Gaussian Random Fields: Strong Local Nondeterminism and Fine Properties, II

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Lecture 4 Strong local nondeterminism and and fine properties, II

- Spectral condition for strong local nondeterminism
- Application to stochastic heat equation
- A comparison theorem

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field in $\mathbb{R}$ and let I be a compact interval.

How can we establish the following conditions?

(C3). $\exists$ a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\big(X(u) \,\big|\, X(t^1), \ldots, X(t^n)\big) \geq c \sum_{j=1}^{N} \min_{1\leq k\leq n} \big|u_j - t_j^k\big|^{2H_j}.$$

(C4). $\exists$ a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\big(X(u)\,\big|\,X(t^1), \ldots, X(t^n)\big) \geq c \min_{1\leq k\leq n} \rho(u,\, t^k)^2,$$

where

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s, t \in \mathbb{R}^N.$$

4.1 Sectorial local nondeterminism

Condition (C3) is satisfied by Gaussian fields with tensor product-type covariance functions, such as the Brownian sheet, fractional Brownian sheets.

Theorem 4.1 [Wu and X. (2007)]

Let $W^H = \{W^H(t), t \in \mathbb{R}^N\}$ be a fractional Brownian sheet with $H = (H_1, \ldots, H_N)$, i.e., W^H has mean 0 and

$$\mathbb{E}\left[W^H(s)W^H(t)\right] = \prod_{j=1}^{N} \frac{1}{2}\left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\right).$$

Then for any $\varepsilon > 0$, there exists a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, 1]^N$,

$$\mathrm{Var}\big(W^H(u) \,\big|\, W^H(t^1), \ldots, W^H(t^n)\big) \geq c \sum_{j=1}^{N} \min_{1 \leq k \leq n} \left|u_j - t_j^k\right|^{2H_j}.$$

The proof of this result is based on stochastic integral representation of W^H and an analytic argument that we will explain below. We omit the details here.

4.2 Spectral condition for strong local nondeterminism

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and $X(0) = 0$.
For any $h \in \mathbb{R}^N$ we have

$$\mathbb{E}\big(X(t+h) - X(t)\big)^2 = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\, \Delta(d\lambda),$$

where $\Delta(d\lambda)$ is the spectral measure of X, which satisfies

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1 + |\lambda|^2}\, \Delta(d\lambda) < \infty.$$

It follows that X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} \left(e^{i\langle t,\lambda\rangle} - 1\right) \widetilde{W}(d\lambda),$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $\widetilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

If $Y = \{Y(t), t \in \mathbb{R}^N\}$ is a stationary Gaussian field, let

$$X(t) = Y(t) - Y(0), \quad \forall\, t \in \mathbb{R}^N.$$

Then $X = \{X(t), t \in \mathbb{R}^N\}$ has stationary increments and has the same spectral measure as that of Y.

The spectral measure Δ can be

- absolutely continuous with density $f(\lambda)$, or
- singular with fractal support (e.g., a self-similar measure), or
- singular with a discrete support.

Theorem 4.2 [Xue and X., 2011]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field with stationary increments and spectral density $f(\lambda)$. If there are constants $H_1, \cdots, H_N \in (0,1]^N$ and $K > 0$ such that

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{2+Q}}, \qquad \lambda \in \mathbb{R}^N, \ |\lambda| \geq K, \qquad (1)$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$, then $\exists$ a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\mathrm{Var}\Big(X(u) \,\Big|\, X(t^1), \ldots, X(t^n)\Big) \geq c \min_{0 \leq k \leq n} \rho(u, t^k)^2,$$

where $t^0 = 0$.

Remarks

- Because of (1), we observe that the behavior of $f(\lambda)$ near 0 is not needed for studying local properties.
- The behavior of $f(\lambda)$ at 0 is related to the long range dependence of X, and determines asymptotic properties of X at $|t| \to \infty$.

We will make use of the following lemma.

Lemma 4.1

Assume (1) is satisfied, then for any fixed constant $T > 0$, there exists a positive and finite constant c_1 such that for all functions g of the form

$$g(\lambda) = \sum_{k=1}^{n} a_k \big(e^{i\langle t^k, \lambda\rangle} - 1\big), \qquad (2)$$

where $a_k \in \mathbb{R}$ and $t^k \in [-T, T]^N$, we have

$$|g(\lambda)| \le c_1 \, |\lambda| \Big(\int_{\mathbb{R}^N} |g(\xi)|^2 f(\xi) d\xi \Big)^{1/2} \qquad (3)$$

for all $\lambda \in \mathbb{R}^N$ that satisfy $|\lambda| \le K$.

