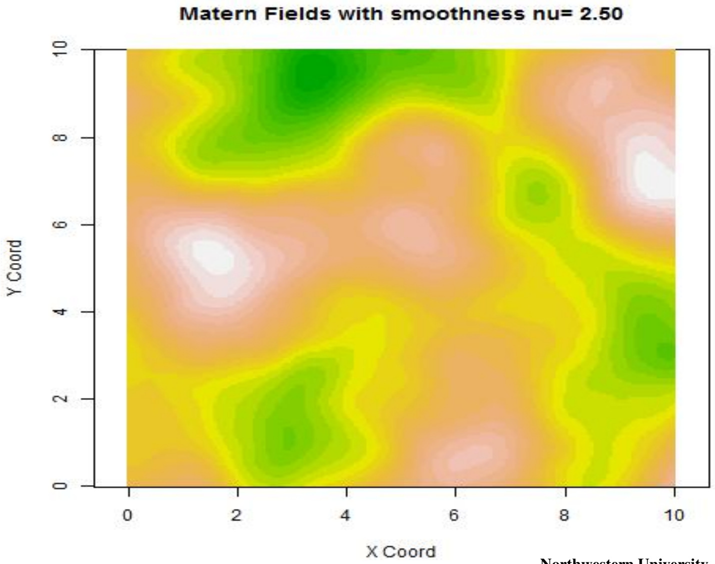
$$\Big(\Delta + \frac{\rho^2}{2\nu}\Big)^{\frac{\nu}{2} + \frac{N}{4}} X(t) = \dot{W}(t),$$

where $\Delta = \frac{\partial^2}{dt_1^2} + \cdots + \frac{\partial^2}{dt_N^2}$ is the N-dimensional Laplacian, and $\dot{W}(t)$ is the white noise.

A smooth Gaussian field: $N = 2, \nu = .25$



A smooth Gaussian field: $N = 2, \nu = 2.5$



Recent extensions:

- Random fields on a spatial-temporal domain
  - In statistics, one needs to consider random fields defined on the spatial-temporal domain $\mathbb{R}^N \times \mathbb{R}$. It is often not reasonable to assume that these random fields are isotropic. Various anisotropic random fields have been constructed (Cressie and Huang 1999, Stein 2005; Biermé, et al. 2007; X. 2009; Li and X. 2011)
- Multivariate (stationary) random fields
- Random fields on the spheres and other manifolds.

1.2.2 Gaussian fields with stationary increments

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. Yaglom (1954) showed that, if $R(s,t) = \mathbb{E}\big[X(s)X(t)\big]$ is continuous, then $R(s,t)$ can be written as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{i\langle s,\lambda\rangle} - 1)(e^{-i\langle t,\lambda\rangle} - 1)\Delta(d\lambda),$$

where $\Delta(d\lambda)$ is a Borel measure which satisfies

$$\int_{\mathbb{R}^N} (1 \wedge |\lambda|^2)\, \Delta(d\lambda) < \infty. \tag{1}$$

The measure Δ is called the *spectral measure* of X.

It follows that

$$\mathbb{E}\big[(X(s)-X(t))^2\big] = 2\int_{\mathbb{R}^N} \big(1-\cos\langle s-t,\lambda\rangle\big)\Delta(d\lambda);$$

and X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle}-1\big)\,\widetilde{W}(d\lambda),$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $\widetilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

Gaussian fields with stationary increments can be constructed by choosing spectral measures Δ.

Two examples

Example 1 If Δ has a density function

$$f_H(\lambda) = c(H,N)|\lambda|^{-(2H+N)},$$

where $H \in (0,1)$ and $c(H,N) > 0$, then X is fractional Brownian motion with index H.

It can be verified that (for proper choice of $c(H,N)$),

$$\begin{aligned}\mathbb{E}\big[(X(s)-X(t))^2\big] &= 2c(H,N)\int_{\mathbb{R}^N}\frac{1-\cos\langle s-t,\lambda\rangle}{|\lambda|^{2H+N}}\,d\lambda\\ &= |s-t|^{2H}.\end{aligned}$$

For the last identity, see, e.g., Schoenberg (1939).

