Lecture 2: Hausdorff dimension results and hitting probabilities

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Outline

1. Fractal properties of Gaussian random fields
2. Hitting probabilities and Riesz-type capacity
3. Brownian motion and thermal capacity

Geometric properties of Gaussian fields

Given an (N, d)-random field $X = \{X(t), t \in \mathbb{R}^N\}$, it is of interest to study the geometric properties of the following random sets:

- Range $X\left([0,1]^N\right) = \left\{X(t) : t \in [0,1]^N\right\}$.
- Graph $\mathrm{Gr}X\left([0,1]^N\right) = \left\{(t, X(t)) : t \in [0,1]^N\right\}$.
- Level set $X^{-1}(x) = \left\{t \in \mathbb{R}^N : X(t) = x\right\}$.
- Excursion set $X^{-1}(F) = \left\{t \in \mathbb{R}^N : X(t) \in F\right\}$, $\forall F \subset \mathbb{R}^d$.
- Multiple points, etc.

Excursion sets

Let X be real-valued. For any $u \in \mathbb{R}$,

$$A_u(X,T) = \{t \in T : X(t) > u\}$$

is called an *excursion set* of X above the level u.
This is closely related to the exceedance probability

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) > u\Big\},$$

which is very important in many applications.

- If $X(t)$ is smooth, we use integral geometry to characterize the topological structures of sets generated by X. See the books by Adler and Taylor (2007), Azaïs and Wschebor (2009).

- If $X(t)$ is not smooth, one uses fractal geometry to study the random sets generated by X.

2.1 Hausdorff dimension and Capacity

For any metric ρ on $\mathbb{R}^N$, any $\beta > 0$ and $E \subseteq \mathbb{R}^N$, the β-dimensional Hausdorff measure in the metric ρ of E is defined by

$$\mathcal{H}_\rho^\beta(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{n=1}^{\infty} (2r_n)^\beta : E \subseteq \bigcup_{n=1}^{\infty} B_\rho(r_n),\ r_n \le \delta \right\},$$

where $B_\rho(r)$ denotes an open ball of radius r in the metric space $(\mathbb{R}^N, \rho)$.

The corresponding Hausdorff dimension of E is defined by

$$\dim_{\rm H}^\rho E = \inf \left\{ \beta > 0 : \mathcal{H}_\rho^\beta(E) = 0 \right\}.$$

The Bessel-Riesz type capacity of order α on the metric space $(\mathbb{R}^N, \rho)$ is defined by

$$\mathcal{C}^\alpha_\rho(E) = \left[\inf_{\mu \in \mathcal{P}(E)} \int \int f_\alpha\big(\rho(u,v)\big)\mu(du)\mu(dv) \right]^{-1},$$

where $\mathcal{P}(E)$ is the family of probability measures carried by E and the function $f_\alpha : (0,\infty) \to (0,\infty)$ is defined by

$$f_\alpha(r) = \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \log\left(\frac{e}{r \wedge 1}\right) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \qquad (1)$$

ρ will be omitted if it is the Euclidean metric.

2.1 Hausdorff dimensions of the range and graph

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{2}$$

where $X_1, \ldots, X_d$ are independent copies of a centered GF X_0.

Recall that the canonical metric of X_0 is

$$d_{X_0}(s,t) = \sqrt{\mathbb{E}\big(X_0(s) - X_0(t)\big)^2}.$$

Many fractal properties of X are determined by the behavior of $d_{X_0}(s,t)$ as $s \to t$.

We will make use of the following condition:
(C1). $\exists$ positive constants c_1 and c_2 such that for all $s, t \in I(= [\varepsilon, 1]^N)$,

$$c_1 \sum_{j=1}^{N} |s_j - t_j|^{2H_j} \le d_{X_0}(s,t)^2 \le c_2 \sum_{j=1}^{N} |s_j - t_j|^{2H_j},$$

where $0 < H_1 \le \ldots \le H_N \le 1$ are constants.
Hence the canonical metric $d_{X_0}(s,t) \asymp \sum_{j=1}^{N} |s_j - t_j|^{H_j}$, which will be denoted by $\rho(s,t)$ below.

