Gaussian Random Fields: Excursion Probabilities

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Lecture 5 Excursion Probabilities

1. Some classical results on excursion probabilities
   - A large deviation result
   - Upper bounds via the entropy method
   - Asymptotic results (the double sum method)
2. Smooth GFs: Excursion probability
   - The expected Euler characteristic approximation
3. Vector-valued Gaussian fields
   - Smooth case
   - Non-smooth case

Let $X = \{X(t), t \in T\}$ be a real-valued Gaussian random field, where T is the index set. The excursion probability

$$\mathbb{P}\Big\{ \sup_{t \in T} X(t) \geq u \Big\}, \quad (u > 0)$$

is important in probability, statistics and their applications.

When $T \subset \mathbb{R}^N$ and $N = 1$, only in very few special cases, the exact formulae for the excursion probability are available. When $N > 1$, no exact formula is known.

5.1 Some classical results

Theorem 5.1 (Landau and Shepp, 1970; Marcus and Shepp, 1972)

If $\{X(t), t \in T\}$ is a centered GRF and $\sup_{t\in T} X(t) < \infty$ a.s., then

$$\lim_{u\to\infty} \frac{1}{u^2} \log \mathbb{P}\Big\{ \sup_{t\in T} X(t) \geq u \Big\} = -\frac{1}{2\sigma_T^2},$$

where $\sigma_T^2 = \sup_{t\in T} \mathbb{E}(X(t)^2)$.

The Borell-TIS inequality

Theorem 5.2 [Borell, 1975; Tsirelson, Ibragimov and Sudakov, 1976]

Let $X = \{X(t), t \in T\}$ be a centered Gaussian process with a.s. bounded sample paths. Let $\|X\| = \sup_{t \in T} X(t)$. Then $\mathbb{E}(\|X\|) < \infty$ and for all $\lambda > 0$,

$$\mathbb{P}\big(\big| \|X\| - \mathbb{E}(\|X\|) \big| > \lambda \big) \leq 2 \exp\Big(- \frac{\lambda^2}{2\sigma_T^2} \Big).$$

Proof of Theorem 5.1

The Borell-TIS inequality implies immediately the upper bound in Theorem 5.1:

$$\lim_{u\to\infty} \frac{1}{u^2} \log \mathbb{P}\Big\{ \sup_{t\in T} X(t) \ge u \Big\} \le -\frac{1}{2\sigma_T^2}.$$

The lower bound in Theorem 5.1 is easy.

Remark The Borell-TIS inequality, combined with a partitioning argument, can lead to improved non asymptotic upper bounds, as shown by the following result.

Upper bounds using entropy method

For $\delta > 0$, set

$$T_\delta = \{t \in T : \mathbb{E}(X(t)^2) \geq \sigma_T^2 - \delta\}.$$

Theorem 5.3 (Samorodnitsky, 1991; Talagrand, 1994)

If $\exists\, v \geq w \geq 1$ such that

$$N(T_\delta, d_X, \varepsilon) \leq K\delta^w \varepsilon^{-v},$$

where d_X is the canonical metric of X, then for $u \geq 2\sigma_T w$,

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \leq K\Big(\frac{u}{\sigma_T^2}\Big)^{v-w} \Phi\Big(\frac{u}{\sigma_T}\Big).$$

Pickands' asymptotic theorem

Theorem 5.4 (Pickands, '69; Qualls and Watanabe, '73)

Let $\{X(t), t \in [0, L]^N\}$ be a centered stationary Gaussian field with

$$\mathbb{E}(X(s)X(t)) = 1 - |s-t|^\alpha + o(|s-t|^\alpha)$$

for a constant $\alpha \in (0, 2]$, then

$$\lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in[0,L]^N} X(t) \ge u\}}{\psi(u)u^{2N/\alpha}} = H_\alpha L^N, \qquad (1)$$

where $\psi(u) = (2\pi)^{-1}u^{-1}\exp(-u^2/2)$ and H_α is Pickands' constant.

Recall that Pickands' constant is defined as

$$H_\alpha = \lim_{A\to\infty} \frac{1}{A^N} \int_0^\infty e^s \cdot \mathbb{P}\Big(\sup_{t\in[0,A]^N} \big(\chi(t) - |t|^\alpha\big) > s\Big)\, ds,$$

where χ is a centered Gaussian field with covariance function $\mathbb{E}[\chi(t)\chi(s)] = |t|^\alpha + |s|^\alpha - |t-s|^\alpha$ (FBM).