- FBm X has stationary increments: for any $b \in \mathbb{R}^N$,

$$\left\{X(t+b) - X(b),\ t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{X(t),\ t \in \mathbb{R}^N\right\},$$

  where $\stackrel{d}{=}$ means equality in finite dimensional distributions.
- FBm X is H-self-similar: for every constant $c > 0$,

$$\left\{X(ct),\ t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c^H X(t),\ t \in \mathbb{R}^N\right\}.$$

Example 2 A large class of Gaussian fields can be obtained by letting spectral density functions satisfy (1) and

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \geq 1, \qquad (2)$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and $\gamma > \sum_{j=1}^{N} \frac{1}{\beta_j}$.
More conveniently, we re-write (2) as

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{H_j}\right)^{Q+2}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \geq 1, \quad (3)$$

where $H_j = \frac{\beta_j}{2}\left(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\right)$ and $Q = \sum_{j=1}^{N} H_j^{-1}$.

1.2.3 The Brownian sheet and fractional Brownian sheets

The Brownian sheet $W = \{W(t),\ t \in \mathbb{R}_+^N\}$ is a centered (N, d)-Gaussian field whose covariance function is

$$\mathbb{E}\big[W_i(s)W_j(t)\big] = \delta_{ij} \prod_{k=1}^{N} s_k \wedge t_k.$$

- When $N = 1$, W is Brownian motion in $\mathbb{R}^d$.
- W is $N/2$-self-similar, but it does not have stationary increments.

- Fractional Brownian sheet $W^{\vec{H}} = \left\{ W^{\vec{H}}(t), t \in \mathbb{R}^N \right\}$ is a mean zero Gaussian field in $\mathbb{R}$ with covariance function

$$\mathbb{E}\left[W^{\vec{H}}(s) W^{\vec{H}}(t) \right] = \prod_{j=1}^{N} \frac{1}{2} \left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j} \right)$$

where $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$.
For all constants $c > 0$,

$$\left\{ W^{\vec{H}}(c^E t), \, t \in \mathbb{R}^N \right\} \stackrel{d}{=} \left\{ c \, W^{\vec{H}}(t), \; t \in \mathbb{R}^N \right\},$$

where $E = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/(NH_i)$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \neq j$.

1.2.4 Linear stochastic heat equation

As an example, we consider the solution of the linear stochastic heat equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + \sigma\, \dot{W} \\ u(0,x) &\equiv 0, \qquad \forall x \in \mathbb{R}. \end{aligned} \tag{4}$$

It follows from Walsh (1986) that the mild solution of (4) is the mean zero Gaussian random field $u = \{u(t,x), t \geq 0], x \in \mathbb{R}\}$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \widetilde{G}_{t-r}(x-y)\, \sigma W(drdy), \quad t \geq 0, x \in \mathbb{R},$$

where $\widetilde{G}_t(x)$ is the Green kernel given by

$$\widetilde{G}_t(x) = (4\pi t)^{-1/2} \exp\left(- \frac{|x|^2}{4t} \right), \qquad \forall\, t > 0, x \in \mathbb{R}.$$

One can verify that

- for every fixed $x \in \mathbb{R}$, the process $\{u(t,x), t \in [0,T]\}$ is a bi-fractional Brownian motion.
- For every fixed $t > 0$, the process $\{u(t,x), x \in \mathbb{R}\}$ is stationary with an explicit spectral density function.

This allows to study the properties of $u(t,x)$ in the time and space-variables either separately or jointly.

1.3 Regularity of Gaussian random fields

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a random field. For each $\omega \in \Omega$, the function $X(\cdot, \omega) : \mathbb{R}^N \to \mathbb{R}^d$:

$$t \mapsto X(t, \omega)$$

is called a sample function of X.

The following are natural questions:

(i) When are the sample functions of X bounded, or continuous?

(ii) When are the sample functions of X differentiable?

(iii) How to characterize the analytic and geometric properties of $X(\cdot)$ precisely?

Let $X = \{X(t), t \in T\}$ be a centered Gaussian process with values in $\mathbb{R}$, where (T, τ) is a metric space; e.g., $T = [0, 1]^N$, or $T = \mathbb{S}^{N-1}$.
We define a pseudo metric $d_X(\cdot, \cdot) : T \times T \to [0, \infty)$ by

$$d_X(s, t) = \left\{\mathbb{E}[X(t) - X(s)]^2\right\}^{\frac{1}{2}}.$$

(d_X is often called the canonical metric for X.)
Let $D = \sup_{t,s \in T} d_X(s, t)$ be the diameter of T, under the pseudo metric d_X.

For any $\varepsilon > 0$, let $N(T, d_X, \varepsilon)$ be the minimum number of d_X-balls of radius ε that cover T.

Dudley's Theorem

$H(T, \varepsilon) = \sqrt{\log N(T, d_X, \varepsilon)}$ is called the **metric entropy** of T.