2.1 Hausdorff dimensions the range and graph

Theorem 2.1 (Ayache and X. 2005; X. 2009)

Assume Condition (C1) holds. Then almost surely

$$\dim_{\mathrm{H}} X\left([0,1]^N\right) = \min\left\{d;\ \sum_{j=1}^{N} \frac{1}{H_j}\right\},$$

$$\dim_{\mathrm{H}} \mathrm{Gr} X\left([0,1]^N\right)$$

$$= \min_{1\le k\le N}\left\{\sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d;\ \sum_{j=1}^{N} \frac{1}{H_j}\right\}.$$

The proof of these results rely on regularity properties of X and the connection of Hausdorff dimension and capacity.

- The above results are significantly different from the isotropic case, as well as the space-anisotropic case.
- If $d = 1$, then

$$\dim_{\rm H} {\rm Gr} X \left([0,1]^N\right) = N + 1 - H_1 \quad \text{a.s.}$$

Hence one needs $d > 1$ to recover all the parameters $H_1, \ldots, H_N$.

- To determine $\dim_{\rm H} X(E)$ for an arbitrary Borel set $E \subset \mathbb{R}^N$, one needs to use a different Hausdorff-type dimension; see Wu and X. (2007), X. (2009).

2.2 Hitting probabilities

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. We consider the following intersection problems:

(1) For Borel sets $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$, when can one has

$$\mathbb{P}(X(E) \cap F \neq \emptyset) > 0? \tag{3}$$

(2) [k-multiple points] Given disjoint sets $E_1, \ldots, E_k \subseteq \mathbb{R}^N$, when does

$$\mathbb{P}(X(E_1) \cap \cdots \cap X(E_k) \cap F \neq \emptyset) > 0? \tag{4}$$

Some history about (1) and (2)

In the case when $E = [a, b]$, $(a, b \in \mathbb{R}^N)$, a necessary and sufficient condition for **(1)** in terms of certain kind of capacity of F has been established for X being

- Brownian motion (Kakutani, 1944)
- Lévy processes (Port and Stone, 1971)
- Some multiparameter Markov processes (Fitzsimmons and Salisbury, 1989)
- The Brownian sheet (Khoshnevisan and Shi, 1999)
- Additive Lévy processes (Khoshnevisan and Xiao, 2002, 2003, 2009)
- SPDEs (Dalang and Nualart, 2004, Dalang, Khoshnevisan and Nualart, 2007, ...)

Question **(1)** include intersections of the graph set and level sets:

- Let $\mathrm{Gr}X(E) = \{(t, X(t)) : t \in E\}$ be the graph of X on E. Then **(1)** is equivalent to

$$\mathbb{P}(\mathrm{Gr}X(E) \cap (E \times F) \neq \emptyset) > 0.$$

- Take $F = \{0\}$, then (1) is equivalent to

$$\mathbb{P}(X^{-1}(0) \cap E \neq \emptyset) > 0.$$

- For general $E \subseteq \mathbb{R}$ and $F \subseteq \mathbb{R}^d$, a necessary and sufficient condition for (1) in terms of "thermal capacity" of $E \times F$ was established for Brownian motion $W = \{W(t), t \geq 0\}$ by Watson (1978). This is the only known complete characterization in this generality.

- The Hausdorff dimension $W(E) \cap F$ is recently determined by Khoshnevisan and Xiao (2015).

- In the special case of $F = \{0\}$, the hitting probability is characterized by Khoshnevisan and Xiao (2002) for a large class of additive Lévy processes, and by Khoshnevisan and Xiao (2007) for the Brownian sheet.

Question (2) is related to existence of self-intersections.

