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On Ruelle's Probability Cascades and an Abstract Cavity Method

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Abstract: We construct in this work a Markov process which describes a clustering mechanism through which equivalence classes on $\mathbb{N}$ are progressively lumped together. This clustering process gives a new description of Ruelle's continuous probability cascades. It also enables to introduce an abstract cavity method, which mimicks certain features of the cavity method developed by physicists in the context of the Sherrington Kirkpatrick model.

0. Introduction

We construct in this article a continuous time Markov process, $(\Gamma_u)_{u\geq 0}$, with state space the set E of equivalence relations on $\mathbb{N}$. We call it the "clustering process"; it describes an evolution in which Γ_0-equivalence classes are "lumped together" to form at a later time u the collection of Γ_u-equivalence classes. The trace of the clustering process on the set E_I of equivalence relations of an arbitrary finite subset I of $\mathbb{N}$, is a pure jump process with generator:

$$(L^I f)(\Gamma) = \sum_{\Gamma'} a_{\Gamma,\Gamma'} f(\Gamma') - (N-1) f(\Gamma), \quad \text{for } \Gamma \in E_I, \tag{0.1}$$

where $a_{\Gamma,\Gamma'}$ is 0 unless Γ' is obtained by collapsing $k \geq 2$ of the N distinct equivalence classes of Γ into a single class, in which case $a_{\Gamma,\Gamma'} = 1/\Big[(N-1)\binom{N-2}{k-2}\Big]$. The precise mechanism of clustering is described in Sect. 1 below.

This process is instrumental for the abstract cavity method we develop in this work. It offers a concrete representation of the "continuous probability cascades" constructed by Ruelle in [10]. It also provides an example of a coalescent Markov process, in the spirit of Kingman [1, 2]. For further developments around the clustering process, see also Pitman [9].

Let us briefly recall what the "probability cascades" are. For $x \in (0,1)$, we denote by $\mathbb{P}_x$ the law of the Poisson point process on $(0,\infty)$, with intensity $x\eta^{-x-1}d\eta$. If M_p stands for the set of simple pure point Radon measures on $(0,\infty)$, the law $\mathbb{P}_x$ is concentrated on:

$$M = \{m \in M_p; m((0,1]) = \infty, \text{ and } |m| < \infty\}, \text{ where} \tag{0.2}$$

$$|m| = \int_{(0,\infty)} \eta \, dm(\eta). \tag{0.3}$$

Each $m \in M$ can uniquely be written in the form

$$m = \sum_{\ell \geq 0} \delta_{\eta_\ell(m)}, \text{ where } \eta_\ell(m), \ell \geq 0, \text{ is a strictly decreasing sequence which tends to 0 as } \ell \text{ tends to } \infty. \tag{0.4}$$

The probability cascades come as follows. For any finite sequence:

$$0 < x_1 < \cdots < x_K < 1, \tag{0.5}$$

one considers a collection of random variables $\eta^k_{(i_1,\dots,i_k)}$, $i_1,\dots,i_k \geq 0$, $k \in [1,K]$, such that the sequences

$$(\eta^k_{(i_1,\dots,i_{k-1},j)}, j \geq 0), \text{ for } k \in [1,K], i_1,\dots,i_{k-1} \geq 0, \text{ are independent and respectively distributed as } (\eta_j(m))_{j\geq 0} \text{ under } \mathbb{P}_{x_k}. \tag{0.6}$$

Then the random weights

$$\pi_{i_1,\dots,i_K} = \eta^1_{i_1} \dots \eta^K_{i_1,\dots,i_K}, \tag{0.7}$$

are a.s. summable:

$$C = \sum_{i_1,\dots,i_K \geq 0} \pi_{i_1,\dots,i_K} < \infty, \text{ a.s.}, \tag{0.8}$$

and one can recursively define the

$$(\overline{\pi}_{i_1}, \overline{\pi}_{i_1,i_2}, \dots, \overline{\pi}_{i_1,\dots,i_K})_{i_1,\dots,i_K \geq 0}, \text{ via} \tag{0.9}$$

$$\overline{\pi}_{i_1,\dots,i_K} = \pi_{i_1,\dots,i_K}/C, \text{ and } \overline{\pi}_{i_1,\dots,i_{k-1}} = \sum_{j\geq 0} \overline{\pi}_{i_1,\dots,i_{k-1},j}, \text{ for } k \in [1,K]. \tag{0.10}$$

Ruelle introduces in [8] "unordered families", for which one only keeps track of the "tree structure of the labels" in (0.8). He shows a consistency property of the resulting distributions as K and the finite sequence $x_1 < \cdots < x_K$ vary. The "continuous probability cascades" are then constructed in [10] by means of an abstract projective limit argument.

It turns out that the clustering obtained by looking backwards from the last component of (0.9), clumping together points which have common ancestor on level $K-1$, then on level $K-2$ etc., has a Markovian structure. In fact, it is the discrete skeleton of a continuous time Markov process, which is a time change of the clustering process essentially defined by (0.1). It is therefore possible to define the continuous cascades directly from the clustering process. The precise connection with Ruelle's cascades is presented in Sect. 2, Theorem 2.2. Other links of the Ruelle's cascades with continuous branching processes have also been discussed in Neveu [7]. The clustering process enables to introduce the variables

$$\tau_{\ell,\ell'} = \inf\{u \geq 0, (\ell,\ell') \in \Gamma_u\},\ \ell,\ell' \geq 0, \tag{0.11}$$

which represent the time at which ℓ and ℓ' are "lumped together".

For an initial distribution concentrated on the "trivial" equality relation on $\mathbb{N}$, the variables $\tau_{\ell,\ell'}$ naturally define a random ultrametric distance on $\mathbb{N}$, see (1.32) below. In fact when $\ell \neq \ell'$, $\tau_{\ell,\ell'}$ are standard exponential variables.

A second goal of the present article is to develop an "abstract cavity method", which mimicks features of the cavity method for the Sherrington–Kirkpatrick model, as presented in Chapter 5 of Mézard–Parisi–Virasoro [6]. Quite a number of quantities, which appear in the physicists' prediction of the large N behavior of the SK model, naturally arise in our context.

The basic ingredients for the abstract cavity method are:

$$\text{a sequence of normalized random weights } \nu_\ell = \frac{\eta_\ell(m)}{|m|},\ \ell \geq 0, \tag{0.12}$$

where m is $\mathbb{P}_{x_M}$-distributed for a given $x_M \in (0,1)$,

$$\text{an independent standard clustering process } \Gamma_\cdot, \tag{0.13}$$

$$\text{a collection } y^\ell(x),\ x \in [0,x_M],\ \ell \geq 0, \text{ of stochastic processes}, \tag{0.14}$$

which conditional on the normalized weights and the clustering process are centered Gaussian with covariance:

$$\mathrm{cov}(y^\ell(x), y^{\ell'}(x')) = q(x \wedge x' \wedge X_{\ell,\ell'}),\ x,x' \in [0,x_M],\ \ell,\ell' \geq 0, \tag{0.15}$$

where $q(\cdot) : [0,x_M] \to [0,q_M]$ is an increasing C^1-diffeomorphism, and

$$X_{\ell,\ell'} = x_M\, e^{-\tau_{\ell,\ell'}},\ \ell,\ell' \geq 0, \tag{0.16}$$

$$\text{a function } \psi : \mathbb{R} \to \mathbb{R} \text{ in the class } C_b^4. \tag{0.17}$$

In the language of Mézard–Parisi–Virasoro [6], the coefficients ν_ℓ mimick the Gibbsian weights in decreasing order of the "pure states", whereas the clustering process $\Gamma_\cdot$, with the help of the variables $X_{\ell,\ell'}$ induces an "ultrametric structure" on the "pure states", and the $y^\ell(x_M)$, play the role of the "mean cavity field" inside the "pure state" with weight ν_ℓ, i.e. up to relabelling the $h_{\alpha(N)}$ variables of [6], p. 67.

In the Mézard–Parisi–Virasoro picture, the addition of a new spin variable σ induces a change $\sigma y^\ell(x_M)$ of the Hamiltonian in "pure state" ℓ. Summing on this spin variable, the added energy is $\psi(y^\ell(x_M))$, where $\psi(x) = \log\cosh(\beta x)$, β being the inverse temperature. Of crucial importance for the cavity method is the effect of this energy change on the Gibbsian weights of the countably many pure states. This effect can be described in an abstract setup, where for technical reasons, we assume that ψ is bounded. Reshuffling occurs as one multiplies the individual weights ν_ℓ by a factor $e^{\psi(y^\ell(x_M))}$, thereby changing the relative rank of importance of the weights. One thus introduces a random permutation of $\mathbb{N}$, $\widetilde{\sigma}(\cdot)$, with inverse $\sigma(\cdot)$, such that for $\ell \geq 0$:

$$\begin{aligned}&\widetilde{\ell} = \widetilde{\sigma}(\ell) \text{ is the rank of } \mu_\ell = \nu_\ell\ \exp\{\psi(y^\ell(x_M))\}, \text{ among}\\ &\text{the collection } \mu_{\ell'} = \nu_{\ell'}\ \exp\{\psi(y^{\ell'}(x_M))\},\ \ell' \geq 0.\end{aligned} \tag{0.18}$$

The reshuffling operation is the replacement of $\Big((\nu_\ell),\ (X_{\ell,\ell'}),\ (y^\ell(\cdot))\Big)$ by $((\nu^{(R)}_{\widetilde{\ell}}),\ (X^{(R)}_{\widetilde{\ell},\widetilde{\ell}'}),\ (y^{\widetilde{\ell}}_{(R)}(\cdot)))$, where for $\widetilde{\ell},\widetilde{\ell}' \geq 0$:

$$\nu^{(R)}_{\widetilde{\ell}} = \frac{\mu_{\sigma(\widetilde{\ell})}}{\sum_\ell \mu_\ell}\,,\ X^{(R)}_{\widetilde{\ell},\widetilde{\ell}'} = X_{\sigma(\widetilde{\ell}),\sigma(\widetilde{\ell}')},\ \ y^{\widetilde{\ell}}_{(R)}(\cdot) = y^{\sigma(\widetilde{\ell})}(\cdot). \tag{0.19}$$

In other words, the relative importance of the weights is changed, but the initial ultrametric structure and marking processes $y(\cdot)$ are carried along the reshuffling operation.