Theorem 1.3.1 [Dudley, 1967]

Assume $N(T, d_X, \varepsilon) < \infty$ for every $\varepsilon > 0$. If

$$\int_0^D \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon < \infty.$$

Then there exists a modification of X, still denoted by X, such that

$$\mathbb{E}\Big(\sup_{t \in T} X(t)\Big) \le 16\sqrt{2} \int_0^{\frac{D}{2}} \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon. \qquad (5)$$

The proof of Dudley's Theorem is based on a chaining argument, which is similar to that of Kolmogorov's continuity theorem. See Talagrand (2005), Marcus and Rosen (2007).

Example: For a Gaussian random field $\{X(t), t \in T\}$ satisfying

$$d_X(s,t) \asymp \left(\log \frac{1}{|s-t|} \right)^{-\gamma},$$

its sample functions are continuous if $\gamma > 1/2$.

- Fernique (1975) proved that (5) is also necessary if X is a Gaussian process with stationary increments.
- In general, (5) is not necessary for sample boundedness and continuity.

Majorizing measure

For a general Gaussian process, Talagrand (1987) proved the following necessary and sufficient for the boundedness and continuity.

Theorem 1.3.2 [Talagrand, 1987]

Let $X = \{X(t), t \in T\}$ be a centered Gaussian process with values in $\mathbb{R}$. Suppose $D = \sup_{t,s\in T} d_X(s,t) < \infty$. Then X has a modification which is bounded on T if and only if there exists a probability measure μ on T such that

$$\sup_{t\in T} \int_0^D \left(\log \frac{1}{\mu(B(t,u))} \right)^{1/2} du < \infty. \qquad (6)$$

Majorizing measure

Theorem 1.3.2 (Continued)

There exists a modification of X with bounded, uniformly continuous sample functions if and only if there exists a probability measure μ on T such that

$$\lim_{\varepsilon \to 0} \sup_{t \in T} \int_0^{\varepsilon} \left(\log \frac{1}{\mu(B(t,u))} \right)^{1/2} du = 0. \qquad (7)$$

Uniform modulus of continuity

Theorem 1.3.3

Under the condition of Theorem 1.3.1, there exists a random variable $\eta \in (0, \infty)$ and a constant $K > 0$ such that for all $0 < \delta < \eta$,

$$\omega_{X,d_X}(\delta) \le K \int_0^\delta \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon,$$

where $\omega_{X,d_X}(\delta) = \sup_{s,t \in T,\, d_X(s,t) \le \delta} |X(t) - X(s)|$ is the modulus of continuity of $X(t)$ on (T, d_X).

Corollary 1.3.4

Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be a fractional Brownian motion with index $H \in (0,1)$. Then B^H has a modification, still denoted by B^H, whose sample functions are almost surely continuous. Moreover,

$$\limsup_{\varepsilon \to 0} \frac{\max_{t\in[0,1]^N, |s|\le \varepsilon} |B^H(t+s) - B^H(t)|}{\varepsilon^H \sqrt{\log 1/\varepsilon}} \le K, \quad a.s.$$

Proof: Recall that $d_{B^H}(s,t) = |s-t|^H$ and $\forall\, \varepsilon > 0$,

$$N\left([0,1]^N, d_{B^H}, \varepsilon\right) \le K \left(\frac{1}{\varepsilon^{1/H}}\right)^N.$$

It follows from Theorem 1.3.3 that $\exists$ a random variable $\eta > 0$ and a constant $K > 0$ such that

$$\omega_{B^H}(\delta) \le K \int_0^\delta \sqrt{\log\left(\frac{1}{\varepsilon^{1/H}}\right)}\, d\varepsilon \le K\,\delta \sqrt{\log \frac{1}{\delta}} \quad \text{a.s.}$$

Returning to the Euclidean metric and noticing

$$d_{B^H}(s,t) \le \delta \iff |s-t| \le \delta^{1/H},$$

yields the desired result.

Later on, we will prove that there is a constant $K \in (0, \infty)$ such that

$$\limsup_{\varepsilon \to 0} \frac{\max_{t \in [0,1]^N, |s| \le \varepsilon} |B^H(t+s) - B^H(t)|}{\varepsilon^H \sqrt{\log 1/\varepsilon}} = K, \quad a.s.$$

This is an analogue of Lévy's uniform modulus of continuity for Brownian motion.

Differentiability

(i). Mean-square differentiability: the mean square partial derivative of X at t is defined as

$$\frac{\partial X(t)}{\partial t_j} = \text{l.i.m}_{h\to 0}\frac{X(t+he_j)-X(t)}{h},$$

where e_j is the unit vector in the j-th direction.
For a Gaussian field, sufficient conditions can be given in terms of the differentiability of the covariance function (Adler, 1981).
(ii). Sample path differentiability: the sample function $t \mapsto X(t)$ is differentiable. This is much stronger and more useful than (i).