- When $F = \mathbb{R}^d$, then (2) gives existence of k-multiple points.
  - Lévy processes (Khoshnevisan and Xiao, 2005): $F = \mathbb{R}^d$, general $E_1, \ldots, E_k$
  - The Brownian sheet: Dalang et al (2012), Dalang and Mueller (2014): $F = \mathbb{R}^d$, $E_1, \ldots, E_k$ are intervals.
- No results for general $F, E_1, \ldots, E_k$.

In the rest of this talk, we present some results from the following three papers:

- Biermé, H., Lacaux, C. and Xiao, Y. (2009). Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. *Bull. London Math. Soc.* **41**, 253–273.

- Khoshnevisan, D. and Xiao Y. (2015). Brownian motion and thermal capacity. *Ann. Probab.* **43**, 405–434.

- Dalang, R., Mueller, C. and Xiao Y. (2015). Polarity of points for Gaussian random fields. *http://arxiv.org/pdf/1505.05417v1.pdf*.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined by (2) that satisfies the following conditions:

(C1). $\exists$ positive constants c_1 and c_2 such that for all $s, t \in I (= [\varepsilon, 1]^N)$,

$$c_1\, \rho(s,t) \le d_{X_0}(s,t) \le c_2\, \rho(s,t),$$

where $0 < H_1 \le \ldots \le H_N \le 1$ are constants,

(C2). $\exists$ $c_3 > 0$ such that for all $s, t \in I$,

$$\mathrm{Var}\left(X_0(t)\big|X_0(s)\right) \ge c_3 \sum_{j=1}^{N} |s_j - t_j|^{2H_j}.$$

Hitting probabilities and Riesz capacity

The following result was motivated by Dalang, Khoshnevisan and Nualart (2007) and X. (1999) for fractional Brownian motion.

Theorem 2.2 [Biermé, Lacaux and X. (2009)]

If X is defined by (2) such that X_0 satisfies Conditions (C1) and (C2) on I. Then $\forall$ Borel set $F \subset \mathbb{R}^d$,

$$c_4\, \mathcal{C}^{d-Q}(F) \le \mathbb{P}\big\{X(I) \cap F \neq \emptyset\big\} \le c_5\, \mathcal{H}^{d-Q}(F), \qquad (5)$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$, $\mathcal{C}^{d-Q}$ is $(d-Q)$-dimensional Riesz capacity and $\mathcal{H}^{d-Q}$ is $(d-Q)$-dimensional Hausdorff measure.

Proof of Theorem 2.2: the upper bound

For proving the upper bound in (5), we make use of a covering argument and the following lemma.

Lemma 2.1 [Biermé, Lacaux and X. (2009)]

Assume the conditions of Theorem 2.2 hold. For any constant $M > 0$, there exist positive constants c and δ_0 such that for all $r \in (0, \delta_0)$, $t \in I$ and all $x \in [-M, M]^d$,

$$\mathbb{P}\Big\{ \inf_{s \in B_\rho(t,r) \cap I} \big\|X(s) - x\big\| \le r \Big\} \le c\, r^d. \qquad (6)$$

In the above $B_\rho(t, r) = \{s \in \mathbb{R}^N : \rho(s, t) \le r\}$ denotes the closed ball of radius r in the metric ρ in $\mathbb{R}^N$.

Proof of the upper bound

Only the case of $d > Q$ needs a proof. Choose and fix an arbitrary constant $\gamma > \mathcal{H}_{d-Q}(F)$. By the definition of $\mathcal{H}_{d-Q}(F)$, there is a sequence of balls $\{B(y_j, r_j), j \geq 1\}$ in $\mathbb{R}^d$ such that

$$F \subseteq \bigcup_{j=1}^{\infty} B(y_j, r_j) \quad \text{and} \quad \sum_{j=1}^{\infty} (2r_j)^{d-Q} \leq \gamma. \tag{7}$$

Notice that

$$\{F \cap X(I) \neq \varnothing\} \subseteq \bigcup_{j=1}^{\infty} \{B(y_j, r_j) \cap X(I) \neq \varnothing\}. \tag{8}$$

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Proof of the upper bound

For every $j \geq 1$, we divide the interval I into $c\, r_j^{-Q}$ intervals of side-lengths r_j^{-1/H_ℓ} $(\ell = 1, \ldots, N)$. Hence I can be covered by at most $c\, r_j^{-Q}$ many balls of radius r_j in the metric ρ.

