Gaussian Random Fields: Strong Local Nondeterminism and Fine Properties, I

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Lecture 3 Strong local nondeterminism and fine properties, I

- Properties of strong local nondeterminism
- Analytic properties of Gaussian fields
- Fractal properties of Gaussian fields: exact Hausdorff measure functions

3.1 Properties of strong local nondeterminism

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field in $\mathbb{R}$.

Given constants $0 < H_1 \leq \ldots \leq H_N \leq 1$, define a metric ρ on $\mathbb{R}^N$ by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s, t \in \mathbb{R}^N.$$

Let $I \subset \mathbb{R}^N$ be a compact interval. As in Lecture 2, we assume that X satisfies the following condition:

(C1). $\exists$ positive constants c_1 and c_2 such that

$$c_1\, \rho(s,t) \leq d_X(s,t) \leq c_2\, \rho(s,t), \quad \forall s, t \in I.$$

Definition 3.1

X is said to have the property of sectorial local nondeterminism on I if there exists a constant $c_3 > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\Big(X(u) \,\Big|\, X(t^1), \ldots, X(t^n)\Big) \geq c_3 \sum_{j=1}^{N} \min_{1 \leq k \leq n} \big|u_j - t_j^k\big|^{2H_j}. \tag{1}$$

Definition 3.2

X is said to have the property of strong local nondeterminism on I if there exists a constant $c_4 > 0$ such that $\forall\, n \geq 1$ and $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\Big(X(u)\,\Big|\,X(t^1), \ldots, X(t^n)\Big) \geq c_4 \min_{1\leq k\leq n} \rho(u,\, t^k)^2, \qquad (2)$$

where $\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}$.

Remarks

The concept of local nondeterminism (LND, in short) of a Gaussian process was first introduced by Berman (1973) to for studying local times of Gaussian processes.

A Gaussian process $Y = \{Y(t), t \in \mathbb{R}\}$ is called *locally nondeterministic* on I if for every integer $m \geq 2$,

$$\lim_{\varepsilon \to 0} \inf_{t_m - t_1 \leq \varepsilon} V_m > 0, \tag{3}$$

where V_m is the relative prediction error:

$$V_m = \frac{\mathrm{Var}\big(Y(t_m) - Y(t_{m-1}) | Y(t_1), \ldots, Y(t_{m-1})\big)}{\mathrm{Var}\big(Y(t_m) - Y(t_{m-1})\big)}$$

and the infimum in (3) is taken over all ordered points $t_1 < t_2 < \cdots < t_m$ in I with $t_m - t_1 \leq \varepsilon$.

(3) is equivalent to the following property: For every integer $m \geq 2$, there exist positive constants $C(m)$ and ε (both may depend on m) such that

$$\mathrm{Var}\Big(\sum_{k=1}^{m} a_k \left(Y(t_k) - Y(t_{k-1})\right)\Big) \geq C(m) \sum_{k=1}^{m} a_k^2 \,\mathrm{Var}\big(Y(t_k) - Y(t_{k-1})\big) \tag{4}$$

for all ordered points $t_1 < t_2 < \cdots < t_m$ in I with $t_m - t_1 < \varepsilon$ and $a_k \in \mathbb{R}$ $(k = 1, \ldots, m)$.

- Pitt (1978) used (4) to define local nondeterminism of a Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$ by introducing a partial order among $t_1, \ldots, t_m \in \mathbb{R}^N$.
- In particular, he proved that fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ has the following property: For any $u \in \mathbb{R}^N \backslash \{0\}$, and and $r \in (0, |u|)$,

$$\mathrm{Var}\big(B^H(u) \,\big|\, B^H(t), \, |t-u| \geq r\big) = c_5 \, r^{2H},$$

  where $c_5 > 0$ is a constant. This implies that B^H satisfies the strong local nondeterminism on any compact interval $I \subset \mathbb{R}^N \backslash \{0\}$.
- Cuzick and DuPreez (1982) introduced strong local ϕ-nonderterminism and showed its usefulness in studying local times.