Sample path differentiability of $X(t)$ can be proved by using criteria for continuity.

Consider a centered Gaussian field with stationary increments whose spectral density function satisfies

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N, \ \ |\lambda| \geq 1, \qquad (8)$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and

$$\gamma > \sum_{j=1}^{N} \frac{1}{\beta_j}.$$

Differentiability

Theorem 1.3.5 (Xue and X. 2011)

(i) If

$$\beta_j\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) > 2, \qquad (9)$$

then the partial derivative $\partial X(t)/\partial t_j$ is continuous almost surely. In particular, if (9) holds for all $1 \le j \le N$, then almost surely $X(t)$ is continuously differentiable.

(ii) If

$$\max_{1\le j\le N} \beta_j\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) \le 2, \qquad (10)$$

then $X(t)$ is not differentiable in any direction.

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1.4 A review of Hausdorff measure and dimension

Let Φ be the class of functions $\varphi : (0, \delta) \to (0, \infty)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \le K \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

A function φ in Φ is often called a *measure function* or *gauge function*.

For example, $\varphi(s) = s^\alpha$ $(\alpha > 0)$ and $\varphi(s) = s^\alpha \log\log(1/s)$ are measure functions.

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Given $\varphi \in \Phi$, the *φ-Hausdorff measure* of $E \subseteq \mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon \right\}, \tag{11}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The sequence of balls satisfying the two conditions on the right-hand side of (11) is called an ε-covering of E.

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Theorem 1.4.1 [Hausdorff, etc]

- φ-m is a Carathéodory outer measure.
- The restriction of φ-m to $\mathcal{B}(\mathbb{R}^d)$ is a [Borel] measure.
- If $\varphi(s) = s^d$, then $\varphi\text{-}m\big|_{\mathcal{B}(\mathbb{R}^d)} = c\times$ Lebesgue measure on $\mathbb{R}^d$.

A function $\varphi \in \Phi$ is called *an exact (or a correct) Hausdorff measure function* for E if $0 < \varphi\text{-}m(E) < \infty$.

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If $\varphi(s) = s^\alpha$, we write φ-$m(E)$ as $\mathcal{H}_\alpha(E)$. The following lemma is elementary.

Lemma

1. *If* $\mathcal{H}_\alpha(E) < \infty$, *then* $\mathcal{H}_{\alpha+\delta}(E) = 0$ *for all* $\delta > 0$.
2. *If* $\mathcal{H}_\alpha(E) = \infty$, *then* $\mathcal{H}_{\alpha-\delta}(E) = \infty$ *for all* $\delta \in (0\,,\alpha)$.
3. *For any* $E \subset \mathbb{R}^d$, *we have* $\mathcal{H}_{d+\delta}(E) = 0$.

The *Hausdorff dimension* of E is defined by

$$\dim_{\rm H} E = \inf\left\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = 0\right\} = \sup\left\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = \infty\right\},$$

Convention: $\sup \varnothing := 0$.

Lemma

1. $E \subset F \subset \mathbb{R}^d \ \Rightarrow \dim_{\rm H} E \le \dim_{\rm H} F \le d$.
2. *(σ-stability)* $\dim_{\rm H}\left(\bigcup_{j=1}^\infty E_j\right) = \sup_{j\ge 1} \dim_{\rm H} E_j$.

Example: Cantor's set

- Let C denote the standard ternary Cantor set in $[0\,,1]$. At the nth stage of its construction, C is covered by 2^n intervals of length/diameter 3^{-n} each.
- Therefore, for $\alpha = \log_3 2$,

$$\mathcal{H}_\alpha(C) \le \lim_{n\to\infty} 2^n \cdot 3^{-n\alpha} = 1.$$

- Thus, we obtain $\dim_{\rm H} C \le \log_3 2 \approx 0.6309$.
- We will prove later that this is an equality and

$$0 < \mathcal{H}_{\log_3 2}(C) < \infty.$$

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$$0 < \mathcal{H}_{\log_3 2}(C) < \infty.$$

Example: Cantor's set

- Let C denote the standard ternary Cantor set in $[0\,,1]$. At the nth stage of its construction, C is covered by 2^n intervals of length/diameter 3^{-n} each.
- Therefore, for $\alpha = \log_3 2$,

$$\mathcal{H}_\alpha(C) \le \lim_{n\to\infty} 2^n \cdot 3^{-n\alpha} = 1.$$

- Thus, we obtain $\dim_{\rm H} C \le \log_3 2 \approx 0.6309$.
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Example: Images of Brownian motion