It follows from Lemma 2.1 that

$$\mathbb{P}\big\{B(y_j, r_j) \cap X(I) \neq \varnothing\big\} \leq c\, r_j^{d-Q}. \tag{9}$$

Combining (8) and (9) we derive that $\mathbb{P}\big\{F \cap X(I) \neq \varnothing\big\} \leq c\gamma$. Since $\gamma > \mathcal{H}_{d-Q}(F)$ is arbitrary, the upper bound in (5) follows.

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Proof of Theorem 2.2: the lower bound

We make use of the following lemma.

Lemma 2.2 [Biermé, Lacaux and X. (2009)]

There exists a positive and finite constant c such that for all $\varepsilon \in (0,1)$, $s,t \in I$ and $x,y \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^{2d}} \exp\Big(-\frac{1}{2}(\xi,\,\eta)\,\big(\varepsilon I_{2d} + \mathrm{Cov}\big(X(s),X(t)\big)\big)\,(\xi,\,\eta)^T\Big) e^{-i(\langle \xi,x\rangle + \langle \eta,y\rangle)} d\xi d\eta \;\le\; \frac{c}{\big(\max\{\rho(s,t),\ \|x-y\|\}\big)^d}.$$

Here I_{2d} denotes the identity matrix of order $2d$, $\mathrm{Cov}\big(X(s),X(t)\big)$ denotes the covariance matrix of the random vector $(X(s),X(t))$.

Proof of the lower bound

Without loss of generality, we assume $\mathcal{C}_0(F) > 0$ and F is compact. Let $M > 0$ be a constant such that $F \subseteq [-M, M]^d$.

We only consider the critical case of $d = Q$. By definition of capacity, there is a Borel probability measure ν_0 on F such that

$$\begin{aligned}\mathcal{E}_0(\nu_0) &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log\left(\frac{e}{\|x-y\| \wedge 1}\right)\nu_0(dx)\nu_0(dy) \\ &\le \frac{2}{\mathcal{C}_0(F)}.\end{aligned} \tag{10}$$

For all integers $n \geq 1$, we consider a family of random measures ν_n on I defined by

$$\int_I g(t)\,\nu_n(dt)$$
$$= \int_I \int_{\mathbb{R}^d} (2\pi n)^{d/2} \exp\left(- \frac{n\left\|X(t) - x\right\|^2}{2} \right) g(t)\,\nu_0(dx)\,dt$$
$$= \int_I \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(- \frac{\|\xi\|^2}{2n} + i\langle \xi, X(t) - x\rangle \right) g(t)\,d\xi \nu_0(dx)dt$$

where g is an arbitrary measurable, nonnegative function on I.

Denote the total mass of ν_n by $\|\nu_n\| := \nu_n(I)$. We verify the following two inequalities hold:

$$\mathbb{E}\big(\|\nu_n\|\big) \ge c_4 \quad \text{and} \quad \mathbb{E}\big(\|\nu_n\|^2\big) \le c_5 \mathcal{E}_0(\nu_0), \tag{11}$$

where the constants c_4 and c_5 are independent of ν_0 and n.

By (11) and the Paley-Zygmund inequality, one can show that there is an event Ω_0 with probability at least

$$c_4^2/(2c_5\,\mathcal{E}_0(\nu_0))$$

such that for every $\omega \in \Omega_0$, $\{\nu_n(\omega), n \geq 1\}$ has a subsequence that converges weakly to a finite positive measure ν which is supported on $X^{-1}(F) \cap I$.

Then, we have

$$\mathbb{P}\Big\{X(I) \cap F \neq \varnothing\Big\} \geq \mathbb{P}\{\|\nu\| > 0\} \geq \frac{c_4^2}{2c_5\,\mathcal{E}_0(\nu_0)}.$$

This proves the lower bound.