Remarks

- It is not hard to see that the Brownian sheet W does not satisfy "strong local nondeterminism" with $H_1 = \cdots = H_N = 1/2$. This caused difficulties in studies of some sample path properties of W; cf. Mountford (1989a, 1989b).
- The "sectorial local nondeterminism" was first discovered by Khoshnevisan and X. (2007) for the Brownian sheet; and extended to fractional Brownian sheets by Wu and X. (2007).
- X. (2009), Luan and X. (2012) proved "strong local nondeterminism" for a large class of Gaussian fields with stationary increments.

For simplicity, the conditions in Definitions 3.1 and 3.2 are called (C3) and (C4), respectively.

The difference between (C3) or (C4) and LND of Berman is that the constants c_3 and c_4 are independent of n.

One of the basic advantage of (C3) or (C4) is to give information on the joint distribution of $(X(t^1), \ldots, X(t^n))$ for all $n \geq 2$, by providing good bounds on

$$\mathrm{detCov}(X(t^1), \ldots, X(t^n));$$

and

$$\mathbb{P}(|X(t^1)| \leq a_1, \ldots, |X(t^n)| \leq a_n).$$

3.2 Analytic properties of Gaussian fields

For a Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$, we study:

(i) Uniform modulus of continuity

(ii) Local modulus of continuity: law of the iterated logarithm

(iii) Chung's law of the iterated logarithm

(iv) Modulus of non-diffenerability

(i) Uniform modulus of continuity

Theorem 3.1 [Meerschaert, Wang and X., 2013]

If $X = \{X(t), t \in \mathbb{R}^N\}$ satisfies (C1) and (C3) on $I = [0, 1]^N$, then its exact modulus of continuity is given by

$$\limsup_{|h|\to 0} \frac{\sup_{t\in I, s\in[0,h]} |X(t+s) - X(t)|}{\rho(0,h)\sqrt{\log(1+\rho(0,h)^{-1})}} = \kappa_1, \qquad (5)$$

where $\kappa_1 > 0$ is a constant. Recall that

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s, t \in \mathbb{R}^N.$$

The proof of Theorem 3.1 has three parts:

(a). $$\limsup_{|h|\to 0} \frac{\sup_{t\in I, s\in[0,h]} |X(t+s)-X(t)|}{\rho(0,h)\sqrt{\log(1+\rho(0,h)^{-1})}} \le c_6 < \infty, \quad \text{a.s.}$$

(b). $$\limsup_{|h|\to 0} \frac{\sup_{t\in I, s\in[0,h]} |X(t+s)-X(t)|}{\rho(0,h)\sqrt{\log(1+\rho(0,h)^{-1})}} \ge c_7 > 0, \quad \text{a.s.}$$

(c). Eq. (5) follows from **(a)**, **(b)** and a zero-one law.

The proof of **(a)** relies on the following estimate of the tail probability: For $\varepsilon > 0$ and $x \geq c_8 \varepsilon \sqrt{\log(1 + \varepsilon^{-1})}$,

$$\mathbb{P}\Bigg\{ \sup_{\substack{s,t \in [0,1]^N, \\ \rho(s,t) \leq \varepsilon}} |X(t) - X(s)| \geq x \Bigg\} \leq \exp\Big(- c_9 \frac{x^2}{\varepsilon^2} \Big).$$

and a standard Borel-Cantelli argument.

Or one can apply Theorem 1.3.3 (after Dudley's theorem) in Lecture 1 directly.

Proof of (b). For any $n \geq 2$, we choose a sequence of points $\{t_{n,k}, 1 \leq k \leq L_n\}$ in $[0,1]^N$ such that

$$\rho(t_{n,k}, t_{n,k-1}) = 2^{-n},$$

and for some direction i,

$$|t^i_{n,k} - t^i_{n,k-1}| \geq 2^{-n/H_i}, \quad \forall\, 2 \leq k \leq L_n.$$

We take $L_n = \min\{2^{n/H_i}\}$.
We will prove the following stronger statement:

$$\liminf_{n\to\infty} \frac{\max_{2\leq k\leq L_n} |X(t_{n,k}) - X(t_{n,k-1})|}{2^{-n}\sqrt{n}} \geq c_7 > 0, \quad \text{a.s.} \tag{6}$$