Our main result Theorem 4.2 describes the effect of reshuffling. It shows that the joint law of the normalized weights and of the ultrametric structure is left invariant by this operation. On the other hand, the conditional law of the $y^{\widetilde{\ell}}_{(R)}(\cdot)$, $\widetilde{\ell} \geq 0$, still preserves the tree structure, but is not Gaussian anymore. For instance, the conditional law of a component $y(\cdot)$, can be represented as that of a time changed process: $z_{q(x)}$, $x \in [0, x_M]$, where z_q, solves the SDE:

$$\begin{aligned} dz_q &= dB_q + x(q)\, m(q, z_q)\, dq, \quad 0 \leq q \leq q_M\,,\\ z_0 &= 0, \end{aligned} \tag{0.20}$$

with $x(\cdot)$ the inverse of the function $q(\cdot)$, $B_.$ a Brownian motion and $m(q,y) = \partial_y\, f(q,y)$, for $f(q,y)$ the unique $C_b^{1,2}$ solution of

$$\partial_q f + \tfrac{1}{2}\,\partial_y^2 f + \frac{x(q)}{2}\,(\partial_y f)^2 = 0, \quad \text{on} \quad (0, q_M)\times\mathbb{R},\ f(q_M,\cdot) = \psi(\cdot). \tag{0.21}$$

Expressions like (0.20), (0.21) can for instance be found in [6], p. 45 or in Parisi [8], see [6], p. 163, as part of the prediction of the large N behavior of the SK model. The boundedness assumption on $\psi(\cdot)$ in (0.17), though technically convenient, excludes the natural choice $\psi(\cdot) = \log(\cosh(\beta\cdot))$, with $\beta > 0$ the inverse temperature, in the context of the SK model. In the case of a non-constant, symmetric function ψ, we can further define an "abstract iteration" procedure, which to $q(\cdot)$ associates a new $q^{(R)}(\cdot)$, see Theorem 5.4. The fixed point equation $q^{(R)}(\cdot) = q(\cdot)$ corresponds to the so-called "selfconsistency equation", for the SK model, see [6] (III.63), p. 45.

Let us now describe how the article is organized: In Sect. 1, we construct the clustering process and derive some of its properties. Section 2 develops the connection between the clustering process and Ruelle's probability cascades. In Sect. 3, we prepare the ground for Sect. 4 and investigate an approximate reshuffling operation. Section 4 contains the main result Theorem 2.2 of the abstract cavity method, which describes the effect of reshuffling. In Sect. 5, we give some applications of the abstract cavity method, to calculations on "single and double replicas". This enables for a non-degenerate symmetric function $\psi(\cdot)$ the definition of an iteration mechanism for the function $q(\cdot)$, see Theorem 5.4.

This work grew out of our efforts to decipher and unravel the probabilistic structure underlying the prediction of the large N behavior of the SK model at low temperature, as presented in the book of Mézard–Parisi–Virasoro [6]. We wish to thank M. Aizenman for helpful discussions in this matter, as well as J.F. Le Gall, J. Pitman, and D. Ruelle for all their comments.

1. The Clustering Process

In this section we shall construct the clustering process. It is a continuous time Markov process with state space

$$E = \{\Gamma \subset \mathbb{N} \times \mathbb{N};\ \Gamma \text{ defines an equivalence relation on } \mathbb{N}\}. \tag{1.1}$$

The set E is endowed with the canonical σ-field $\mathcal{E}$ generated by all events of the form: $\{\Gamma \in E; (a, b) \in \Gamma\}$, for $a, b \in \mathbb{N}$. Remark that E can be viewed as a closed and therefore compact subset of $\{0, 1\}^{\mathbb{N}\times\mathbb{N}}$, the latter being equipped with the product topology. When I is a non-empty subset of $\mathbb{N}$, it is convenient to consider the set E_I and the σ-field $\mathcal{E}_I$, which are defined analogously to E and $\mathcal{E}$, with $\mathbb{N}$ replaced by I. The process we shall introduce describes a "clustering mechanism". Its trajectories $\Gamma_u, u \geq 0$, are non-decreasing E-valued functions, (for the inclusion relation on E).

We keep the notations introduced in the Introduction. The set M_p of simple pure point measures on $(0, \infty)$ is endowed with its canonical σ-field $\mathcal{M}_p$ generated by the applications $m \in M_p \to m(A) \in \mathbb{N} \cup \{\infty\}$ for $A \in \mathcal{B}((0, \infty))$. The measurable subsets M in (0.2), and

$$M_1 = \{m \in M;\ |m| = 1\}, \tag{1.2}$$

are endowed with the respective trace σ-fields $\mathcal{M}$ and $\mathcal{M}_1$.

For $x \in (0, 1)$, we shall denote by $\overline{\mathbb{P}}_x$ the image on M_1 of the Poisson law $\mathbb{P}_x$ under the normalization map:

$$\mathcal{N} : M \to M_1,\ \mathcal{N}\Big(\sum_{\ell\geq 0} \delta_{\eta_\ell(m)}\Big) = \sum_{\ell\geq 0}\ \delta_{\frac{\eta_\ell(m)}{|m|}}. \tag{1.3}$$

We now define for each non-empty $I \subseteq \mathbb{N}$, and $u \geq 0$, a probability kernel R^I_u on E_I. When $u = 0$, $R^I_0(\Gamma, d\Gamma')$ is simply the Dirac mass at $\Gamma \in E_I$. On the other hand when $u > 0$, and $\Gamma \in E_I$, $R^I_u(\Gamma, d\Gamma')$ is defined as follows. We consider the at most denumerable collection $\mathcal{C}_\Gamma$ of Γ-equivalence classes on I. The space $M_1 \times \mathbb{N}^{\mathcal{C}_\Gamma}$ is endowed with the canonical σ-field and the probability

$$Q_x = \overline{\mathbb{P}}_x(dm) \otimes \bigotimes_{C\in\mathcal{C}_\Gamma} \Big(\sum_{\ell\geq 0} \eta_\ell(m)\, \delta_\ell(y_C)\Big), \quad \text{where} \quad x = e^{-u}, \tag{1.4}$$

and y_C, $C \in \mathcal{C}_\Gamma$, are the canonical coordinates on $\mathbb{N}^{\mathcal{C}_\Gamma}$. In other words, conditional to $m = \sum_{\ell\geq 0} \delta_{\eta_\ell(m)} \in M_1$, the variables y_C, $C \in \mathcal{C}_\Gamma$, are independent $\sum_{\ell\geq 0} \eta_\ell\, \delta_\ell$-distributed.

We now "lump together" Γ-equivalences C, which possess the same mark y_C, and obtain a random equivalence relation Γ' on I. Formally, for $(m, (y_C, C \in \mathcal{C}_\Gamma)) \in M_1 \times \mathbb{N}^{\mathcal{C}_\Gamma}$, the collection of subsets:

$$C'_\ell = \bigcup_{y_C=\ell} C,\ \ell \geq 0, \tag{1.5}$$

defines a partition of I, which uniquely determines an equivalence relation $\Gamma' \supset \Gamma$, on I, with equivalence classes the non-empty C'_ℓ, $\ell \geq 0$. We then define

$$R^I_u(\Gamma, d\Gamma') = \text{the law of the } E_I\text{-valued variable } \Gamma', \text{ under } Q_x. \tag{1.6}$$

When $J \subset I$ are non-empty subsets of $\mathbb{N}$, we denote by $r_{I,J}$ the measurable restriction map from E_I to E_J:

$$r_{I,J}(\Gamma) = \Gamma \cap (J \times J). \tag{1.7}$$

When $I = \mathbb{N}$, we simply write r_J in place of $r_{I,J}$.