Conjecture: $\mathcal{H}^{d-Q}(F)$ in (5) can be replaced by $\mathcal{C}^{d-Q}(F)$.

Recently, Dalang , Mueller and Xiao (2015) verified this for the case of $F = \{x\}$. They proved that, if $d = Q$, then for every $x \in \mathbb{R}^d$,

$$\mathbb{P}\Big\{X(I) \cap \{x\} \neq \emptyset\Big\} = \mathbb{P}\Big\{\exists t \in I : X(t) = x\Big\} = 0.$$

Their method is based on an argument of Talagrand (1998).

For any Borel set $F \subseteq \mathbb{R}^d$, consider the inverse image

$$X^{-1}(F) = \{t \in \mathbb{R}^N : X(t) \in F\}.$$

Theorem 2.3 [Biermé, Lacaux and X. (2009)]

Let X be as in Theorem 2.2 and let $F \subseteq \mathbb{R}^d$ be a Borel set such that $\sum_{j=1}^N \frac{1}{H_j} > d - \dim_{\rm H} F$. Then with positive probability,

$$\dim_{\rm H}\big(X^{-1}(F) \cap I\big) = \min_{1\le k\le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F) \right\}.$$

Theorem 2.3 [Biermé, Lacaux and X. (2009)]

In particular, if t $\sum_{j=1}^N \frac{1}{H_j} > d$. Then for every $x \in \mathbb{R}^d$,

$$\dim_{\rm H}\big(X^{-1}(x) \cap I\big) = \min_{1 \le k \le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d \right\}$$

with positive probability.

2.3 Brownian images and thermal capacity

Let $W := \{W(t)\}_{t\geq 0}$ denote standard d-dimensional Brownian motion where $d \geq 1$, and let E and F be compact subsets of $(0, \infty)$ and $\mathbb{R}^d$, respectively.

We consider the following questions:

1. When is $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$?
2. What is $\dim_{\mathrm{H}}(W(E) \cap F)$?

Note that

$$\{W(E) \cap F \neq \emptyset\} = \{(t, W(t)) \in E \times F \text{ for some } \ t > 0\}.$$

Problem 1 is an interesting problem in probabilistic potential theory.

Conditions for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$

Necessary and sufficient condition in terms of "thermal capacity" for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$ were proved by Waston (1978) and Doob (1984).

Waston and Taylor (1985) provided a simple-to-use condition:

$$\mathbb{P}(W(E) \cap F \neq \emptyset) \begin{cases} > 0, & \text{if } \dim_{\mathrm{H}}(E \times F\,; \varrho) > d, \\ = 0, & \text{if } \dim_{\mathrm{H}}(E \times F\,; \varrho) < d. \end{cases}$$

In the above, $\dim_{\mathrm{H}}(E \times F\,; \varrho)$ is the Hausdorff dimension of $E \times F$ using the metric

$$\varrho\left((s\,,x)\,;(t\,,y)\right) := \max\left(|t-s|^{1/2},\ \|x-y\|\right).$$

As a by-product of our main result, we obtain an improved version of the result of Waston (1978) and Doob (1984).

Theorem 2.4

Suppose $F \subset \mathbb{R}^d$ is compact and has Lebesgue measure 0. Then

$$\mathbb{P}\{W(E) \cap F \neq \emptyset\} > 0 \iff \exists\, \mu \in \mathcal{P}_d(E \times F) \text{ such that } \mathcal{E}_0(\mu) < \infty,$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$ and the energy $\mathcal{E}_0(\mu)$ is defined by

$$\mathcal{E}_0(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2}}\, \mu(ds\, dx)\, \mu(dt\, dy).$$

Hausdorff dimension of $\dim_{\rm H}(W(E) \cap F)$

- If $F = \mathbb{R}^d$, then $\dim_{\rm H} W(E) = \min\{d, 2\dim_{\rm H} E\}$ a.s.
- If $E = \mathbb{R}_+$, then

$$\dim_{\rm H}(W(\mathbb{R}_+) \cap F) = \begin{cases} \dim_{\rm H} F & \text{if } d = 1; \\ 2 + \dim_{\rm H} F - d & \text{if } d \geq 2. \end{cases}$$