The only known values of H_α are

$$H_1 = 1, \qquad H_2 = \frac{1}{\sqrt{\pi}}.$$

Ideas for the proof of Theorem 5.4

- Divide $[0, L]^N$ into N_u small cubes C_j of side-length $u^{-2/\alpha}$. So $N_u = L^N u^{2N/\alpha}$.
  Observe that

$$\mathbb{P}\Big\{ \sup_{t \in [0,L]^N} X(t) \geq u \Big\} = \mathbb{P}\Big\{ \bigcup_{j=1}^{N_u} \sup_{t \in C_j} X(t) \geq u \Big\}$$

$$\leq \sum_{j=1}^{N_u} \mathbb{P}\Big\{ \sup_{t \in C_j} X(t) \geq u \Big\}$$

and

$$\mathbb{P}\Big\{ \sup_{t\in[0,L]^N} X(t) \ge u \Big\} \ge \sum_{j=1}^{N_u} \mathbb{P}\Big\{ \sup_{t\in C_j} X(t) \ge u \Big\}$$

$$- \sum_{i=1}^{N_u} \sum_{j=1}^{N_u} \mathbb{P}\Big\{ \sup_{t\in C_i} X(t) \ge u,\ \sup_{t\in C_j} X(t) \ge u \Big\}.$$

Ideas for the proof

Prove that $\sum_{j=1}^{N_u} \mathbb{P}\Big\{ \sup_{t\in C_j} X(t) \ge u \Big\}$ is the main term and the double sum is negligible.

We recall one important step in the proof. Write

$$\mathbb{P}\Big\{ \sup_{t\in C_j} X(t) \ge u \Big\} = \mathbb{P}\big\{ X(0) \ge u \big\} + \int_{-\infty}^{u} \mathbb{P}\Big\{ \max_{t\in C_j} X(t) \ge u \Big| X(0) = x \Big\} \phi(x) dx,$$

where ϕ is the density of standard normal $N(0, 1)$.

For any $a > 0$ and integer vector $\mathbf{n}$, let

$$I_u[a\mathbf{n}/u^{2/\alpha}] = \left\{ \frac{a\mathbf{k}}{u^{2/\alpha}} : 0 \le \mathbf{k} \le \mathbf{n} \right\} := I_u.$$

One can show that

$$\lim_{u\to\infty} \frac{\mathbb{P}\Big\{ \max_{t\in I_u} X(t) \ge u \Big\}}{\psi(u)} = 1 + \int_0^\infty e^y \mathbb{P}\Big\{ \max_{\mathbf{k}\le\mathbf{n}} \chi(\mathbf{k}) > y \Big\} dy.$$

where $\psi(u) = \mathbb{P}\{N(0,1) > u\}$.

Nonstationary case: A result for fBm

Theorem 5.5 (Talagrand, 1988)

Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be a fBm with index $H \in (0, 1)$. If $H > 1/2$, then for any $L > 0$,

$$\lim_{u\to\infty} \frac{\mathbb{P}\{ \sup_{t\in[0,L]^N} B^H(t) \ge u\}}{\mathbb{P}\{B^H(\langle L\rangle) \ge u\}} = \lim_{u\to\infty} \frac{\mathbb{P}\{ \sup_{t\in[0,L]^N} B^H(t) \ge u\}}{\psi(u/(L\sqrt{N})^H)} = 1.$$

This is clearly different from (1). The reason is that $\mathbb{E}(B^H(t)^2)$ has a unique maximum at $t = \langle L\rangle$.

5.2 Asymptotic expansion for smooth Gaussian fields

- The Rice method initiated by Rice (1944) and developed by many others: see Adler (1981), Azais and Wschebor (2009).
- The Euler characteristic method by Worsley (1995), Taylor, Takemura and Adler (2005), Taylor and Adler (2007).

The Euler characteristic method

Let $A_u = \{t \in T : X(t) \geq u\}$ be the excursion set.

A general conjecture is that the mean Euler characteristic of A_u gives the behavior of $\mathbb{P}\big\{ \sup_{t\in T} X(t) \geq u \big\}$.

This conjecture is referred to as "the Expected Euler Characteristic Heuristic", which has proven to be true for some cases.

Before we give any details, let us recall the notion of Euler characteristic of a set.

The Euler characteristic method

Let $A \subset \mathbb{R}^N$ be a finite union of "basic" sets. The EC $\varphi(A)$ can be defined as the unique function which satisfies the following properties:

- $\varphi(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \neq \emptyset \text{ is "basic" or "ball like".} \end{cases}$
- $\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B)$.