Proof. By (1), we can find positive constants C and η, such that

$$f(\lambda) \geq \frac{C}{|\lambda|^\eta}, \quad \forall \lambda \in \mathbb{R}^N \text{ with } |\lambda| \text{ large enough.}$$

Let $\mathcal{G}$ be the collection of the functions $g(z)$ defined by (2) with $a_k \in \mathbb{R}$, $s^k \in [-T, T]^N$ and $z \in \mathbb{C}^N$. Since each $g \in \mathcal{G}$ is an entire function, it follows from Proposition 1 of Pitt (1975) that for any given constant K,

$$c_1 = \sup_{\substack{g \in \mathcal{G} \\ z \in U(0,K)}} \left\{ |g(z)| : \int_{\mathbb{R}^N} |g(\lambda)|^2 f(\lambda)\, d\lambda \leq 1 \right\} < \infty,$$

where $U(0,K) = \{z \in \mathbb{C}^N : |z| < K\}$ is the open ball of radius K in $\mathbb{C}^N$.
Since $g(0) = 0$ and g is analytic in $U(0,K)$, Schwartz's lemma implies

$$\big|g(z)\big| \le c_1 K^{-1}\,|z| \Big(\int_{\mathbb{R}^N} |g(\xi)|^2 f(\xi) d\xi\Big)^{1/2}$$

for all $z \in U(0,K)$. This finishes the proof.

Proof of Theorem 4.2

Denote $r \equiv \min_{0 \le k \le n} \rho(u, t^k)$. It is sufficient to prove that for all $a_k \in \mathbb{R}$ $(1 \le k \le n)$,

$$\mathbb{E}\left(X(u) - \sum_{k=1}^{n} a_k X(t^k)\right)^2 \ge c\, r^2. \tag{4}$$

By the stochastic integral representation of X, the left hand side of (4), up to a constant, can be written as

$$\begin{aligned} &\mathbb{E}\left(X(u) - \sum_{k=1}^{n} a_k X(t^k)\right)^2 \\ &= \int_{\mathbb{R}^N} \left| e^{i\langle u, \lambda\rangle} - 1 - \sum_{k=1}^{n} a_k \left(e^{i\langle t^k, \lambda\rangle} - 1\right)\right|^2 f(\lambda)\, d\lambda. \end{aligned} \tag{5}$$

Hence, we only need to show

$$\int_{\mathbb{R}^N} \left| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k \, e^{i\langle t^k,\,\lambda\rangle} \right|^2 f(\lambda)\, d\lambda \geq c\, r^2, \qquad (6)$$

where $t^0 = 0$ and $a_0 = -1 + \sum_{k=1}^{n} a_k$.

Let $\delta(\cdot) : \mathbb{R}^N \to [0, 1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open ball $B_\rho(0, 1)$.

Denote by $\widehat{\delta}$ the Fourier transform of δ. Then $\widehat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ and decays rapidly as $|\lambda| \to \infty$.

Let A be the diagonal matrix with $H_1^{-1}, \ldots, H_N^{-1}$ on its diagonal and let $\delta_r(t) = r^{-Q}\delta(r^{-A}t)$. By the inverse Fourier transform,

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i\langle t,\lambda\rangle}\, \widehat{\delta}(r^A\lambda)\, d\lambda.$$

Since $\min\{\rho(u, t^k) : 0 \le k \le n\} = r$, we have

$$\delta_r(u - t^k) = 0 \quad \text{for} \quad k = 0, 1, \ldots, n.$$

Hence,

$$
\begin{aligned}
I &= \int_{\mathbb{R}^N} \left(e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle t^k,\lambda\rangle} \right) e^{-i\langle u,\lambda\rangle}\, \widehat{\delta}(r^A \lambda)\, d\lambda \\
&= (2\pi)^N \left(\delta_r(0) - \sum_{k=0}^{n} a_k\, \delta_r(u - t^k) \right) \\
&= (2\pi)^N\, r^{-Q}.
\end{aligned}
\tag{7}
$$