Let $\eta > 0$ be a constant whose value will be chosen later. Consider the events

$$A_n = \left\{ \max_{2 \le k \le L_n} \left| X(t_{n,k}) - X(t_{n,k-1}) \right| \le \eta 2^{-n} \sqrt{n} \right\}$$

and write

$$\begin{aligned} &\mathbb{P}(A_n) \\ &= \mathbb{P}\left\{ \max_{2 \le k \le L_n - 1} \left| X(t_{n,k}) - X(t_{n,k-1}) \right| \le \eta 2^{-n} \sqrt{n} \right\} \\ &\qquad \times \mathbb{P}\left\{ \left| X(t_{n,L_n}) - X(t_{n,L_n-1}) \right| \le \eta 2^{-n} \sqrt{n} \middle| \widetilde{A}_{L_n-1} \right\}, \end{aligned} \tag{7}$$

where

$$\widetilde{A}_{L_n-1} = \left\{ \max_{2 \le k \le L_n - 1} \left| X(t_{n,k}) - X(t_{n,k-1}) \right| \le \eta 2^{-n} \sqrt{n} \right\}.$$

The conditional distribution of the Gaussian random variable $X(t_{n,L_n}) - X(t_{n,L_n-1})$ under $\widetilde{A}_{L_n-1}$ is still Gaussian and, by Condition (C3), its conditional variance satisfies

$$\mathrm{Var}\Big(X(t_{n,L_n}) - X(t_{n,L_n-1})\Big|\widetilde{A}_{L_n-1}\Big) \geq c\, 2^{-2n}.$$

This and Anderson's inequality (1955) imply

$$\mathbb{P}\Big\{\big|X(t_{n,L_n}) - X(t_{n,L_n-1})\big| \leq \eta 2^{-n}\sqrt{n}\Big|\tilde{A}_{L_n-1}\Big\}$$

$$\leq \mathbb{P}\Big\{N(0,1) \leq c\,\eta\sqrt{n}\Big\} \qquad \text{(use Mill's ratio)}$$

$$\leq 1 - \frac{1}{c\eta\sqrt{n}} \exp\Big(-\frac{c^2\eta^2 n}{2}\Big) \qquad (\text{use } 1 - x \leq e^{-x} \text{ for } x > 0)$$

$$\leq \exp\bigg(-\frac{1}{c\eta\sqrt{n}} \exp\Big(-\frac{c^2\eta^2 n}{2}\Big)\bigg).$$

Iterating this procedure in (7) for L_n times, we obtain

$$\mathbb{P}(A_n) \le \exp\bigg(- \frac{1}{c\eta\sqrt{n}} L_n \exp\Big(- \frac{c^2\eta^2 n}{2}\Big)\bigg).$$

By taking $\eta > 0$ small enough, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty.$$

Hence the Borel-Cantelli lemma implies (6).

(ii) Local modulus of continuity: law of the iterated logarithm

Theorem 3.2 [Meerschaert, Wang and X., 2013]

For every $t_0 \in \mathbb{R}^N$,

$$\limsup_{r \to 0} \frac{\max_{\rho(0,h) \le r} |X(t_0 + h) - X(t_0)|}{r(\log \log 1/r)^{1/2}} = \kappa_2, \qquad \text{a.s.,}$$

where $0 < \kappa_2 < \infty$ is a constant.

This describes the largest local oscillation of $X(t)$.

Research Problem

For any $\lambda > 0$, define the set of "fast points"

$$F(\lambda) = \left\{ t \in [0,1]^N : \limsup_{r \to 0} \frac{|X(t+h) - X(t)|}{\rho(0,h)\sqrt{\log \frac{1}{\rho(0,h)}}} \geq \lambda \right\}.$$

Questions:

- What is the Hausdorff dimension of $F(\lambda)$?
- For a given set $E \subset [0,1]^N$, when is

$$\mathbb{P}\{F(\lambda) \cap E \neq \emptyset\} > 0?$$

(iii) Chung's law of the iterated logarithm

Theorem 3.4 [Luan and X., 2010]

For every $t_0 \in \mathbb{R}^N$,

$$\liminf_{r \to 0} \frac{\max_{\rho(0,h) \le r} |X(t_0 + h) - X(t_0)|}{r(\log \log 1/r)^{-1/Q}} = \kappa_3, \qquad a.s.,$$

where $0 < \kappa_3 < \infty$ is a constant.