Proposition 1.1.

$$R^I_u, u \geq 0, \textit{ is a Feller semigroup on } E_I. \tag{1.8}$$

For $J \subset I$ non-empty subsets of $\mathbb{N}$, $u \geq 0$, $\Gamma \in E_I$, one has the compatibility relation

$$R^J_u(r_{I,J}(\Gamma), \cdot) \ \textit{ is the image of } R^I_u(\Gamma, \cdot) \textit{ under } r_{I,J}. \tag{1.9}$$

When $C_1, \dots, C_k$, are $k \geq 2$ distinct Γ-equivalence classes on I:

$$\begin{aligned} &R^I_u\,(\Gamma, \{C_1, \dots, C_k \textit{ are in the same } \Gamma'\textit{-class}\}) = \\ &\frac{(k-1-e^{-u})(k-2-e^{-u})\dots(1-e^{-u})}{(k-1)!}, \ \ \textit{for } u > 0. \end{aligned} \tag{1.10}$$

Proof. The compatibility relation (1.9) is a direct consequence of the definition of R^I_u and R^J_u. Let us now prove that R^I_u, $u \geq 0$, are Feller semigroups. Observe that R^I_u, for $u \geq 0$, preserves the space of continuous functions on E_I. Indeed, in view of Stone–Weierstrass' theorem it suffices to prove the continuity of the map

$$\Gamma \in E_I \to R^I_u(\Gamma, A) \in [0,1], \ \ \text{for} \ \ u \geq 0, \tag{1.11}$$

when A has the form

$$A = \bigcap_1^n \{\Gamma' \in E_I; (a_i, b_i) \in \Gamma'\}, \tag{1.12}$$

with $a_i, b_i \in I$, for $i = 1, \dots, n$. For such an A, we can apply (1.9) with $J = \{a_i, b_i, i = 1, \dots, n\} \subseteq I$. We are therefore reduced to the case of a finite set I, where the continuity of the map in (1.11) is obvious. As a consequence of (1.10), with $k = 2$, which is proven below, R^I_u tends to the identity when I is finite and u tends to 0. By a similar argument as above, it follows that for arbitrary I and f continuous on E_I, $R^I_u\, f$ tends uniformly to f as u tends to 0.

We now come to the proof of the semigroup property. For notational simplicity, we assume $I = \mathbb{N}$, although this plays no role in the proof. Given $\Gamma \in E$, $u, v > 0$, we can construct the law $R_u\, R_v(\Gamma, \cdot)$ on E, as follows. We consider on some auxiliary space $(\Omega, \mathcal{A}, P)$, $(m_1, (y_C, C \in \mathcal{C}_\Gamma))$ independent of $(m_2, (Z_\ell, \ell \geq 0))$ such that m_1 is $\mathbb{P}_{x_1}$-distributed with $x_1 = e^{-u}$, conditionally on m_1, the variables $y_C, C \in \mathcal{C}_\Gamma$, are i.i.d. $\sum_{\ell \geq 0} \frac{\eta_\ell(m_1)}{|m_1|}\, \delta_\ell$-distributed; m_2 is $\mathbb{P}_{x_2}$-distributed with $x_2 = e^{-v}$, conditionally on m_2, the variables $Z_\ell, \ell \geq 0$, are i.i.d. $\sum_{\ell' \geq 0} \frac{\eta_{\ell'}(m_2)}{|m_2|}\, \delta_{\ell'}$-distributed.

We can define variable y'_C, for $C \in \mathcal{C}_\Gamma$, via:

$$y'_C = Z_{y_C}. \tag{1.13}$$

The formula:

$$C''_{\ell'} = \bigcup_{C \in \mathcal{C}_\Gamma : y'_C = \ell'} C\,, \ \ \text{for} \ \ \ell' \geq 0, \tag{1.14}$$

defines a partition on $\mathbb{N}$, which naturally determines an equivalence relation Γ'', which is precisely $R_u\, R_v(\Gamma, \cdot)$-distributed.

We shall now construct a suitable random permutation τ of $\mathbb{N}$ such that the variables $\tau(y'_C)$, under P have the same joint distribution as the variables y_C, under Q_x in (1.4), with $x = e^{-(u+v)}$. This will complete the proof of the semigroup property. Conditionally on $m_1, m_2, Z_\ell, \ell \geq 0$, the variables $y'_C, C \in \mathcal{C}_\Gamma$, are independent with common distribution:

$$P[y'_C = \ell' | m_1, m_2, (Z_\ell)_{\ell \geq 0}] = \sum_{\ell : Z_\ell = \ell'} \eta^1_\ell / \sum_{\ell \geq 0} \eta^1_\ell, \tag{1.15}$$

where we write for simplicity $\eta_\ell(m_1) = \eta^1_\ell$ and $\eta_{\ell'}(m_2) = \eta^2_{\ell'}$. Taking a closer look at the numerator of the fraction in (1.15), we observe that conditional on m_2:

$$\sum_{\ell \geq 0} 1\{Z_\ell = \ell'\}\, \delta_{\eta^1_\ell}\,,\ \ell' \geq 0, \tag{1.16}$$

are independent Poisson point processes on $(0, \infty)$ with respective intensities $(\eta^2_{\ell'} / \sum_{k \geq 0} \eta^2_k)\, x_1\, \eta^{-x_1 - 1} d\eta$. Using scaling (i.e. (A.7) with a constant function g), we see that conditional on m_2:

$$m^{\ell'} \stackrel{\text{def}}{=} \sum_{\ell \geq 0} 1\{Z_\ell = \ell'\}\, \delta_{C^{-1}_{\ell'} \eta^1_\ell}\,,\ \ \ell' \geq 0, \text{ with} \tag{1.17}$$

$$C_{\ell'} = \Big(\eta^2_{\ell'} / \sum_{k \geq 0} \eta^2_k\Big)^{\frac{1}{x_1}}, \tag{1.18}$$

are i.i.d. $\mathbb{P}_{x_1}$-distributed. Coming back to (1.15), P-a.s.

$$\begin{aligned} &P[y'_C = \ell' | m_1, m_2, (Z_\ell)_{\ell \geq 0}] = C_{\ell'} |m^{\ell'}| / \Big(\sum_k C_k |m^k|\Big) \\ &= (\eta^2_{\ell'})^{\frac{1}{x_1}}\, |m^{\ell'}| / \Big(\sum_{k \geq 0} (\eta^2_k)^{\frac{1}{x_1}} |m^k|\Big),\ \ell' \geq 0. \end{aligned} \tag{1.19}$$

From (A.2), we know that $\sum_{\ell' \geq 0} \delta_{(\eta^2_{\ell'})^{\frac{1}{x_1}}}$ is $\mathbb{P}_{x_1 x_2}$-distributed. Since the $m^{\ell'}, \ell' \geq 0$, are i.i.d. independent from the $(\eta^2_{\ell'})^{\frac{1}{x_1}}, \ell' \geq 0$, and $E[|m^{\ell'}|^{x_1 x_2}] < \infty$, by (A.3), it follows from (A.7) that

$$m_3 \stackrel{\text{def}}{=} \mathcal{N}\Big(\sum_{\ell' \geq 0} \delta_{(\eta^2_{\ell'})^{\frac{1}{x_1}} |m^{\ell'}|}\Big) \text{ is } \overline{\mathbb{P}}_{x_1 x_2}\text{-distributed.} \tag{1.20}$$

The formula

$$\tau(\ell') = j, \quad \text{on the set } \Big\{\eta_j(m_3) = (\eta^2_{\ell'})^{\frac{1}{x_1}} |m^{\ell'}| / \sum_{k \geq 0} (\eta^2_k)^{\frac{1}{x_1}} |m^k|\Big\}; \tag{1.21}$$

$\mathbb{P}$-a.s. defines a $\sigma(m_1, m_2, Z_\ell, \ell \geq 0)$-measurable permutation of $\mathbb{N}$. We can thus consider the variables

$$\widetilde{y}_C = \tau(y'_C), \quad \text{for } C \in \mathcal{C}_\Gamma. \tag{1.22}$$

Considering (1.15), (1.19), we see that conditional on $m_1, m_2, Z_\ell, \ell \geq 0$, the $\widetilde{y}_C, C \in \mathcal{C}_\Gamma$, are i.i.d. with common distribution:

$$P[\widetilde{y}_C = j\, | m_1, m_2, Z_\ell, \ell \geq 0] = \eta_j(m_3),\ j \geq 0.$$

This conditional distribution only depends on m_3. Thus conditional on m_3, $\widetilde{y}_C, C \in \mathcal{C}_\Gamma$, are i.i.d., $\sum_{j \geq 0} \eta_j(m_3)\, \delta_j$-distributed. Since the collection $\bigcup_{C \in \mathcal{C}_\Gamma, \widetilde{y}_C = j} C,\ j \geq 0$, defines up to relabelling the same partition of $\mathbb{N}$, as (1.14), and m_3 is $\overline{\mathbb{P}}_{x_1 x_2}$-distributed, we have proved that Γ'' is $R_{u+v}(\Gamma, \cdot)$-distributed.

Let us finally prove (1.10). The left member of (1.10) equals $E^{\overline{\mathbb{P}}_x}[\sum_{\ell \geq 0} \eta^k_\ell] \stackrel{(A.4)}{=} \frac{1}{(k-1)!}\, (k - 1 - x) \dots (1 - x)$, with $x = e^{-u}$. This concludes the proof of Proposition 1.1. □

The canonical space for the "clustering process" will be

$$T: \text{the set of non-decreasing right continuous } E\text{-valued, } \Gamma_u, u \geq 0. \tag{1.23}$$

Observe that E_I is finite when I is finite. Thus the right continuous non-decreasing function $r_I(\Gamma_u), u \geq 0$, is a step function, which only takes finitely many values.