We split the integral in (7) over $\{\lambda : |\lambda| < K\}$ and $\{\lambda : |\lambda| \geq K\}$ and denote the two integrals by I_1 and I_2, respectively. It follows from Lemma 4.1 that

$$\begin{aligned} I_1 &\leq \int_{|\lambda|<K} \left| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k,\lambda\rangle} \right| |\hat{\delta}(r^A \lambda)| d\lambda \\ &\leq c_1 \left[\int_{\mathbb{R}^N} \left| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k,\lambda\rangle} \right|^2 f(\lambda)\, d\lambda \right]^{1/2} \\ &\qquad \times \int_{|\lambda|<K} |\lambda| |\hat{\delta}(r^A \lambda)| d\lambda \\ &\leq c_2 \left[\mathbb{E}\left(X(u) - \sum_{k=0}^{n} a_k X(t^k) \right)^2 \right]^{1/2}. \end{aligned} \tag{8}$$

On the other hand, the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
I^2 \le \int_{|\lambda|\ge K} \Big| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle t^k,\lambda\rangle}\Big|^2 f(\lambda)\, d\lambda \\
\times \int_{|\lambda|\ge K} \frac{\big|\widehat{\delta}(r^A\lambda)\big|^2}{f(\lambda)}\, d\lambda \\
\le \mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2 \cdot r^{-Q} \int_{\mathbb{R}^N} \frac{\big|\widehat{\delta}(\lambda)\big|^2}{f(r^{-A}\lambda)}\, d\lambda \\
\le c\,\mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2 \cdot r^{-2Q-2}.
\end{aligned}
$$

We square both sides of (7) and use the above to obtain

$$(2\pi)^{2N}\, r^{-2Q} \le c\, r^{-2Q-2}\, \mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2.$$

This proves (6) and hence the theorem.

Remarks

- This method can be modified to prove sectorial local nondeterminism.
- Recently, the method is applied in Lan, Marinucci and X. (2016) to prove strong local nondeterminism for isotropic Gaussian random fields on the sphere $\mathbb{S}^2$.

4.3 An application to SHE

As an application of Theorem 4.2, we consider the stochastic heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \dot W \\ &u(0,x) \equiv 0, \qquad \forall x \in \mathbb{R}, \end{aligned} \tag{9}$$

where $\dot W$ is a space-time white noise in $\mathbb{R}$ with covariance given by

$$\mathbb{E}\left[\dot W(t,x)\dot W(s,y)\right] = \delta(x-y)\delta(t-s).$$

The mild solution of (9) is the mean zero Gaussian field $u = \{u(t,x), t \geq 0,\ x \in \mathbb{R}\}$ with values in $\mathbb{R}$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-r}(x-y)\, W(drdy), \quad t \geq 0, x \in \mathbb{R},$$

where $G_t(x)$ is the Green kernel given by

$$G_t(x) = \frac{1}{(4\pi t)^{1/2}} \exp\left(- \frac{|x|^2}{4t} \right), \qquad \forall\, t > 0, x \in \mathbb{R}.$$

One can verify that for any $t \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}\left(u(t,x)u(s,x)\right) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s} - \sqrt{|t-s|}\right);$$

and

$$\begin{aligned}\mathbb{E}\left(u(t,x)u(t,y)\right) &= \int_0^t \int_{\mathbb{R}} G_{t-r}(x-z)G_{t-r}(x-y)\, drdz \\ &= \int_{\mathbb{R}} e^{i(x-y)\xi}\, \frac{1-e^{-|\xi|^2}}{\xi^2}\, d\xi.\end{aligned}$$

Consequently,

(i) for every fixed $x \in \mathbb{R}$, the process $\{u(t,x), t \geq 0\}$ is a bi-fractional Brownian motion introduced by Houdré and Villa (2003). Its properties are studied by

- Russo and Tudor (2005), Swanson (2007), Tudor and X. (2007), Lei and Nualart (2010)

Many of the properties of $\{u(t,x), t \geq 0\}$ are similar to those of a fractional Brownian motion with index $1/4$.