If $N = 1$, then the Euler characteristic of A is $\varphi(A)$ = # of disjoint intervals in A.

If $N = 2$, then
$\varphi(A) = \#$ of its connected components $- \#$ of holes.

The Euler characteristic method

- When $T = [0, L]$, $\varphi(A_u)$ is like the number of upcrossings of the level u by the process $X(t)$ and $\mathbb{E}\{\varphi(A_u)\}$ is similar to the Rice formula, which has long been used to approximate the excursion probability.
- If $T = [0, L]^N$ and $N \geq 2$, it is difficult to define "upcrossings of the level u". The Euler characteristic becomes a natural choice.
- One can also use other quantities such as the "expected number of local maxima" to approximate the excursion probability.

Euler characteristic method

Theorem 5.6 (Taylor, Takemura and Adler, 2005)

Let $X = \{X(t) : t \in T\}$ be a unit-variance smooth Gaussian field parameterized on a manifold T. Under certain conditions on the regularity of X and topology of T, there exists $\alpha_0 > 0$ such that as $u \to \infty$,

$$\mathbb{P}\Big\{ \sup_{t\in T} X(t) \geq u \Big\} = \mathbb{E}\{\varphi(A_u(X,T))\}\big(1 + o\big(e^{-\alpha_0 u^2}\big)\big),$$

where $\varphi(A_u(X,T))$ is the Euler characteristic of the excursion set

$$A_u(X,T) = \{t \in T : X(t) \geq u\}.$$

- $\mathbb{E}\{\varphi(A_u(X,T))\}$ can be computed via the Kac-Rice formula [cf. Adler and Taylor (2007)],

$$\mathbb{E}\{\varphi(A_u(X,T))\} = C_0\Psi(u) + \sum_{j=1}^{\dim(T)} C_j u^{j-1} e^{-u^2/2},$$

  where C_j are constants depending on X and T.
- Compared with Pickands' approximation, this expansion is much more accurate since the error decays exponentially fast. In fact, Pickands' approximation only contains one of the terms involving $u^{N-1}e^{-u^2/2}$ in $\mathbb{E}\{\varphi(A_u(X,T))\}$.

Example 5.1 Let X be a smooth isotropic Gaussian field with unit variance and $T = [0, L]^N$, then

$$\mathbb{E}\{\varphi(A_u(X,T))\} = \Psi(u) + \sum_{j=1}^{N} \frac{\binom{N}{j} L^j \lambda^{j/2}}{(2\pi)^{(j+1)/2}} H_{j-1}(u) e^{-u^2/2},$$

where $\lambda = \mathrm{Var}(X_i(t))$ and $H_{j-1}(u)$ are Hermite polynomials.

The constant-variance and isotropy conditions are sometimes too restrictive for many applications.

Adler (2000, section 7.3) listed "non-stationary random fields" as one of the main future research directions.

We study the following questions:

- For Gaussian fields with stationary increments, how to compute the mean Euler characteristic of their excursion sets?
- Can they still be used to approximate the excursion probabilities?
- What about excursion probabilities of vector-valued Gaussian random fields?

We have obtained some results about these questions and they are presented in the following papers.

- D. Cheng and Y. Xiao. Mean Euler characteristic approximation to excursion probability of Gaussian random fields. *Ann. Appl. Probab.* **26** (2016), 722–759.
- D. Cheng and Y. Xiao. Excursion probability of smooth vector-valued Gaussian random fields. *Preprint*, 2016.
- Y. Zhou and Y. Xiao. Tail asymptotics of extremes for bivariate Gaussian random fields. *Bernoulli*, to appear.

In the following, we present some results from the first two papers.

5.3 Smooth Gaussian fields with stationary increments

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and $X(0) = 0$. It is represented by

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1)\widetilde{W}(d\lambda),$$

where $\widetilde{W}$ is a complex-valued Gaussian random measure with control measure Δ, which satisfies

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\Delta(d\lambda) < \infty.$$

Sufficient conditions for sample path differentiability in terms of the spectral measure of X are known.
For example, if the spectral density $f(\lambda)$ satisfies

$$f(\lambda) = O\left(|\lambda|^{-(2H+N+2k)}\right) \quad \text{as } |\lambda| \to \infty,$$

where $k \geq 1, H \in (0,1)$, then X has a version $\widetilde{X}$ such that $\widetilde{X}(\cdot) \in C^k(\mathbb{R}^N)$ almost surely.