We endow T with the canonical σ-field $\mathcal{T}$ generated by the canonical E-valued coordinates, and with the filtration:

$$\mathcal{T}_u = \sigma(\Gamma_v, 0 \leq v \leq u),\ u \geq 0. \tag{1.24}$$

Theorem 1.2 (The Clustering Process). *There is a unique collection of probabilities P_Γ on $(T, \mathcal{T})$, $\Gamma \in E$, such that $(T, \mathcal{T}, (\Gamma_u)_{u\geq 0}, (\mathcal{T}_u)_{u \geq 0}, P_\Gamma)$ is a Markov process with semigroup $R_u, u \geq 0$.*

Proof. Uniqueness is obvious, we shall thus only explain the construction of the P_Γ probabilities. As shown in Proposition 1.1, R_u^I, $u \geq 0$, is a strongly continuous semigroup on the finite state space E_I, when $I \neq \emptyset$ is finite. Thus for a given $\Gamma \in E$, with the help of the compatibility relation (1.9), we can construct on some auxiliary probability space a sequence $\Gamma_u^n, u \geq 0$, of right continuous E_{I_n}-valued processes when $I_n = [0, n]$, such that:

$$\Gamma_u^n,\ u \geq 0, \text{ is a Markov process with semigroup } R_u^{I_n}, u \geq 0, \text{ and } \Gamma_0^n = r_{I_n}(\Gamma), \tag{1.25}$$

$$r_{I_m, I_n}(\Gamma_u^m) = \Gamma_u^n, \quad \text{for} \quad m \geq n,\ u \geq 0. \tag{1.26}$$

We simply define $\Gamma_u^\infty = \bigcup_{n \geq 0} \Gamma_u^n$, for $u \geq 0$. It is straightforward using (1.9) to see that $\Gamma_u^\infty, u \geq 0$, is a Markov process with semigroup R_u. Moreover $\Gamma_\cdot^\infty$ is a $(T, \mathcal{C})$ valued random variable, and we define P_Γ to be its law. $\square$

As already mentioned in (0.11), it is convenient to introduce on T the variables:

$$\tau_{\ell,\ell'} = \inf\{u \geq 0, (\ell, \ell') \in \Gamma_u\},\ \ell, \ell' \geq 0. \tag{1.27}$$

It is immediate to check that for ℓ, ℓ', ℓ'':

$$\tau_{\ell,\ell} = 0,\ \tau_{\ell,\ell'} = \tau_{\ell',\ell},\ \tau_{\ell,\ell''} \leq \max\{\tau_{\ell,\ell'}, \tau_{\ell',\ell''}\}. \tag{1.28}$$

The variables $\tau_{\ell,\ell'}$ are P_Γ-a.s. finite for any $\Gamma \in E$, since either $(\ell, \ell') \in \Gamma$, in which case $\tau_{\ell,\ell'} = 0$, P_Γ-a.s. or from (1.10)

$$P_\Gamma(\tau_{\ell,\ell'} > u) = e^{-u},\ u \geq 0, \quad \text{when} \quad (\ell, \ell') \notin \Gamma, \tag{1.29}$$

i.e. $\tau_{\ell,\ell'}$ is a standard exponential variable. Observe also that $\Gamma_u, u \geq 0$, is a measurable function of the variables $\tau_{\ell,\ell'}, \ell, \ell' \geq 0$, since

$$\Gamma_u = \{(\ell, \ell') \in \mathbb{N} \times \mathbb{N},\ \tau_{\ell,\ell'} \geq u\}. \tag{1.30}$$

When Γ is the equality relation on $\mathbb{N}$, we shall simply write P in place of P_Γ. Note that

$$P\text{-a.s. } \tau_{\ell,\ell'}, \ell, \ell' \geq 0, \text{ defines an ultrametric distance on } \mathbb{N}. \tag{1.31}$$

We shall now close this section with a description of the pure jump process associated to the semigroups R_u^I, $u \geq 0$, for finite I. Although not explicitly needed for the sequel, this will provide further insights in the structure of the clustering process. We denote by L^I the generator of the semigroup R^I_u, $u \geq 0$, so that for f a function on E_I:

$$L^I f(\Gamma) = \sum_{\Gamma' \in E_I} a^I_{\Gamma,\Gamma'} \, f(\Gamma'), \quad \Gamma \in E_I. \tag{1.32}$$

Proposition 1.3 (I finite). *Let $N \geq 1$ denote the number of distinct equivalence classes of $\Gamma \in E_I$. If $N = 1$ all $a^I_{\Gamma,\Gamma'} = 0$ (trivially), if $N \geq 2$, $a^I_{\Gamma,\Gamma'} = 0$ unless Γ' is obtained by lumping together $k \geq 2$ distinct classes of Γ, in which case:*

$$a^I_{\Gamma,\Gamma'} = \frac{1}{(N-1)\binom{N-2}{k-2}}, \tag{1.33}$$

or $\Gamma' = \Gamma$, in which case:

$$a^I_{\Gamma,\Gamma'} = 1 - N. \tag{1.34}$$

Proof. One can compute the intensity of the second moments of the point processes $\overline{\mathbb{P}}_x$, by the same technique as in Proposition 2.1 of [10], see also [6] p. 55, and see that

$$E^{\overline{\mathbb{P}}_x}\Big[\sum_{\ell \neq \ell'} \eta^2_\ell \, \eta^2_{\ell'}\Big] = o(u), \text{ as } u \to 0, \text{ with } x = e^{-u}.$$

As a result $a^I_{\Gamma,\Gamma'}$ vanishes unless Γ' is obtained by lumping together one subcollection of $\mathcal{C}_\Gamma$. From (1.10), we deduce that when $\Gamma \in E_I$ and $C_1, \dots, C_k$ are $k \geq 2$ distinct equivalence classes of Γ:

$$L^I(1_{\{C_1,\dots,C_k \text{ are lumped together}\}})(\Gamma) = \frac{1}{k-1}. \tag{1.35}$$

It follows from an "inclusion exclusion" argument that:

$$\begin{aligned}
&L^I(1_{\{C_1,\dots,C_k \text{ form an equivalence class}\}})(\Gamma) = \sum_{p=0}^{N-k} \frac{1}{k-1+p} \, (-1)^p \binom{N-k}{p} \\
&= \int_0^1 \sum_{p=0}^{N-k} \binom{N-k}{p} \, (-1)^p \, t^{k-2+p} dt = \int_0^1 t^{k-2}(1-t)^{N-k} dt \\
&= \frac{\Gamma(k-1)\,\Gamma(N-k+1)}{\Gamma(N)} = \Big[(N-1)\binom{N-2}{k-2}\Big]^{-1}.
\end{aligned}$$

This proves (1.33). As for (1.34), it follows immediately from the identity $a^I_{\Gamma,\Gamma'} = -\sum_{\Gamma' \neq \Gamma} a^I_{\Gamma,\Gamma'}$. $\square$

The pure jump process attached to $R^I_u, u \geq 0$, when I is finite, is now easy to describe. It has a finite number of jump times: $0 < \tau_1 < \tau_1 + \tau_2 < \cdots < \tau_1 + \cdots + \tau_\lambda < \infty$. If the initial condition Γ has $N \geq 2$ classes, then τ_1 is exponentially distributed with expectation $(N-1)^{-1}$. At time τ_1, the process jumps to an equivalence relation $\widetilde{\Gamma}_1$ by collapsing x_1 classes of Γ, where the distribution of x_1 is

$$P[x_1 = k] = \frac{N}{N-1} \; \frac{1}{k(k-1)}, \quad 2 \leq k \leq N. \tag{1.36}$$

Conditionally on $\{x_1 = k\}$, $\widetilde{\Gamma}_1$ is chosen uniformly among the $\binom{N}{k}$ possibilities. After that, τ_2 is chosen with $\widetilde{\Gamma}_1$ as the new starting element, etc. After a finite number of jumps, the final state with one class is reached. It should also be remarked that the integer valued process which counts the number of equivalence relations is Markovian as well, with downwards jumps and transition kernel essentially described by (1.36).

There is in fact a simple explicit expression for the semigroup R_u^I, which we now provide.