- For compact sets $E \subset (0, \infty)$ and $F \subset \mathbb{R}$ ($d = 1$), Kaufman (1972) obtained $\|\dim_{\rm H}(W^{-1}(F) \cap E)\|_{L^\infty(\mathbb{P})}$, where $\| \cdot \|_{L^\infty(\mathbb{P})}$ denotes the $L^\infty(\mathbb{P})$-norm. However, this does not provide information on $\dim_{\rm H}(W(E) \cap F)$.
- Hawkes (1978) considered the problem for an α-stable Lévy process in $\mathbb{R}$ with $0 < \alpha < 1$.
- We solve this problem completely for Brownian motion (and Lévy stable processes).

Hausdorff dimension of $\dim_{\mathrm{H}}(W(E) \cap F)$

To compute $\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})}$, we distinguish two cases: $|F| > 0$ and $|F| = 0$, where $|\cdot|$ denotes the Lebesgue measure.

Theorem 2.5 [Khoshnevisan and X. (2015)]

If $F \subset \mathbb{R}^d$ $(d \geq 1)$ is compact and $|F| > 0$, then

$$\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} = \min\{d\,, 2\dim_{\mathrm{H}} E\}. \tag{12}$$

If $\dim_{\mathrm{H}} E > \frac{1}{2}$ and $d = 1$, then $\mathbb{P}\{|W(E) \cap F| > 0\} > 0$.

Proof of Theorem 2.5

1. Thanks to the uniform Hölder continuity of $W(t)$ on bounded sets, we have

$$\dim_{\rm H}\left(W(E)\cap F\right) \le \min\{d\,, 2\dim_{\rm H} E\}, \qquad \text{a.s.}$$

   This implies the upper bound in (12).
2. For proving the lower bound in (12), we construct a random measure on $W(E)\cap F$ and use the capacity argument.
3. The last part is proved by showing that the constructed random measure on $W(E)\cap F$ has a density function almost surely.

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Theorem 2.6 [Khoshnevisan and X. (2015)]

If $F \subset \mathbb{R}^d$ $(d \geq 1)$ is compact and $|F| = 0$, then

$$\left\| \dim_{\rm H} \left(W(E) \cap F \right) \right\|_{L^\infty(\mathbb{P})} = \sup \left\{ \gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\}, \tag{13}$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and

$$\mathcal{E}_\gamma(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2} \cdot \|y-x\|^\gamma}\, \mu(ds\, dx)\, \mu(dt\, dy). \tag{14}$$

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The co-dimension argument

Recall that the two common ways to compute the Hausdorff dimension of a set are

- Use a covering argument for obtaining an upper bound and a capacity argument for lower bound;
- The co-dimension argument.

The "co-dimension argument" was initiated by S.J. Taylor (1966) for computing the Hausdorff dimension of the multiple points of a stable Lévy process in $\mathbb{R}^d$. His method was based on potential theory of Lévy processes.

Let $Z_\alpha = \{Z_\alpha(t), t \in \mathbb{R}_+\}$ be a (symmetric) stable Lévy process in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$ and let $F \subset \mathbb{R}^d$ be a Borel set. Then

$$\mathbb{P}(Z_\alpha((0,\infty)) \cap F \neq \varnothing) > 0 \iff \text{Cap}_{d-\alpha}(F) > 0,$$

where $\text{Cap}_{d-\alpha}$ is the Riesz-Bessel capacity of order $d - \alpha$.

The co-dimension argument

The above result and Frostman's theorem lead to the *stochastic co-dimension argument*: If $\dim_H F \ge d-2$, then

$$\begin{aligned}\dim_H F &= \sup\{d-\alpha : Z_\alpha((0,\infty)) \cap F \neq \varnothing\}\\ &= d - \inf\big\{\alpha > 0 : \ F \ \text{is not polar for } Z_\alpha\big\}.\end{aligned}$$

[The restriction $\dim_H F \ge d-2$ is caused by the fact that $Z_\alpha((0,\infty)) \cap F = \emptyset$ if $\dim_H F < d-2$.]