- We will consider the case $k = 2$ and use the following notations

$$X_i(t) = \frac{\partial X(t)}{\partial t_i}, \quad \nabla X(t) = (X_1(t), \ldots, X_N(t)),$$
$$X_{ij}(t) = \frac{\partial^2 X(t)}{\partial t_i \partial t_j}, \quad \nabla^2 X(t) = (X_{ij}(t))_{1 \le i,j \le N}.$$

For Gaussian fields with stationary increments, we have $\mathbb{E}\{X_i(t)X_{jk}(t)\} = 0$ for all $t \in \mathbb{R}^N$, i.e. $X_i(t)$ and $X_{jk}(t)$ are independent.

- Let $T = \prod_{i=1}^{N} [a_i, b_i]$ be an N-dimensional rectangle.
- A face J of dimension k, is defined by fixing a subset $\sigma(J) \subset \{1, \cdots, N\}$ of size k and a subset $\varepsilon(J) = \{\varepsilon_j, j \notin \sigma(J)\} \subset \{0, 1\}^{N-k}$ of size $N - k$, so that

$$J = \{t \in T : a_j < t_j < b_j \text{ if } j \in \sigma(J), \\ t_j = (1 - \varepsilon_j)a_j + \varepsilon_j b_j \text{ if } j \notin \sigma(J)\}.$$

- $k = 0$ means $\sigma(J) = \emptyset$, the faces are the vertices.
- $k = N$, then there is only one face which is $\overset{\circ}{T}$.

Let $\partial_k T$ be the collection of faces of dimension k in T, then $\overset{\circ}{T} = \partial_N T$ and

$$\partial T = \cup_{k=0}^{N-1} \cup_{J \in \partial_k T} J.$$

5.3.1 Mean Euler Characteristic

Morse's Theorem (cf. Adler and Taylor, 2007) gives a formula for the Euler characteristic of the excursion set of X.

Theorem 5.7 [Morse's theorem]

Let $X(t)$ be a Morse function a.s. Then

$$\varphi(A_u(X,T)) = \sum_{k=0}^{N} \sum_{J\in\partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(J) \text{ a.s.,}$$

where

$$\mu_i(J) = \#\big\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0,$$
$$\text{index}(\nabla^2 X_{|J}(t)) = i, \varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\big\}.$$

Example 5.3 Let $T = [0,1] = \{0\} \cup \{1\} \cup (0,1)$ and a smooth function $X(t)$, we have

$$\begin{aligned}\varphi(A_u(X,T)) &= \mathbb{1}_{\{X(0)\geq u, X'(0)\leq 0\}} + \mathbb{1}_{\{X(1)\geq u, X'(1)\geq 0\}}\\ &+ \#\{t\in(0,1):\ X(t)\geq u, X'(t)=0, X''(t)<0\}\\ &- \#\{t\in(0,1):\ X(t)\geq u, X'(t)=0, X''(t)>0\}.\end{aligned}$$

Mean Euler Characteristic

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and spectral density $f(\lambda)$. Assume

H1: $f(\lambda) = O\left(|\lambda|^{-(2H+N+4)}\right)$ for some $H \in (0, 1)$.

H2: $\forall t \in T$, $(X(t), \nabla X(t), \nabla^2 X(t))$ has nondegenerate distribution.

Notation:

$$(\mathbb{E}\{X_i(t)X_j(t)\})_{i,j=1,\cdots,N} = (\lambda_{ij})_{i,j=1,\cdots,N} = \Lambda,$$
$$(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j=1,\cdots,N} = (\lambda_{ij}(t) - \lambda_{ij})_{i,j=1,\cdots,N} = \Lambda(t) - \Lambda,$$

where

$$\lambda_{ij} = \int_{\mathbb{R}^N} \lambda_i \lambda_j f(\lambda)\, d\lambda, \quad \lambda_{ij}(t) = \int_{\mathbb{R}^N} \lambda_i \lambda_j \cos\langle t, \lambda\rangle f(\lambda)\, d\lambda.$$

Define $\Lambda_J = (\lambda_{ij})_{i,j\in\sigma(J)}$, $\Lambda_J(t) = (\lambda_{ij}(t))_{i,j\in\sigma(J)}$ and

$$\gamma_t^2 = \mathrm{Var}(X(t)|\nabla X(t)) = \frac{\det\mathrm{Cov}(X(t), \nabla X(t))}{\det\mathrm{Cov}(\nabla X(t))}.$$

For $J \in \partial_k T$, we denote

$$\{1, \cdots, N\}\backslash\sigma(J) = \{J_1, \cdots, J_{N-k}\}$$

and let

$$E(J) = \Big\{(t_{J_1}, \cdots, t_{J_{N-k}}) \in \mathbb{R}^{N-k} : \\ t_j\varepsilon_j^* > 0,\ j = J_1, \cdots, J_{N-k}\Big\}.$$

Let $C_j(t)$ be the $(1, j+1)$ entry of $(\mathrm{Cov}(X(t), \nabla X(t)))^{-1}$.