Proposition 1.4 (I finite). *If Γ has $N \geq 2$ classes, and Γ' is obtained by respective clumpings of $m_1, m_2, \dots, m_k \geq 1$, classes of Γ, with $\sum_j m_j = N$, then*

$$R_u^I(\Gamma, \Gamma') = \frac{(k-1)!}{(N-1)!}\, e^{-(k-1)u} \prod_{i=1}^{k} g_{m_i}(u), \tag{1.37}$$

where $g_1(u) = 1$, and for $m \geq 2$,

$$g_m(u) = (m-1-e^{-u})(m-2-e^{-u})\dots(1-e^{-u}). \tag{1.38}$$

Proof. It is convenient to set $x = e^{-u}$, $u \geq 0$, and $f_x(s) = s^x$, for $s > 0$. If $f_x^{(m)}$ denotes the m^{th} derivative of f with respect to s, then the right-hand side of (1.37) equals

$$(-1)^{N-k}\, \frac{(k-1)!}{(N-1)!}\, \frac{1}{x} \prod_{j=1}^{k} f_x^{(m_j)}(1) \overset{\text{def}}{=} \widetilde{R}_u^I(\Gamma, \Gamma'). \tag{1.39}$$

This will be helpful in order to check the backward equation

$$\frac{d}{du}\, \widetilde{R}_u^I = L^I\, \widetilde{R}_u^I\,,\ u \geq 0. \tag{1.40}$$

Since $\widetilde{R}_0^I$ is obviously the identity matrix, our claim (1.37) will follow. Observe that for $m \geq 1$:

$$\begin{aligned}\frac{\partial}{\partial u}\, f_x^{(m)}(1) &= \frac{\partial^m}{\partial s^m}\, (-x(\log s)\, f_x(s))\big|_{s=1} \\ &= x \sum_{j=1}^{m} (-1)^j \binom{m}{j} (j-1)!\, f_x^{(m-j)}(1).\end{aligned}$$

Using the identity $x\, f_x^{(\ell)}(1) = f_x^{(\ell+1)}(1) + \ell\, f_x^{(\ell)}(1)$, we get

$$\begin{aligned}\frac{\partial}{\partial u}\, f_x^{(m)}(1) &= \sum_{j=1}^{m} (-1)^j \binom{m}{j} (j-1)!\, [f_x^{(m-j+1)}(1) + (m-j)\, f_x^{(m-j)}(1)] \\ &= m! \sum_{j=2}^{m} (-1)^{j-1}\, \frac{1}{j(j-1)(m-j)!}\, f_x^{(m-j+1)}(1) - m\, f_x^{(m)}(1),\end{aligned} \tag{1.41}$$

after regrouping, and the above sum over j is 0 if $m = 1$. We use this expression to differentiate $\widetilde{R}_u^I(\Gamma, \Gamma')$ with respect to u. We find:

$$
\begin{aligned}
\frac{\partial}{\partial u}\, \widetilde{R}^I_u(\Gamma,\Gamma') = (-N+1)\, \widetilde{R}^I_u(\Gamma,\Gamma') \\
&+ (-1)^{N-k}\, \frac{(k-1)!}{(N-1)!} \sum_{i:m_i\geq 2} \sum_{\ell=2}^{m_i} (-1)^{\ell-1} \\
&\frac{m_i!}{(m_i-\ell)!\,\ell(\ell-1)}\; \frac{1}{x}\; f_x^{(m_i-\ell+1)}(1) \prod_{j\neq i} f_x^{(m_j)}(1) \\
&= (-N+1)\, \widetilde{R}^I_u(\Gamma,\Gamma) + \sum_{i:m_i\geq 2} \sum_{\ell=2}^{m_i} a_{N,\ell} \binom{m_i}{\ell} (-1)^{N-k-\ell+1} \\
&\frac{(k-1)!}{(N-\ell)!}\; \frac{1}{x}\; f_x^{(m_i-\ell+1)}(1) \prod_{j\neq i} f_x^{(m_j)}(1) \\
&= \sum_{\Gamma'':\Gamma\preceq\Gamma''\preceq\Gamma'} a^I_{\Gamma,\Gamma''}\, \widetilde{R}^I_u(\Gamma'',\Gamma'),
\end{aligned}
\tag{1.42}
$$

where we have used the notation $a_{N,\ell} = [(N-1)\binom{N-2}{\ell-2}]^{-1}$. This finishes the proof of (1.37). $\square$

2. Clustering Process and Ruelle's Probability Cascades

We shall present in this section the precise connection between the clustering process constructed in the previous section and Ruelle's cascades as defined in [8]. We consider a sequence

$$0 < x_1 < x_2 < \cdots < x_K < 1,\ K \geq 1,\ \text{ as well as} \tag{2.1}$$

$$u_1 = \log\left(\frac{x_K}{x_1}\right) > u_2 = \log\left(\frac{x_K}{x_2}\right) > \cdots > u_K = \log\left(\frac{x_K}{x_K}\right) = 0. \tag{2.2}$$

The main object of this section is to give an alternative description of the law on $M_1 \times E^K$ of $(m, \Gamma_{u_K}, \Gamma_{u_{K-1}}, \ldots, \Gamma_{u_1})$ under $\overline{\mathbb{P}}_{x_k} \times P$. We first introduce some notations.

We denote by $\mathcal{I}_k$, for $k \geq 1$, the set $\mathbb{N}^k$ of multi-indices $i = (i_1, \ldots, i_k)$ of length k. As a convention we also define $\mathcal{I}_0 = \{\emptyset\}$. If $i \in \mathcal{I}_k$, $i' \in \mathcal{I}_{k'}$, with $k, k' \geq 0$, $i.i'$, denotes the concatenation of i and i'. Furthermore, when $k \leq k'$, $i \preceq i'$ means that i' extends i, whereas $[i']_k$ stands for the truncation to order k of i'. We now introduce an auxiliary probability space $(S, \mathcal{S}, Q)$, endowed with a family of $(0,\infty)$-valued random variables, $\eta^k_i, i \in \mathcal{I}_k$, $1 \leq k \leq K$, satisfying (0.6). For $i \in \mathcal{I}_k$, $1 \leq k \leq K$, it is convenient to introduce the generalization of (0.7):

$$\pi_i = \eta^1_{[i]_1} \cdot \eta^2_{[i]_2} \ldots \eta^k_{[i]_k} \quad \text{(where of course } [i]_k = i\text{)}. \tag{2.3}$$

The following lemma is actually part of the results proved in Sect. 3 of Ruelle [6]. Nevertheless its simple proof is included for the reader's convenience.

Lemma 2.1.

$$\omega = \sum_{i\in\mathcal{I}_K} \delta_{\pi_i} \textit{ is } Q\textit{-a.s. } M\textit{-valued and} \tag{2.4}$$

$$\overline{\omega} = \mathcal{N}(\omega) \textit{ is } \overline{\mathbb{P}}_{x_K}\textit{-distributed (see (1.3) for the notation).} \tag{2.5}$$

Proof. We use induction on K. When $K = 1$, (2.4), (2.5) are immediate. Consider $K > 1$. Conditional on $\eta^1_\cdot, \eta^2_\cdot, \dots, \eta^{K-1}_\cdot$,

$$\sum_{j\geq 0} \delta_{\pi_i\, \eta^K_{i\cdot j}}, \quad \text{for } i \in \mathcal{I}_{K-1}, \tag{2.6}$$

are independent Poisson point processes on $(0,\infty)$, with respective intensities $\pi_i^{x_K} x_K\, \eta^{-x_K-1} d\eta$. It also follows from (A.2) that the collection of variables

$$\widetilde{\eta}^k_i = (\eta^k_i)^{x_K},\ i \in \mathcal{I}_k,\ k \in [1, K-1], \tag{2.7}$$

satisfy (0.6) relative to the sequence:

$$0 < \widetilde{x}_1 = \frac{x_1}{x_K} < \widetilde{x}_2 = \frac{x_2}{x_K} < \cdots < \widetilde{x}_{K-1} = \frac{x_{K-1}}{x_K}. \tag{2.8}$$

Defining $\widetilde{\pi}_i$ analogously to (2.3) in terms of the $\widetilde{\eta}$ variables, the induction hypothesis implies that

$$C \stackrel{\text{def}}{=} \sum_{\mathcal{I}_{K-1}} \pi_i^{x_K} < \infty, \quad Q\text{–a.s..} \tag{2.9}$$

Coming back to (2.6), we see that conditional on $\eta^1_\cdot, \dots, \eta^{K-1}_\cdot$, the variable ω is distributed as a Poisson point process on $(0,\infty)$ with intensity $C\, x_K\, \eta^{-x_K-1}\, d\eta$. Using the scaling relation (A.7) (when g is constant), we see that

$$\widetilde{\omega} = \sum_{\mathcal{I}_K} \delta_{C^{-\frac{1}{x_K}}\,\pi_i} \tag{2.10}$$

is $\mathbb{P}_{x_K}$-distributed and independent of $\eta^1_\cdot, \dots, \eta^{K-1}_\cdot$. It now follows that ω is Q-a.s. M-valued and $\mathcal{N}(\omega) = \mathcal{N}(\widetilde{\omega})$ is $\overline{\mathbb{P}}_{x_K}$-distributed. This concludes the proof of the induction step. □

With the help of Lemma 2.1, we introduce on a set of full Q-probability a measurable bijection $i(\cdot) : \mathbb{N} \to \mathcal{I}_K$, such that:

$$\pi_{i(\ell)} = \eta_\ell(\omega), \quad \text{for } \ell \geq 0. \tag{2.11}$$

In other words $\pi_{i(\ell)}$ has rank ℓ among the collection $\pi_i, i \in \mathcal{I}_K$. Furthermore, we consider a decreasing sequence of E-valued variables:

$$\overline{\Gamma}_k = \{(\ell,\ell') \in \mathbb{N};\ [i(\ell)]_k = [i(\ell')]_k\}, \quad \text{so that } \overline{\Gamma}_1 \supseteq \overline{\Gamma}_2 \supseteq \cdots \supseteq \overline{\Gamma}_K. \tag{2.12}$$