(ii) For every fixed $t > 0$, the process $\{u(t,x), x \in \mathbb{R}\}$ is stationary with the following representation

$$u(t,x) = \int_{\mathbb{R}} e^{ix\xi} \frac{\sqrt{1 - e^{-|\xi|^2}}}{|\xi|} \widetilde{W}(d\xi),$$

where $\widetilde{W}$ is a complex-valued Gaussian random measure with Lebesgue measure as its control measure.

Many of the properties of $\{u(t,x), x \in \mathbb{R}\}$ are similar to those of Brownian motion.

(iii) We are interested in the behavior of the sample function $(t,x) \mapsto u(t,x)$.

Let $\{U(t,x),\ t \ge 0, x \in \mathbb{R}\}$ be a real-valued random string process:

$$U(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(G_{(t-r)_+}(x-y) - G_{(-r)_+}(y) \right) W(drdy), \tag{10}$$

where $a_+ = \max\{a, 0\}$. It can be written as

$$U(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-r}(x-y)\, W(drdy) + \int_{-\infty}^0 \int_{\mathbb{R}} \Big(G_{t-r}(x-y) - G_{-r}(y) \Big) W(drdy).$$

Then $\{U(t,x), t \geq 0, x \in \mathbb{R}\}$ has stationary increments [Mueller and Tribe (2002)].

By deriving a harmonizable representation for $U(t,x)$ and applying Theorem 4.2 above, we prove that $U(t,x)$ has the property of strong local nondeterminism.

Corollary 4.1

There exists a constant $c > 0$ such that for all $n \geq 1$ and $(t,x), (s_1,y_1) \ldots, (s_n,y_n) \in [0,1] \times [-1,1]$,

$$\begin{aligned}\mathrm{Var}\left(U(t,x) \,\middle|\, U(s_1,y_1), \ldots, U(s_n,y_n)\right) \\ \geq c \min_{0 \leq k \leq n} \left(|t - s_i|^{1/2} + |x - y_i|\right),\end{aligned}$$

where $(s_0,y_0) = (0,0)$.

Applying Corollary 4.1, together with Theorem 3.1 in Lecture 3, we obtain

Corollary 4.2

Let $\{U(t,x), t \geq 0, x \in \mathbb{R}\}$ be a real valued random string process. Then

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{(t,x),(s,y) \in I \\ \sigma((t,x),(s,y)) \leq \varepsilon}} \frac{|U(t,x) - U(s,y)|}{\sigma((t,x),(s,y))\sqrt{|\log \sigma((t,x),(s,y))|}} = \kappa,$$

where κ is a positive and finite constant, and

$$\sigma((t,x),(s,y)) := |t-s|^{1/4} + |x-y|^{1/2}.$$

4.4 A comparison theorem

For any $\lambda \in \mathbb{R}^N$ and $h > 0$, denote by $C(\lambda, h)$ the cube with side-length $2h$ and center λ, i.e.,

$$C(\lambda, h) = \left\{x \in \mathbb{R}^N : |x_j - \lambda_j| \le h, \, j = 1, \cdots, N\right\}.$$

Let $L^2(C(0,T))$ be the subspace of $g \in L^2(\mathbb{R}^N)$ whose support is contained in $C(0,T)$.

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Let $L^2(C(0,T))$ be the subspace of $g \in L^2(\mathbb{R}^N)$ whose support is contained in $C(0,T)$.

Theorem 4.3 [Luan and X., 2012]

Let $\{Y(t), t \in \mathbb{R}^N\}$ be a real, centered Gaussian field with stationary increments and $Y(0) = 0$. If for some $h > 0$ the spectral measure Δ of Y satisfies

$$\begin{aligned} 0 < \liminf_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta(C(\lambda,h)) \\ \le \limsup_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta(C(\lambda,h)) < \infty, \end{aligned} \tag{11}$$

then for any $T > 0$ such that $ThN < \log 2$, for all $u, t^1, \ldots, t^n \in C(0,T)$,

$$\mathrm{Var}\Big(Y(u) \,\big|\, Y(t^1), \ldots, Y(t^n)\Big) \ge c \min_{0\le k\le n} \rho(u,\, t^k)^2.$$

Proof of Theorem 4.3

Lemma 4.2 (Pitt, 1975)

Let $\widetilde{\Delta}(d\lambda)$ be a positive measure on $\mathbb{R}^N$. If, for some constant $h > 0$, $\widetilde{\Delta}(d\lambda)$ satisfies

$$0 < \liminf_{\|\lambda\|\to\infty} \widetilde{\Delta}(C(\lambda, h)) \le \limsup_{\|\lambda\|\to\infty} \widetilde{\Delta}(C(\lambda, h)) < \infty.$$

Then for every $T > 0$ satisfying $ThN < \log 2$, we have

$$\int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 \widetilde{\Delta}(d\lambda) \asymp \int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 d\lambda$$

for all $\psi \in L^2(C(0, T))$.