Theorem 5.8 (Cheng and X. 2016)

$$
\begin{aligned}
\mathbb{E}\{\varphi(A_u)\} &= \sum_{t\in\partial_0 T} \mathbb{P}\Big(X(t) \geq u, \nabla X(t) \in E(\{t\})\Big) \\
&\quad + \sum_{k=1}^{N} \sum_{J\in\partial_k T} \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k}\, dt \int_u^\infty dx \\
&\quad \times \int_{E(J)} H_k\Big(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t) y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t) y_{J_{N-k}}\Big) \\
&\quad \times p_t(x, y_{J_1}, \cdots, y_{J_{N-k}} | 0, \cdots, 0)\, dy_{J_1} \cdots dy_{J_{N-k}},
\end{aligned}
$$

where p_t is the conditional density of

$$(X(t), X_{J_1}(t), \cdots, X_{J_{N-k}}(t) | \nabla X_{|J}(t) = 0).$$

Remarks

- The proof relies strongly on two properties of Gaussian fields with stationary increments:
  (i) $X_i(t)$ and $X_{jk}(t)$ are independent.
  (ii) $\Big(\mathbb{E}\{X(t)X_{ij}(t)\}\Big)_{i,j} = \Lambda(t) - \Lambda$ are negative definite.
- In many cases, the formula can be simplified with only a super-exponentially small difference.

5.3.2 Approximation to the excursion probability

Define the number of extended outward maxima above level u by

$$M_u^E(J) \triangleq \#\Big\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0,$$
$$\text{index}(\nabla^2 X_{|J}(t)) = k, \varepsilon_j^* X_j(t) > 0 \text{ for all } j \notin \sigma(J)\Big\}.$$

Recall $T = \cup_{k=0}^N \partial_k T = \cup_{k=0}^N \cup_{J\in\partial_k T} J$, it can be shown that

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{P}\Big\{\bigcup_{k=0}^{N}\bigcup_{J\in\partial_k T}\{M_u^E(J) \geq 1\}\Big\},$$

see Azais and Delmas (2002).

By the Bonferroni inequality and Piterbarg (1996),

$$\sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\} \geq \mathbb{P}\Big\{ \sup_{t \in T} X(t) \geq u \Big\}$$

$$\geq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \big(\mathbb{E}\{M_u^E(J)\} - \mathbb{E}\{M_u^E(J)(M_u^E(J) - 1)\} \big)$$

$$- \sum_{J \neq J'} \mathbb{E}\{M_u^E(J) M_u^E(J')\}.$$

Extending the method in Azais and Delmas (2002), we prove

Lemma 5.1

Under conditions in Theorem 5.8, there exists some $\alpha > 0$ such that

$$\sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\} = \mathbb{E}\{\varphi(A_u)\} + o\left(e^{-\alpha u^2 - u^2/2\sigma_T^2}\right),$$

where $\sigma_T^2 \triangleq \sup_{t \in T} \operatorname{Var}\big(X(t)\big)$.

The following theorem shows that the "Expected Euler Characteristic Heuristic" holds more generally.

Theorem 5.9 (Cheng and X., 2016)

Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments satisfying **H1**, **H2** and **H3**: For all $t \neq s \in \mathbb{R}^N$,

$$(X(t), \nabla X(t), X_{ij}(t), X(s), \nabla X(s), \nabla^2 X(s), 1 \leq i \leq j \leq N)$$

have nondegenerate distributions.

Then there exists $\alpha > 0$ such that

$$\mathbb{P}\Big\{ \sup_{t \in T} X(t) \geq u \Big\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2\sigma_T^2}).$$

Corollary 5.1

Under the conditions in Theorem 5.9 and an extra condition, $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ equals

$$\sum_{t\in\partial_0 T} \Psi\Big(\frac{u}{\sigma_t}\Big) + \sum_{k=1}^{N} \sum_{J\in\partial_k T} \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}}$$
$$\times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/2\theta_t^2}\, dt + o\big(e^{-\alpha u^2 - u^2/2\sigma_T^2}\big),$$

where for $t \in J$,

$$\theta_t^2 = \mathrm{Var}(X(t)|\nabla X_{|J}(t)) = \frac{\det \mathrm{Cov}(X(t), \nabla X_{|J}(t))}{\det \mathrm{Cov}(\nabla X_{|J}(t))}.$$

5.4. Vector-valued Gaussian fields

Consider a multivariate random field $\mathbf{X} = \{X(t),\ t \in \mathbb{R}^N\}$ taking values in $\mathbb{R}^p$ defined by

$$\mathbf{X}(t) = (X_1(t), \cdots, X_p(t)), \qquad t \in \mathbb{R}^N. \tag{2}$$

Their key features are:

- the components $X_1, \ldots, X_p$ are dependent.
- $X_1, \ldots, X_p$ may have different smoothness properties.