This method determines $\dim_H F$ by intersecting F using a family of testing random sets.

Hawkes (1971) applied the co-dimension method for computing $\dim_H X^{-1}(F)$ of a stable Lévy process.

The co-dimension argument

Families of testing random sets:

- ranges of symmetric stable Lévy processes;
- fractal percolation sets [Peres (1996, 1999)];
- ranges of additive Lévy processes [Khoshnevisan and X. (2003, 2005, 2009), Khoshnevisan, Shieh and X. (2008)].

Hitting probability of random fields

We prove Theorem 2.6 by checking whether or not $W(E) \cap F$ intersects the range of an additive Lévy stable process.

Let $X^{(1)}, \ldots, X^{(N)}$ be N isotropic stable processes with common stability index $\alpha \in (0, 2]$. We assume that the $X^{(j)}$'s are independent from one another, as well as from the process W, and all take their values in $\mathbb{R}^d$.

We assume also that $X^{(1)}, \ldots, X^{(N)}$ have right-continuous sample paths with left-limits and

$$\mathbb{E}\left[e^{i\langle \xi, X^{(k)}(1)\rangle}\right] = e^{-\|\xi\|^\alpha/2}, \quad \forall\ \xi \in \mathbb{R}^d.$$

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Define the corresponding additive stable process $X_\alpha := \{X_\alpha(t \times t),\ t \times t \in \mathbb{R}_+^N\}$ as

$$X_\alpha(t \times t) := \sum_{k=1}^{N} X^{(k)}(t_k), \quad \forall\, t \times t = (t_1, \ldots, t_N) \in \mathbb{R}_+^N. \tag{15}$$

Khoshnevisan, X. and Zhong (2003) showed that for any Borel set $G \subset \mathbb{R}^d$,

$$\mathbb{P}\big(X_\alpha(\mathbb{R}_+^N) \cap G \neq \varnothing\big) \begin{cases} = 0 & \text{if } \dim_{\mathrm{H}} G < d - \alpha N, \\ > 0 & \text{if } \dim_{\mathrm{H}} G > d - \alpha N. \end{cases} \tag{16}$$

The key ingredient for proving Theorem 2.6

Theorem 2.7

If $d > \alpha N$ and $F \subset \mathbb{R}^d$ has Lebesgue measure 0, then

$$\mathbb{P}\left\{W(E) \cap X_\alpha(\mathbb{R}^N_+) \cap F \neq \emptyset\right\} > 0 \iff \mathcal{C}_{d-\alpha N}(E \times F) > 0.$$

Here $\mathcal{C}_\gamma$ is the capacity corresponding to the energy form (14): for all compact sets $U \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \geq 0$,

$$\mathcal{C}_\gamma(U) := \left[\inf_{\mu \in \mathcal{P}_d(U)} \mathcal{E}_\gamma(\mu)\right]^{-1}. \tag{17}$$

Proof of Theorem 2.6

Lower bound: Denote

$$\Delta := \sup\left\{\gamma \geq 0 : \inf_{\mu\in\mathcal{P}_d(E\times F)} \mathcal{E}_\gamma(\mu) < \infty\right\}. \qquad (18)$$

If $\Delta > 0$ and we choose $\alpha \in (0\,,2]$ and $N \in \mathbf{Z}_+$ $0 < d - \alpha N < \Delta$. Then $\mathcal{C}_{d-\alpha N}(E \times F) > 0$. It follows from Theorem 2.3 and (16) that

$$\mathbb{P}\left\{\dim_{\rm H}(W(E)\cap F) \geq d - \alpha N\right\} > 0. \qquad (19)$$

Because $d - \alpha N \in (0\,,\Delta)$ is arbitrary, we have

$$\|\dim_{\rm H}(W(E)\cap F)\|_{L^\infty(\mathbb{P})} \geq \Delta.$$

Upper bound: Similarly, Theorem 2.7 and (16) imply that

$$d-\alpha N > \Delta \ \Rightarrow \ \dim_{\rm H}\big(W(E)\cap F\big) \le d-\alpha N \qquad \text{a. s.} \quad (20)$$

Hence $\|\dim_{\rm H}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} \le \Delta$ whenever $\Delta \ge 0$. This proves Theorem 2.6.