The connection between the clustering process and Ruelle's cascades comes in the following

Theorem 2.2.

$$\begin{aligned}&(\mathcal{N}(m),\ \Gamma_{u_K}, \dots, \Gamma_{u_1}) \textit{ has the same distribution under}\\ &\mathbb{P}_{x_K} \times P \textit{ as } (\overline{\omega}, \overline{\Gamma}_K, \dots, \overline{\Gamma}_1) \textit{ under } Q.\end{aligned} \tag{2.13}$$

Proof. With the help of (2.5) and the fact that Γ_{u_K} and $\overline{\Gamma}_K$ both almost surely coincide with the equality relation on $\mathbb{N}$, we see that $(\mathcal{N}(m), \Gamma_{u_K})$ and $(\varpi, \overline{\Gamma}_K)$ have the same law. In view of the Markov property asserted in Theorem 1.2, the claim (2.13) will follow when we show that

$$\begin{aligned}&R_{(u_r-u_{r+1})}(\overline{\Gamma}_{r+1},\,\cdot)\text{ is a version of the conditional}\\&\text{law of }\overline{\Gamma}_r\text{ given }(\varpi,\overline{\Gamma}_K,\dots,\overline{\Gamma}_{r+1}),\text{ for }r\in[1,K-1].\end{aligned}\tag{2.14}$$

The argument used in the proof of Lemma 2.1 shows that

$$C \overset{\text{def}}{=} \sum_{\mathcal{I}_r} \pi_i^{x_{r+1}} < \infty\,, \quad Q\text{–a.s.,} \tag{2.15}$$

and conditional to $\eta^1_\cdot, \dots, \eta^r_\cdot$,

$$\omega_i = \sum_{i'\in\mathcal{I}_{r+1}: i'\succeq i} \delta_{C^{-\frac{1}{x_{r+1}}}\pi_{i'}}\,, \quad \text{for } \; i \in \mathcal{I}_r \tag{2.16}$$

are independent Poisson point processes on $(0,\infty)$ with respective intensities $C^{-1}\,\pi_i^{x_{r+1}}$ $x_{r+1}\,\eta^{-x_{r+1}-1}d\eta$. If we now define the point process μ on $(0,\infty)\times\mathcal{I}_r$:

$$\mu = \sum_{i'\in\mathcal{I}_{r+1}} \delta_{(C^{-\frac{1}{x_{r+1}}}\pi_{i'},[i']_r)}, \tag{2.17}$$

we see that conditionally on $\eta^1_\cdot, \dots, \eta^r_\cdot$

$$\begin{aligned}&\mu \text{ is a Poisson point process with intensity}\\&x_{r+1}\,\eta^{-x_{r+1}-1}\,d\eta\otimes\sum_{\mathcal{I}_r}\frac{\pi_i^{x_{r+1}}}{C}\,\delta_i\,, \quad\text{and}\end{aligned}\tag{2.18}$$

$$\omega' \overset{\text{def}}{=} \sum_{i'\in\mathcal{I}_{r+1}} \delta_{C^{-\frac{1}{x_{r+1}}}\pi_{i'}} \quad \text{is independent of } \; \eta^1_\cdot, \dots, \eta^r_\cdot,\; \mathbb{P}_{x_{r+1}}\text{–distributed.} \tag{2.19}$$

Analogously to (2.11), we introduce a Q-a.s. defined measurable bijection $i_{r+1}(\cdot): \mathbb{N} \to \mathcal{I}_{r+1}$, such that:

$$\eta_j(\omega') = C^{-\frac{1}{x_{r+1}}}\pi_{i_{r+1}(j)}, \quad \text{for } \; j \geq 0. \tag{2.20}$$

We further introduce variables $\overline{\eta}^k_i$, $k \in [1, K-r]$, $i \in \mathcal{I}_k$:

$$\overline{\eta}^1_j = \eta_j(\omega'), \quad \text{for } \; j \in \mathbb{N} = \mathcal{I}_1, \tag{2.21}$$

$$\overline{\eta}^k_{(j_1,\dots,j_k)} = \eta^{r+k}_{i_{r+1}(j_1)\cdot(j_2,\dots,j_k)}, \quad \text{for } \; k \in [2, K-r], (j_1,\dots,j_k) \in \mathcal{I}_k. \tag{2.22}$$

The variables $\overline{\eta}^2_\cdot, \dots, \overline{\eta}^{K-r}_\cdot$, are independent of $\eta^1_\cdot, \dots, \eta^r_\cdot, \mu$, and coming back to (2.17), conditionally on $\eta^1_\cdot, \dots, \eta^r_\cdot, \overline{\eta}^1_\cdot, \dots, \overline{\eta}^{K-r}$,

$$\begin{aligned}&\text{the } \mathcal{I}_r\text{-valued variables } [i_{r+1}(j)]_r,\ j \geq 0, \text{ are i.i.d.,}\\&\text{with common law } \sum_{\mathcal{I}_r}\frac{\pi_i^{x_{r+1}}}{C}\delta_i.\end{aligned}\tag{2.23}$$

It is plain that the variables $\overline{\eta}$ satisfy (0.6) relative to the sequence:

$$\overline{x}_1 = x_{r+1} < \cdots < \overline{x}_{K-r} = x_K.$$

Defining $\overline{\pi}_i$, for $i \in \mathcal{I}_k$, $k \in [\underline{1}, K-r]$, in analogy with (2.3), we can introduce a Q-a.s. defined measurable bijection $\overline{i} : \mathbb{N} \to \mathcal{I}_{K-r}$, such that:

$$\overline{\pi}_{\overline{i}(\ell)} = \eta_\ell\Big(\sum_{\mathcal{I}_{K-r}} \delta_{\overline{\pi}_i}\Big) = C^{-\frac{1}{x_{r+1}}}\, \eta_\ell(\omega), \quad \text{for} \quad \ell \geq 0. \tag{2.24}$$

We now find that Q-a.s., for $k \in [r+1, K]$:

$$\overline{\Gamma}_k = \{(\ell, \ell') \in \mathbb{N} \times \mathbb{N}; [\overline{i}(\ell)]_{k-r} = [\overline{i}(\ell')]_{k-r}\}, \tag{2.25}$$

whereas for $k = r$,

$$\overline{\Gamma}_r = \{(\ell, \ell') \in \mathbb{N} \times \mathbb{N};\ [i_{r+1}([\overline{i}(\ell)]_1)]_r = [i_{r+1}([\overline{i}(\ell)]_1)]_r\}. \tag{2.26}$$

In other words the $\overline{\Gamma}_r$-equivalence classes are the

$$\begin{aligned} C_i^r \quad &= \bigcup_{j:[i_{r+1}(j)]_r = \underline{i}} C_j^{r+1}, \quad \text{for} \quad i \in \mathcal{I}_r, \quad \text{where} \\ C_j^{r+1} &= \{\ell \in \mathbb{N};\ [\overline{i}(\ell)]_1 = j\}, \quad \text{for} \quad j \geq 0, \end{aligned} \tag{2.27}$$

are the various $\overline{\Gamma}_{r+1}$-equivalence classes. Observe that (2.25) expresses the $\overline{\Gamma}_k$, $k \in [r+1, K]$, in terms of the $\overline{\eta}^{k-r}$, $k \in [r+1, K]$, and that $\overline{\omega} = \mathcal{N}(\sum_{\mathcal{I}_{K-r}} \delta_{\overline{\pi}_i})$. Furthermore, it follows from (A.2) and (2.9) (when $K = r$), that

$$\mathcal{N}\Big(\sum_{\mathcal{I}_r} \delta_{\pi_i^{x_{r+1}}}\Big) \ \text{is}\ \overline{\mathbb{P}}_{\frac{x_r}{x_{r+1}}}\text{-distributed}, \quad \Big(\text{with}\ \frac{x_r}{x_{r+1}} = e^{-(u_r - u_{r+1})}\Big). \tag{2.28}$$

If we now recall (2.27) and (2.23), it is now routine to deduce (2.14). This concludes our proof of (2.13). $\square$

As an application of Theorem 2.2, we can consider the process of "mass coagulation", $\overline{m}_u$, $u \geq 0$, defined on $M_1 \times T$, as the random pure point measure on $(0, \infty)$:

$$\overline{m}_u = \sum_{C:\Gamma_u\text{-equivalence classes}} \delta_{\sum_{\ell \in C} \eta_\ell(m)}. \tag{2.29}$$

Under $\overline{\mathbb{P}}_x \otimes P$, for $x \in (0,1)$, its law is in fact concentrated on M_1 as follows from the

Corollary 2.3.

$$\textit{Under}\ \overline{\mathbb{P}}_x \otimes P,\ \overline{m}_u\ \textit{is}\ \overline{\mathbb{P}}_{xe^{-u}}\textit{-distributed}. \tag{2.30}$$