Given subsets $T_1, \ldots, T_p$ of $\mathbb{R}^N$, it is of interest to estimate the excursion probability

$$\mathbb{P}\Big\{ \max_{t \in T_1} X_1(t) \ge u_1, \cdots, \max_{t \in T_p} X_p(t) \ge u_p \Big\} \tag{3}$$

for certain threshold values $u_1, \ldots, u_p$.

For $T \subset \mathbb{R}^N$, another type of excursion probability for $\mathbf{X}$ is

$$\mathbb{P}\big\{ \exists t \in T \text{ such that } X_i(t) \ge u_i, \ \forall\, 1 \le i \le p \big\}. \tag{4}$$

We focus on the excursion probabilities in (3) for $p = 2$.

- Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian random field, where T and S are rectangles in $\mathbb{R}^N$.
- We are interested in the joint excursion probability

$$\mathbb{P}\Big\{ \sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u \Big\}.$$

- Only a few results are known, see Piterbarg (2000), Piterbarg and Stamatovic (2005) and Debicki et al. (2010).

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5.4.1 The expected Euler characteristic method

- We decompose T and S into several faces of lower dimensions

$$T = \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} J, \quad S = \bigcup_{l=0}^{N} \bigcup_{L \in \partial_l S} L.$$

- Similarly to the real-valued case,

$$\mathbb{P}\Big\{ \sup_{t \in T} X(t) \geq u, \sup_{s \in S} Y(s) \geq u \Big\}$$

$$= \mathbb{P}\Bigg\{ \bigcup_{k,l=0}^{N} \bigcup_{J \in \partial_k T, L \in \partial_l S} \{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\} \Bigg\}.$$

Upper Bound

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\}$$

$$\leq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\}$$

$$\leq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{E}\{M_u^E(X,J) M_u^E(Y,L)\}.$$

Lower Bound

$$
\begin{aligned}
&\mathbb{P}\Big\{ \sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u \Big\} \\
&\geq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \Big\{ \mathbb{E}\{M_u^E(X,J) M_u^E(Y,L)\} \\
&\quad - \mathbb{E}\{M_u^E(X,J)[M_u^E(X,J)-1] M_u^E(Y,L)\} \\
&\quad - \mathbb{E}\{M_u^E(Y,L)[M_u^E(Y,L)-1] M_u^E(X,J)\} \Big\} \\
&\quad - \text{“crossing terms”}.
\end{aligned}
$$

Smoothness and regularity conditions

- $(\mathbf{H}1')$. $X, Y \in C^2$ a.s. and their second derivatives satisfy the uniform mean-square Hölder condition.
- $(\mathbf{H}2')$. For every $(t, t', s) \in T^2 \times S$ with $t \neq t'$,

$$\begin{aligned}&(X(t), \nabla X(t), \nabla^2 X(t), X(t'), \nabla X(t'), \nabla^2 X(t'),\\ &Y(s), \nabla Y(s), \nabla^2 Y(s), 1 \le i \le j \le N)\end{aligned}$$

  is non-degenerate; and for every $(s, s', t) \in S^2 \times T$ with $s \neq s'$,

$$\begin{aligned}&(Y(s), \nabla Y(s), \nabla^2 Y(s), Y(s'), \nabla Y(s'), \nabla^2 Y(s'),\\ &X(t), \nabla X(t), \nabla^2 X(t), 1 \le i \le j \le N)\end{aligned}$$

  is non-degenerate.