Lemma 4.3 (Luan and X. 2012)

Let $\Delta_1(d\lambda)$ be a measure on $\mathbb{R}^N$ such that for some $h > 0$,

$$0 < \liminf_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_1(C(\lambda,h)) \le \limsup_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_1(C(\lambda,h)) < \infty.$$

Then for any $T > 0$ with $ThN < \log 2$, $\exists$ constants c_3 and c_4 such that

$$c_3 \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{Q+2}} \, d\lambda \le \int_{\mathbb{R}^N} |g(\lambda)|^2 \Delta_1(d\lambda) \le c_4 \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{Q+2}} \, d\lambda$$

for all $g(\lambda)$ as in Lemma 4.1.

Theorem 4.3 follows from Lemma 4.3 and Theorem 4.2.

Examples

Example 4.1. Let $\{\xi_n, n \in \mathbb{Z}^N\}$ and $\{\eta_n, n \in \mathbb{Z}^N\}$ be two independent sequences of i.i.d. $N(0,1)$ random variables. Let

$$Z(t) = \sum_{n \in \mathbb{Z}^N} a_n \big(\xi_n \cos \langle n, t\rangle + \eta_n \sin \langle n, t\rangle \big), \quad t \in \mathbb{R}^N,$$

where $\{a_n, n \in \mathbb{Z}^N\}$ is a sequence of real numbers such that

$$a_n^2 \asymp \frac{1}{\big(\sum_{j=1}^N |n_j|^{H_j}\big)^{Q+2}}.$$

Then the Gaussian field $Y(t) = Z(t) - Z(0)$ has the property of strong local nondeterminism.

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$$a_n^2 \asymp \frac{1}{\big(\sum_{j=1}^N |n_j|^{H_j} \big)^{Q+2}}.$$

Then the Gaussian field $Y(t) = Z(t) - Z(0)$ has the property of strong local nondeterminism.

Example 4.2. Let μ be the measure on $\mathbb{R}$ obtained by "patching" fractal probability measures on $[n, n+1]$, and let the spectral measure Δ be given by

$$\frac{d\mu(\lambda)}{|\lambda|^{1+2H}},$$

then Theorem 4.3 implies that a Gaussian process X spectral measure Δ has the property of SLND which is similar to that of fBm B^H.

More interesting is the following example.

Example 4.3 Let C be the one-third Cantor set and let σ be the uniform probability measure on C. We obtain a symmetric measure ν on $\mathbb{R}$ by

$$\nu(A) = \lim_{n\to\infty} 2^n \sigma(3^{-n}A).$$

Let

$$\Delta(d\lambda) = \frac{1}{|\lambda|^{1+2H}} \nu(d\lambda).$$

Consider the Gaussian process

$$X(t) = \int_{\mathbb{R}} \left(e^{it\lambda} - 1\right) \widetilde{W}(d\lambda),$$

where $\widetilde{W}$ is a complex-valued Gaussian random measure with Δ as its control measure.

Clearly, Δ does not satisfy the condition of Theorem 4.3, so it is not comparable with Gaussian processes which are familiar to us.

Nevertheless, the following properties can be verified:

- semi-self-similarity:

$$\{X(3t), t \in \mathbb{R}\} \stackrel{d}{=} \{3^{1-\frac{\log 2}{2\log 3}} X(t), t \in \mathbb{R}\}.$$

- X has stationary increments with

$$\mathbb{E}\Big(\big|X(t) - X(s)\big|^2\Big) \asymp \big|t - s\big|^{2 - \frac{\log 2}{\log 3}}$$

for $s, t \in [0, 1]$. That is, X satisfies (C1).

Consequently, X shares some sample path properties with fractional Brownian motion B^H with $H = 1 - \frac{\log 2}{2 \log 3}$.

Thank you