Let $B([0,1])$ be the image of Brownian motion in $\mathbb{R}^d$. Lévy (1948) and Taylor (1953) proved that

$$\dim_{\rm H} B([0,1]) = \min\{d, 2\} \qquad \text{a.s.}$$

Ciesielski and Taylor (1962), Ray and Taylor (1964) proved that

$$0 < \varphi_d\text{-}m\big(B([0,1])\big) < \infty \quad \text{a.s.},$$

where

$$\varphi_1(r) = r$$
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A method for determining upper bounds

Lemma

Let $I \subset \mathbb{R}^N$ be a hyper-cube. If there is a constant $\alpha \in (0,1)$ such that for every $\varepsilon > 0$, the function $f : I \to \mathbb{R}^d$ satisfies a uniform Hölder condition of order $\alpha - \varepsilon$ on I, then for every Borel set $E \subset I$

$$\dim_{\rm H} f(E) \le \min\left\{d,\ \frac{1}{\alpha}\dim_{\rm H} E\right\}, \tag{12}$$

$$\dim_{\rm H} {\rm Gr}f(E) \le \min\left\{\frac{1}{\alpha}\dim_{\rm H} E,\ \dim_{\rm H} E + (1-\alpha)d\right\}, \tag{13}$$

where ${\rm Gr}f(E) = \{(t, f(t)) : t \in E\}$.

Proof of (12)

For any $\gamma > \dim_{\rm H} E$, there is a covering $\{B(x_i, r_i)\}$ of E such that

$$\sum_{i=1}^{\infty} (2r_i)^{\gamma} \le 1.$$

For any fixed $\varepsilon \in (0, \alpha)$, $f(B(x_i, r_i))$ is contained in a ball in $\mathbb{R}^d$ of radius $r_i^{\alpha-\varepsilon}$, which yields a covering of $f(E)$. Since

$$\sum_{i=1}^{\infty} \left(r_i^{\alpha-\varepsilon}\right)^{\gamma/(\alpha-\varepsilon)} \le 1,$$

we have $\dim_{\rm H} f(E) \le \gamma/(\alpha - \varepsilon)$. Letting $\varepsilon \downarrow 0$ and $\gamma \downarrow \dim_{\rm H} E$ yield (12).

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A method for determining lower bounds

$\mathcal{P}(E) :=$ all probability measures that are supported in E.

Lemma (Frostman, 1935)

Let E be a Borel subset of $\mathbb{R}^d$ and $\alpha > 0$ be a constant. Then $\mathcal{H}_\alpha(E) > 0$ if and only if there exist $\mu \in \mathcal{P}(E)$ and a constant K such that

$$\mu(B(x,r)) \le K\, r^\alpha \quad \forall\, x \in \mathbb{R}^d, r > 0.$$

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Theorem 1.4.2 [Frostman, 1935]

Let E be a Borel subset of $\mathbb{R}^d$. Suppose there exist $\alpha > 0$ and $\mu \in \mathcal{P}(E)$ such that

$$I_\alpha(\mu) := \iint \frac{\mu(dx)\, \mu(dy)}{|x-y|^\alpha} < \infty.$$

Then, $\dim_{\rm H} E \geq \alpha$.

- $I_\alpha(\mu) :=$ the α-dimensional [Bessel-] Riesz energy of μ.
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Then $\mu(E_\lambda) > 0$ for λ large enough. To show $\dim_{\rm H}(E_\lambda) \ge \alpha$, we take an arbitrary ε-covering $\{B(x_i, r_i)\}$ of E_λ. WLOG, we assume $x_i \in E_\lambda$.

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Using Frostman's lemma, one can also prove that if $\alpha < \dim_{\rm H} E$, then there exists a probability measure μ on E such that $I_\alpha(\mu) < \infty$, so $\mathcal{C}_\alpha(E) > 0$.

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Corollary

Let E be a Borel subset of $\mathbb{R}^d$. Then

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For any Borel set $E \subset \mathbb{R}_+$, $\dim_{\rm H} B(E) = \min\{d\,, 2\dim_{\rm H} E\}$ a.s.

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An upper density theorem

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *upper φ-density* of μ at $x \in \mathbb{R}^d$ is defined as

$$\overline{D}_\mu^\varphi(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

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Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

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Then for every $x \in C$ and any $r \in (0, 1)$,

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