Smoothness and regularity conditions

Let

$$\rho(t,s) = \mathbb{E}\{X(t)Y(s)\}, \quad \rho(T,S) = \sup_{t\in T, s\in S} \rho(t,s).$$

- $(\mathbf{H}3')$. For every $(t,s) \in T \times S$ such that $\rho(t,s) = \rho(T,S)$,

$$(\mathbb{E}\{X_{ij}(t)Y(s)\})_{i,j\in\zeta(t,s)}, \; (\mathbb{E}\{X(t)Y_{i'j'}(s)\})_{i',j'\in\zeta'(t,s)}$$

are both negative semi-definite, where

$$\zeta(t,s) = \{n : \mathbb{E}\{X_n(t)Y(s)\} = 0, 1 \le n \le N\},$$
$$\zeta'(t,s) = \{n : \mathbb{E}\{X(t)Y_n(s)\} = 0, 1 \le n \le N\}.$$

Theorem 5.10 [(Cheng and X. (2016+)]

Under $(\mathbf{H}1')$-$(\mathbf{H}3')$, there exists $\alpha_0 > 0$ such that as $u \to \infty$,

$$\mathbb{P}\Big\{ \sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u \Big\} \\ = \mathbb{E}\{\varphi(A_u(X,T) \times A_u(Y,S))\} \\ + o\bigg(\exp\Big\{ - \frac{u^2}{1+\rho(T,S)} - \alpha_0 u^2 \Big\}\bigg),$$

where

$$A_u(X,T)\times A_u(Y,S) = \{(t,s) \in T\times S : X(t) \geq u, Y(s) \geq u\}.$$

5.5 The double sum method

Consider non-smooth bivariate locally stationary Gaussian field $X(t) = (X_1(t), X_2(t))$.
Define

$$r_{ij}(s,t) := E[X_i(s)X_j(s+t)], \ i,j = 1 \text{ or } 2. \tag{5}$$

Let $|t| := \sqrt{\sum_{j=1}^{N} t_j^2}$ be the l^2-norm of a vector $t \in \mathbb{R}^N$.

Assumptions:

- **i)** $r_{ii}(s,t) = 1 - c_i|t-s|^{\alpha_i} + o(|t-s|^{\alpha_i})$, where $\alpha_i \in (0,2), c_i > 0$ for $i = 1, 2$.
- **ii)** $|r_{ii}(s,t)| < 1$ for all $|t-s| > 0$, $i = 1, 2$.
- **iii)** $r_{12}(s,t) = r_{21}(s,t) := r(|t-s|)$, which means the cross correlation is isotropic.
- **iv)** $r(\cdot) : [0,\infty) \to \mathbb{R}$ attains maximum only at zero with $r(0) = \rho \in (0,1)$, i.e., $|r(t)| < \rho$ for all $t > 0$. Moreover, we assume $r'(0) = 0$, $r''(0) < 0$ and there exists $\eta > 0$, for any $s \in [0,\eta]$, $r''(s)$ exists and continuous.

Let $S,\ T \subset \mathbb{R}^N$ be bounded Jordan measurable sets (that is, the boundary of S and T have Lebesgue measure 0.

Theorem 5.11 [Zhou and X. (2015)]

If $mes_N(S \cap T) \neq 0$, then as $u \to \infty$,

$$\mathbb{P}\Big\{ \max_{s\in S} X_1(s) > u, \max_{t\in T} X_2(t) > u \Big\}$$
$$= (2\pi)^{\frac{N}{2}} (-r''(0))^{-\frac{N}{2}} c_1^{\frac{N}{\alpha_1}} c_2^{\frac{N}{\alpha_2}} (1+\rho)^{-N(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-1)}$$
$$\times\, mes_N(S \cap T) H_{\alpha_1} H_{\alpha_2}\, u^{N(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-1)} \Psi(u,\rho)(1+o(1)),$$

where H_α denotes Pickands constant and $\Psi(u,\rho)$ is

$$\Psi(u,\rho) := \frac{(1+\rho)^2}{2\pi u^2 \sqrt{1-\rho^2}} \exp\left(-\frac{u^2}{1+\rho} \right).$$

Two remarks about Theorem 5.11

- The rate of exponential decay is $-\frac{u^2}{1+\rho}$, where ρ is the **maximum cross correlation** over $S \times T$.
- The extreme tail probability is proportional to **the volume of the set** $\{(s,s) \mid s \in S \cap T\}$, where $(X_1(\cdot), X_2(\cdot))$ attains maximum cross correlation.

If $mes_N(S \cap T) = 0$, the above theorem fails, and result depends on the dimension of $S \cap T$.

Let $S = S_{1,M} \times \prod_{j=M+1}^{N} [a_j, b_j]$ and $T = T_{2,M} \times \prod_{M+1}^{N} [h_j, k_j]$, where $0 \le M \le N - 1$, $S_{1,M}$ and $T_{2,M}$ are M dimensional Jordan sets with

$$mes_M(S_{1,M} \cap T_{2,M}) \neq 0$$

and $a_j \le b_j < h_j$ for $j = M + 1, \ldots, N$.