Proof. We choose $K = 2$, $x_1 = xe^{-u} < x_2 = x$. From Theorem 2.2, $(\mathcal{N}(m), \Gamma_0, \Gamma_u)$ has same law under $\mathbb{P}_x \otimes P$ as $(\overline{\omega}, \overline{\Gamma}_2, \overline{\Gamma}_1)$ under Q. It follows that the law of $\overline{m}_u$ is the same as that of

$$\overline{m} = \mathcal{N}\Big(\sum_{j \geq 0} \delta_{\sum_{j' \geq 0} \eta_j^1\, \eta^2_{(j,j')}}\Big). \tag{2.31}$$

As a consequence of (A.7), $\sum \delta_{\eta_j^1 c_j}$ is $\mathbb{P}_{x_1}$-distributed, if

$$c_j = \sum_{j' \geq 0} \eta^2_{(j,j')} / E^{\mathbb{P}_x}[|m|^{x_1}]^{\frac{1}{x_1}},$$

where $E^{\mathbb{P}_x}[|m|^{x_1}] < \infty$, by (A.3). Coming back to (2.32), we see that $\overline{m}$ is $\overline{\mathbb{P}}_{x_1}$-distributed. This proves our claim. $\square$

With the help of Theorem 1.2, it is straightforward to see that $\overline{m}_u$, $u \geq 0$, under $\overline{\mathbb{P}}_x \otimes P$, with $x \in (0,1)$, is a simple Markov process with semigroup:

$$\overline{R}_u(m,\cdot) = \text{law of } \overline{m}_u \text{ under } \delta_m \otimes P. \tag{2.32}$$

3. Approximate Reshuffling

The main goal of this section is to prepare the ground for the next section, where we shall derive the effect of the true reshuffling operation. We first need to introduce some notations. We suppose we are given $x_M \in (0,1)$, $q_M \in (0,1]$, and a non-decreasing function $q(\cdot)$ such that:

$$\begin{aligned}&q(\cdot) \text{ is a } C^1\text{-diffeomorphism between } [0,x_M] \text{ and } [0,q_M],\ q(x) = q_M,\\ &\text{for } \ x \in [x_M,1].\end{aligned} \tag{3.1}$$

We denote by $x(\cdot)$: $[0,q_M] \to [0,x_M]$ the inverse of $q(\cdot)$. We consider the probability space $(\widetilde{\Sigma},\widetilde{\mathcal{B}},\widetilde{\mathbb{Q}})$, where

$$\widetilde{\Sigma} = M \times T \times C_0(\mathbb{R}_+,\mathbb{R})^{\mathbb{N}}, \tag{3.2}$$

$\widetilde{\mathcal{B}}$ is the canonical product σ-field, and under the law $\widetilde{\mathbb{Q}}$, the canonical coordinates m, $(\Gamma_u, u \geq 0)$, $w^\ell(\cdot)$, $\ell \geq 0$, on $\widetilde{\Sigma}$, are independent, respectively $\overline{\mathbb{P}}_{x_M}$, P, and W-distributed, with W the Wiener measure on $C_0(\mathbb{R}_+,\mathbb{R})$.

We also introduce the $[0,x_M]$-valued variables on T, (and $\widetilde{\Sigma}$ as well):

$$X_{\ell,\ell'} = x_M \ \exp\{-\tau_{\ell,\ell'}\},\ \ell',\ell' \geq 0, \text{ (see (1.27) for the notation).} \tag{3.3}$$

In view of (1.30), Γ_u, $u \geq 0$, is a measurable function of the $X_{\ell,\ell'}$, $\ell,\ell' \geq 0$, and

$$X_{\ell,\ell'} \geq \min(X_{\ell,\ell''},X_{\ell'',\ell'}), \quad \text{for } \ \ell,\ell',\ell'' \geq 0. \tag{3.4}$$

We then come to the construction of the conditionally Gaussian stochastic processes announced in (0.14). We define by induction a sequence $Y^\ell(x)$, $x \in [0,x_M]$, $\ell \geq 0$, of stochastic processes with:

$$\begin{aligned}Y^0(x) \quad &= w^0(q(x)), \quad x \in [0,x_M], \quad \text{and for } \ N \geq 0,\\ Y^{N+1}(x) &= Y^L(x), \quad \text{for } \ x \in [0,X_{N+1,L}],\\ &= w^{N+1}(q(x) - q(X_{N+1,L})) + Y^L(X_{N+1,L}), \quad \text{for } \ x \in [X_{N+1,L},x_M],\end{aligned} \tag{3.5}$$

provided $L \in [0,N]$ is any integer such that:

$$X_{N+1,L} = \max\{X_{N+1,\ell}\,;\ \ell \in [0,N]\}. \tag{3.6}$$

With the help of (3.4) and an induction argument, one readily checks that (3.5)–(3.6) unambiguously defines $Y^{N+1}(\cdot)$. In fact one has:

$$Y^\ell(x) = Y^{\ell'}(x), \quad \text{for } \ x \in [0,X_{\ell,\ell'}], \tag{3.7}$$

and under $\widetilde{\mathbb{Q}}$, conditional to m, Γ_u, $u \geq 0$, the $Y^\ell(x)$, $x \in [0,x_M]$, $\ell \geq 0$, are centered Gaussian processes with covariance

$$E^{\widetilde{\mathbb{Q}}}[Y^\ell(x)\, Y^{\ell'}(x')\,|m,\Gamma_u,u \geq 0] = q(x \wedge x' \wedge X_{\ell,\ell'}),\ x,x' \in [0,x_M],\ell,\ell' \geq 0. \tag{3.8}$$

We shall now introduce a sequence $Y_n^\ell(\cdot)$, for $n \geq 0$, which approximates the processes $Y^\ell(\cdot)$, as n tends to infinity through a discretization of $[0, x_M]$. For $n \geq 0$, $k \in [0, 2^n]$, we define:

$$x_{k,n} = x\Big(\frac{k}{2^n}\, q_M\Big) \tag{3.9}$$

(see (3.1) for the notation), as well as for $n, \ell, \ell' \geq 0$:

$$X^n_{\ell,\ell'} = \sum_{0 \leq k \leq 2^n} x_{k,n}\, 1\{x_{k,n} \leq X_{\ell,\ell'} < x_{k+1,n}\}, \text{ with the convention } x_{2^n+1,n} = 1. \tag{3.10}$$

The processes $Y_n^\ell(\cdot)$ are defined exactly as in (3.5)–(3.6), except that we replace everywhere in the definition $X_{\cdot,\cdot}$ by $X^n_{\cdot,\cdot}$. Then in analogy to (3.8), conditional to m and Γ_u, $u \geq 0$, the $Y_n^\ell(x)$, $x \in [0, x_M]$, $\ell \geq 0$, are centered Gaussian processes with covariance

$$E^{\widetilde{\mathbb{Q}}}[Y_n^\ell(x)\, Y_n^{\ell'}(x')\,|m, \Gamma_u, u \geq 0] = q(x \wedge x' \wedge X^n_{\ell,\ell'}),\ x, x' \in [0, x_M],\ \ell \geq 0. \tag{3.11}$$

It is plain that when $n \to \infty$:

$$X^n_{\ell,\ell'} \uparrow X_{\ell,\ell'}, \text{ for } \ell, \ell' \geq 0, \tag{3.12}$$

$$Y_n^\ell(\cdot) \text{ converges to } Y^\ell(\cdot) \text{ uniformly on } [0, x_M], \text{ for } \ell \geq 0. \tag{3.13}$$

It is useful in view of the calculations on the effect of the approximate reshuffling operation to give an alternate description of the joint law of m, $(X^n_{\ell,\ell'})_{\ell,\ell' \geq 0}$, $(Y_n^\ell(\cdot))_{\ell \geq 0}$, for n fixed. To this end, we consider the situation of Sect. 2, with $K = 2^n$ and $x_k = x_{k,n}$, for $k \in [1, K]$. We assume that on the auxiliary probability space $(S, \mathcal{S}, Q)$, parallel to the variables η_i^k, $i \in \mathcal{I}_k$, $k \in [1, 2^n]$, we also have mutually independent standard Wiener processes, $z_i^k(\cdot)$, $i \in \mathcal{I}_k$, $k \in [1, 2^n]$, which are also independent of the η_i^k variables. For $i \in \mathcal{I}_k$, $k \in [1, 2^n]$, $x \in [0, x_{k,n}]$, we unambiguously define

$$\overline{Y}_i(x) = \sum_{1 \leq k' < k_0} z^{k'}_{[i]_{k'}}\Big(\frac{q_M}{2^n}\Big) + z^{k_0}_{[i]_{k_0}}\Big(q(x) - \frac{(k_0 - 1)}{2^n}\, q_M\Big), \tag{3.14}$$

where $k_0 \in [1, k]$ is such that $x \in [x_{k_0-1,n}, x_{k_0,n}]$. In the case $i \in \mathcal{I}_K$, (recall K stands for 2^n), $\overline{Y}_i(\cdot)$ is thus a continuous process on $[0, x_M]$. We are now ready for