This describes the smallest local oscillation of $X(t)$, which will be useful for estimating the Hausdorff measure of the range and graph of X.

(iv) Modulus of non-differentiability

We consider a class of approximately isotropic Gaussian random fields.

Modulus of non-differentiability (Wang and X. 2016): For any compact rectangle $I \subseteq \mathbb{R}^N$,

$$\lim_{\varepsilon \to 0+} \inf_{t \in I} \frac{\sup_{s \in B(t,r)} |B^H(s) - B^H(t)|}{\varepsilon^H |\log 1/\varepsilon|^{-H/N}} = \kappa_4, \quad \text{a.s.}$$

where κ_4 is a constant related to the small ball probability of B^H.

3.3 Exact Hausdorff measure functions

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d) Gaussian random field defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big).$$

We assume that $X_1, \ldots, X_d$ are independent copies of a real-valued Gaussian field X_0.

We consider first the fractal properties of the range $X([0, 1]^N)$.

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Review: Hausdorff measure

Let Φ be the class of functions $\varphi : (0, \delta) \to (0, \infty)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \le K \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

A function φ in Φ is often called a *measure function* or *gauge function*.

For example, $\varphi(s) = s^\alpha$ $(\alpha > 0)$ and $\varphi(s) = s^\alpha \log\log(1/s)$ are measure functions.

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Given $\varphi \in \Phi$, the *φ-Hausdorff measure* of $E \subseteq \mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \Big\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon \Big\}, \tag{8}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The sequence of balls satisfying the two conditions on the right-hand side of (8) is called an ε-covering of E.

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An upper density theorem

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *upper φ-density* of μ at $x \in \mathbb{R}^d$ is defined as

$$\overline{D}_{\mu}^{\varphi}(x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

Theorem [Rogers and Taylor, 1961]

Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

$$K^{-1}\mu(E) \inf_{x\in E} \left\{\overline{D}_{\mu}^{\varphi}(x)\right\}^{-1} \le \varphi\text{-}m(E) \le K\|\mu\| \sup_{x\in E} \left\{\overline{D}_{\mu}^{\varphi}(x)\right\}^{-1}. \tag{9}$$

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Exact Hausdorff measure of the ranges

Let $\sigma^2(s,t) = \mathbb{E}\left(X_0(s) - X_0(t)\right)^2$ and ρ be the metric on $\mathbb{R}^N$ defined by $\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}$ for $s,t \in \mathbb{R}^N$.

Ayache and X. (2005) and X. (2009) proved that, if

$$c_1\, \rho^2(s,t) \le \sigma^2(s,t) \le c_2\, \rho^2(s,t), \quad \forall s,t \in [0,1]^N,$$

then

$$\dim_{\mathrm{H}} X\left([0,1]^N\right) = \min\left\{d; \quad \sum_{j=1}^N \frac{1}{H_j}\right\}, \quad \text{a.s.}$$

Question: Is there a measure function φ such that

$$0 < \varphi\text{-}m\Big(X([0,1]^N)\Big) < \infty \quad \text{a.s.?}$$

The case of fBm

For $H \in (0,1)$, the fBm $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ with index H is a centered (N,d)-Gaussian field such that

$$\mathbb{E}\left[B_i^H(s)B_j^H(t)\right] = \frac{1}{2}\,\delta_{ij}\,\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

- When $N = 1$ and $H = 1/2$, B^H is Brownian motion.
- B^H is H-self-similar and has stationary increments.

Kahane (1985) proved that

$$\dim_{\rm H} B^H([0,1]^N) = \min\left\{d,\ \frac{N}{H}\right\} \qquad \text{a.s.}$$

Theorem 3.5 [Talagrand, 1995, 1998]

Let $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.

(i). If $N < Hd$, then $K^{-1} \le \varphi_1\text{-}m\big(B^H([0,1]^N)\big) \le K$, a.s., where $\varphi_1(r) = r^{\frac{N}{H}} \log\log(1/r)$.