Theorem 5.12 [Zhou and X. (2015)]

Under the above conditions, we have

$$\mathbb{P}\left(\max_{s\in S} X_1(s) > u,\ \max_{t\in T} X_2(t) > u \right)$$
$$= (2\pi)^{\frac{M}{2}} (-r''(0))^{-\frac{2N-M}{2}} c_1^{\frac{N}{\alpha_1}} c_2^{\frac{N}{\alpha_2}} (1+\rho)^{2N-M-\frac{2N}{\alpha_1}-\frac{2N}{\alpha_2}}$$
$$\times\, mes_M(S_{1,M} \cap T_{2,M}) H_{\alpha_1} H_{\alpha_2}$$
$$\times\, u^{M+N(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-2)} \Psi(u,\rho)(1+o(1)), \text{ as } u \to \infty.$$

Example: The bivariate Matérn field

Multivariate stationary Matérn models $\{\mathbf{X}(t),\ t \in \mathbb{R}^N\}$ in (2) with marginal and cross-covariance functions of the form

$$M(h|\nu, a) := \frac{2^{1-\nu}}{\Gamma(\nu)}(a\|h\|)^{\nu} K_{\nu}(a\|h\|),$$

(with parameters a, ν) have been introduced and studied by Gneiting, Kleiber and Schlather (2010), Apanansovich, Genton and Sun (2012), Kleiber and Nychka (2013).

Sometimes, It is more convenient to work with the spectral density:

$$f(\omega|\nu, a) = \frac{\Gamma(\nu + \frac{N}{2}) a^{2\nu}}{\Gamma(\nu)\pi^{N/2}} \frac{1}{(a^2 + |\omega|^2)^{\nu+(N/2)}}.$$

The bivariate Matérn field

Let $\mathbf{X}(t) = (X_1(t), X_2(t))^{\mathrm{T}}$ be an $\mathbb{R}^2$-valued Gaussian field whose covariance matrix is determined by

$$C(h) = \begin{pmatrix} c_{11}(h) & c_{12}(h) \\ c_{21}(h) & c_{22}(h) \end{pmatrix}, \tag{6}$$

where $c_{ij}(h) := \mathbb{E}[X_i(s+h)X_j(s)]$ are specified by

$$\begin{aligned} c_{11}(h) &= \sigma_1^2 M(h|\nu_1, a_1), \\ c_{22}(h) &= \sigma_2^2 M(h|\nu_2, a_2), \\ c_{12}(h) &= c_{21}(h) = \rho\sigma_1\sigma_2 M(h|\nu_{12}, a_{12}) \end{aligned} \tag{7}$$

with $a_1, a_2, a_{12}, \sigma_1, \sigma_2 > 0$ and $\rho \in (-1, 1)$.

Gneiting, et al. (2010) gave NSC for (6) to give a valid covariance matrix. In particular, if $\rho \neq 0$, one must have

$$\frac{\nu_1 + \nu_2}{2} \leq \nu_{12}.$$

The parameters ν_1 and ν_2 control the smoothness of the sample function $t \mapsto \mathbf{X}(t)$.

- If $\min\{\nu_1, \nu_2\} > 1$, then a.s. the sample function $t \mapsto (X_1(t), X_2)$ is continuously differentiable.
- If $0 < \nu_1 \leq \nu_2 \leq 1$, then a.s. the sample function $t \mapsto (X_1(t), X_2)$ are non-smooth.

Suppose $0 < \nu_1 \le \nu_2 \le 1$, then by Xiao (1995), we have

$$\dim_{\rm H} {\rm Gr}\mathbf{X}([0,1]^N) = \begin{cases} N + 2 - (\nu_1 + \nu_2), & \text{if } \nu_1 + \nu_2 < N, \\ \frac{N+\nu_2-\nu_1}{\nu_2} & \text{if } \nu_1 < N \le \nu_1 + \nu_2, \end{cases}$$

where ${\rm Gr}\mathbf{X}([0,1]^N) = \{(t, X_1(t), X_2(t))^{\rm T} : t \in [0,1]^N\}$ is the graph set of $\mathbf{X}$. Many other any random sets generated by $\mathbf{X}$ are also fractals.

Theorems 5.10, 5.11, and 5.12 can be applied depending on whether $\min\{\nu_1, \nu_2\} > 2$ and $\max\{\nu_1, \nu_2\} < 1$, respectively.

Thank you