Proof of Theorem 2.7

To prove the sufficiency

$$\mathcal{C}_{d-\alpha N}(E \times F) > 0 \implies \mathbb{P}\left\{W(E) \cap X_\alpha(\mathbb{R}_+^N) \cap F \neq \emptyset\right\} > 0,$$

we define, for every $\mu \in \mathcal{P}_d(E \times F)$ and $\varepsilon > 0$, the occupation measure $Z_\varepsilon(\mu)$ by

$$Z_\varepsilon(\mu) = \int_{[1,2]^N} du \int_{E \times F} \mu(ds\,dx)\, \phi_\varepsilon(W(s) - x)\phi_\varepsilon(X_\alpha(u) - x),$$

where

$$\phi_\varepsilon(y) = \frac{1}{\varepsilon^d}\mathcal{I}_{B(0,\varepsilon)}(y).$$

The proof is based on computing $\mathbb{E}[Z_\varepsilon(\mu)]$ and $\mathbb{E}[Z_\varepsilon(\mu)^2]$.

Proof of Theorem 2.7

For proving the necessity, we assume

$$\mathbb{P}\left\{W(E)\cap X_\alpha(\mathbb{R}_+^N)\cap F\neq\varnothing\right\}>0,$$

and construct a probability measure $\mu\in\mathcal{P}_d(E\times F)$ such that $\mathcal{E}_{d-\alpha N}(\mu)<\infty$.

If $W(E)\cap X_\alpha(\mathbb{R}_+^N)$ is replaced by the range of a Lévy process, then we can use a stoping time argument and the strong Markov property.

The current random field case is much harder. We omit the details.

An explicit formula

Theorem 2.8 [Khoshnevisan and X. (2016)]

If $d \geq 2$ and $\dim_{\rm H}(E \times F\,;\varrho) \geq d$, then

$$\big\|\dim_{\rm H}(W(E) \cap F)\big\|_{L^\infty(\mathbb{P})} = \dim_{\rm H}(E \times F\,;\varrho) - d. \quad (21)$$

Remarks

- Eq (21) does not always hold for $d = 1$: For $E := [0\,,1]$ and $F = \{0\}$, we have $\dim_{\rm H}(W(E) \cap F) = 0$ a.s., whereas $\dim_{\rm H}(E \times F\,;\varrho) - d = 1$.
- When $F \subset \mathbb{R}^d$ satisfies $|F| > 0$, it can be shown that

$$\dim_{\rm H}(E \times F\,;\varrho) = 2\dim_{\rm H} E + d.$$

Hence (12) coincides with (21) when $d \geq 2$.

Proof of Theorem 2.8

The proof replies on the following "uniform dimension result" of Kaufman (1968): If $\{W(t), t \in \mathbb{R}_+\}$ is a Brownian motion in $\mathbb{R}^d$ with $d \geq 2$, then

$$\mathbb{P}\{\dim_{\rm H} W(G) = 2\dim_{\rm H} G, \ \forall \text{ Borel sets } \ G \subset \mathbb{R}_+\} = 1.$$

It is sufficient to show that for all compact sets $E \subset (0, \infty)$ and $F \subset \mathbb{R}^d$,

$$\left\| \dim_{\rm H} \left(E \cap W^{-1}(F)\right) \right\|_{L^\infty(\mathbb{P})} = \frac{\dim_{\rm H} (E \times F \, ; \varrho) - d}{2}. \tag{22}$$

When $d = 1$, the lower bound of (22) was found first by Kaufman (1972).

Thank you