(ii). If $N = Hd$, then $\varphi_2\text{-}m\big(B^H([0,1]^N)\big)$ is σ-finite, where

$$\varphi_2(r) = r^d \log(1/r) \log\log\log(1/r).$$

The problems on the exact Hausdorff measure functions for the graph set and level set of B^H were studied by X. (1996, 1997, 1998).

The general case

Theorem 3.6 (Luan and X., 2012)

Assume that X_0 has stationary increments with spectral density $f(\lambda)$ which satisfies

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{2+Q}}, \qquad \lambda \in \mathbb{R}^N, \ |\lambda| \geq 1,$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$.

(i). If $\sum_{j=1}^N \frac{1}{H_j} > d$, then a.s. $X([0,1]^N)$ has positive Lebesgue measure. Moreover, a.s. $X([0,1]^N)$ has interior points.

The general case

Theorem 3.6 (Continued)

(ii). If $\sum_{j=1}^{N} \frac{1}{H_j} < d$, then $\exists$ constants $c_3 > 0$ and $c_4 < \infty$

$$c_3 \le \varphi\text{-}m\left(X\big([0,1]^N\big)\right) \le c_4 \quad \text{a.s.},$$

where $\varphi(r) = r^{\sum_{j=1}^{N} \frac{1}{H_j}} \log\log 1/r$.

Sketch of the proof of φ-$m\left(X\left([0,1]^N\right)\right) \geq c$

Let μ be the occupation measure of X, defined by

$$\mu(B) = \lambda_N\big\{t \in [0,1]^N : X(t) \in B\big\}, \qquad \forall\, B \in \mathcal{B}(\mathbb{R}^d).$$

Lemma 3.1

For every $t_0 \in [0,1]^N$, let $T_{t_0}(r) = \mu\big(B(X(t_0), r)\big)$. Then

$$\limsup_{r\to 0} \frac{T_{t_0}(r)}{r^{\sum_{j=1}^N \frac{1}{H_j}} \log\log 1/r} \leq c < \infty, \qquad \text{a.s.}$$

Sketch of Proof. Since

$$T_{t_0}(r) = \int_{[0,1]^N} \mathbf{1}_{\{|X(t)-X(t_0)|\le r\}}\, dt,$$

we use Fubini's theorem to get

$$\begin{aligned}\mathbb{E}\big(T(r)\big) &= \int_{[0,1]^N} \mathbb{P}\left\{|X(t) - X(t_0)| < r\right\} dt \\ &\le \int_{[0,1]^N} \min\left\{1, \left(\frac{r}{\rho(t, t_0)}\right)^d\right\} dt \\ &\le c\, r^Q,\end{aligned}$$

where $Q = \sum_{j=1}^N H_j^{-1}$.

For all $n \geq 2$,

$$\mathbb{E}\left(T_{t_0}(r)^n\right) = \int_{[0,1]^{Nn}} \mathbb{P}\left\{|X(t^j) - X(t_0)| \leq r,\ 1 \leq j \leq n\right\} dt^1 \cdots dt^n$$

Using the property of strong local nondeterminism, conditioning and induction, we have

$$\mathbb{E}\left(T_{t_0}(r)^n\right) \leq c^n\, n!\, r^{Qn}.$$

This leads to Lemma 3.1.

Sketch of the proof of φ-$m\left(X\left([0,1]^N\right)\right) \le c$

The covering method was invented by Talagrand (1995), which is different from that for Markov processes.
In our case, we show

- For most of points $t_0 \in [0,1]^N$, there is a sequence $r_n \downarrow 0$, such that $X(B_\rho(t_0, r_n))$ can be covered by a ball of radius $cr(\log\log 1/r)^{-1/Q}$. This is the smallest oscillation as suggested by Theorem 3.4. These are good points.
- For bad points $t_0 \in [0,1]^N$, we can cover $X(B_\rho(t_0, r))$ by a ball of radius $c\, r\sqrt{\log 1/r}$.

The property of strong local nondeterminism plays an important role in the proof as well.

Open Problems

Questions:

- What about the case $\sum_{j=1}^{N} \frac{1}{H_j} = d$?
- What are the exact Hausdorff measure functions for the graph and level sets?

Thank you