Lemma 3.1 **($n \geq 0$ is fixed).** *$(m, (X^n_{\ell,\ell'})_{\ell,\ell' \geq 0},\ (Y_n^\ell(\cdot))_{\ell \geq 0})$ has the same law under $\widetilde{\mathbb{Q}}$, as $(\overline{\omega}, (\overline{X}_{\ell,\ell'})_{\ell,\ell' \geq 0},\ (\overline{Y}_{i(\ell)})_{\ell \geq 0})$ under Q, with $i(\cdot)$ defined as in (2.11), and for $\ell, \ell' \geq 0$,*

$$\begin{aligned} &\overline{X}_{\ell,\ell'} = \sup\{x_{k,n};\ k \in [1, K] \text{ such that } [i(\ell)]_k = [i(\ell')]_k\}, \\ &\textit{and the convention } \sup \emptyset = 0. \end{aligned} \tag{3.15}$$

Proof. We shall write x_k instead of $x_{k,n}$, for simplicity. It follows from (3.10) that $\widetilde{\mathbb{Q}}$-a.s., for $\ell, \ell' \geq 0$:

$$X^n_{\ell,\ell'} = \sup\{x_k,\ k \in [1, K] \text{ such that } (\ell, \ell') \in \Gamma_{u_k}\},$$

where u_k is defined as in (2.2), and $\sup \emptyset = 0$, as in (3.15). Applying Theorem 2.2, we thus see that:

$$(m, (X^n_{\ell,\ell'})_{\ell,\ell' \geq 0}) \text{ has same law under } \widetilde{\mathbb{Q}} \text{ as } (\varpi, (\overline{X}_{\ell,\ell'})_{\ell,\ell' \geq 0}) \text{ under } Q. \tag{3.16}$$

Furthermore, conditional on $(m, (X^n_{\ell,\ell'})_{\ell,\ell' \geq 0})$, the processes $Y^\ell_n(\cdot)$ are centered Gaussian processes with covariance as in (3.11). On the other hand, by inspection of (3.14), the processes $\overline{Y}_i(\cdot)$, $i \in \mathcal{I}_K$, are independent of $(\varpi, (\overline{X}_{\ell,\ell'})_{\ell,\ell' \geq 0}, i(\cdot))$, centered Gaussian with covariance

$$\mathrm{cov}(\overline{Y}_i(x), \overline{Y}_{i'}(x')) = \min(q(x), q(x'), \sup\{q(x_k);\ [i]_k = [i']_k\}.$$

It follows that conditional on $(\varpi, (\overline{X}_{\ell,\ell'})_{\ell,\ell' \geq 0}, i(\cdot))$, the processes $\overline{Y}_{i(\ell)}(\cdot)$, $\ell \geq 0$, are centered Gaussian with covariance:

$$\min(q(x), q(x'), q(\overline{X}_{\ell,\ell'})) = q(x \wedge x' \wedge \overline{X}_{\ell,\ell'}), \tag{3.17}$$

which is a measurable function of $(\overline{X}_{\ell,\ell'})_{\ell,\ell' \geq 0}$. This proves that conditional on $(\varpi, (\overline{X}_{\ell,\ell'})_{\ell,\ell' \geq 0})$, the $\overline{Y}_{i(\ell)}(\cdot)$, $\ell \geq 0$, are centered Gaussian processes with covariance as in (3.17). This concludes the proof of Lemma 3.1. □

We shall now define the approximate reshuffling operation. To this end we consider a function

$$\psi(\cdot) : \mathbb{R} \to \mathbb{R}, \quad \text{bounded measurable.} \tag{3.18}$$

The boundedness assumption is here for technical convenience, although it excludes the natural choice $\psi(x) = \log(2 \cosh(\beta x))$ in the context of [6]. For the time being, we keep $n \geq 0$ fixed, and write x_k in place of $x_{k,n}$. We introduce a sequence of functions ψ_k, $k \in [0, 2^n]$, via:

$$\begin{aligned} \psi_{2^n}(\cdot) &= \psi(\cdot), \quad \text{and} \\ \psi_{k-1}(\cdot) &= \frac{1}{x_k} \log P_{\frac{q_M}{2^n}} [e^{x_k \psi_k}](\cdot),\ 1 \leq k \leq 2^n, \end{aligned} \tag{3.19}$$

where P_t, $t \geq 0$, stands for the usual Brownian semigroup:

$$\begin{aligned} P_t\, h(y) &= \int \frac{1}{\sqrt{2\pi t}} \exp\Big\{-\frac{(y'-y)^2}{2t}\Big\}\, h(y') dy', \quad \text{when } t > 0, \\ &= h(y), \text{ for } t = 0,\ y \in \mathbb{R},\ h \text{ bounded measurable.} \end{aligned} \tag{3.20}$$

Functions closely related to the ψ_k appear in Mézard-Virasoro [4], p. 1299. We also introduce a sequence $H_N, N \geq 0$, of random variables on $\widetilde{\Sigma}$:

$$H_0 = \exp\Big\{x_M\, \psi(Y^0_n(x_M)) + \sum_{0 \leq k < 2^n} (x_k - x_{k+1})\, \psi_k(Y^0_n(x_k))\Big\}, \tag{3.21}$$

and for $N \geq 0$,

$$\begin{aligned} H_{N+1} = H_N\ \exp\Big\{ & x_M\, \psi(Y^{N+1}_n(x_M)) - X^n_{N+1,L}\, \psi_{k_0}(Y^{N+1}_n(X^n_{N+1,L})) \\ & + \sum_{k_0 \leq k < 2^n} (x_k - x_{k+1})\, \psi_k(Y^{N+1}_n(x_k))\Big\}, \end{aligned}$$

where $L \in [0, N]$ is such that $X^n_{N+1,L} = \max\{X^n_{N+1,\ell}, 0 \le \ell \le N\}$, and $q(X^n_{N+1,L}) = \frac{k_0}{2^n}$.

The approximate reshuffling operation comes as follows. On a set of full $\widetilde{\mathbb{Q}}$-probability, the sequence

$$\nu_\ell = \eta_\ell(m)\ \exp\{\psi(Y^\ell_n(x_M))\},\ \ell \ge 0, \tag{3.22}$$

has pairwise distinct terms, and is summable. We can thus introduce on a set of full $\widetilde{\mathbb{Q}}$-probability a measurable permutation $\widetilde{\sigma}_n$ of $\mathbb{N}$, with inverse σ_n, such that

$$\widetilde{\ell} = \widetilde{\sigma}_n(\ell) \text{ is the rank of } \nu_\ell \text{ among the } \nu_{\ell'}, \ell' \ge 0. \tag{3.23}$$

The approximate reshuffling corresponds to considering

$$m_{(r)} = \mathcal{N}\Big(\sum_{\ell \ge 0} \delta_{\nu_\ell}\Big),\ X^{(r)}_{\widetilde{\ell},\widetilde{\ell}'} = X^n_{\sigma_n(\widetilde{\ell}),\sigma_n(\widetilde{\ell}')},\ Y^{\widetilde{\ell}}_{(r)}(\cdot) = Y^{\sigma_n(\widetilde{\ell})}_n(\cdot),\ \widetilde{\ell},\widetilde{\ell}' \ge 0, \tag{3.24}$$

in place of m, $X^n_{\ell,\ell'}$, $Y^\ell_n(\cdot)$, $\ell,\ell' \ge 0$, and the dependence on n of the reshuffled objects is omitted from the notation (3.24) for simplicity. Intuitively, one keeps the initial tree structure and marking processes, but relabels according to the new relative importance of the weights ν_ℓ. The next result is crucial for the sequel:

Proposition 3.2. *For $N \ge 0$, $(m_{(r)}, (X^{(r)}_{\cdot,\cdot}), (Y^{\widetilde{\ell}}_{(r)}(\cdot))_{0 \le \widetilde{\ell} \le N})$ has same law under $\widetilde{\mathbb{Q}}$ as*

$$(m, (X^n_{\cdot,\cdot}), (Y^\ell_n(\cdot))_{0 \le \ell \le N}) \textit{ under the probability } H_N \cdot \widetilde{\mathbb{Q}}.$$

Proof. We use Lemma 3.1, and recall that K stands for 2^n. We introduce on a set of full Q-probability a measurable bijection $\tau : \mathbb{N} \to \mathcal{I}_K$, such that:

$$\begin{aligned} &\pi_{\tau(\widetilde{\ell})}\ \exp\{\psi(\overline{Y}_{\tau(\widetilde{\ell})}(x_M))\} \text{ has rank } \widetilde{\ell} \text{ among the collection} \\ &\pi_i\ \exp\{\psi(\overline{Y}_i(x_M))\},\ i \in \mathcal{I}_K. \end{aligned} \tag{3.25}$$

It readily follows from Lemma 3.1 that $(m, (X^n_{\cdot,\cdot}), (Y^\ell_n(\cdot))_{\ell \ge 0}, \widetilde{\sigma}_n)$ has same law under $\widetilde{\mathbb{Q}}$ as $(\overline{\omega}, (\overline{X}_{\cdot,\cdot}), \overline{Y}_{i(\ell)}(\cdot))_{\ell \ge 0}, \tau^{-1} \circ i)$ under Q, (where it should be observed that $\tau^{-1} \circ i$ depends measurably on $\overline{\omega}$ and $(\overline{Y}_{i(\ell)}(x_M))_{\ell \ge 0}$). As a result $(m_{(r)}, (X^{(r)}_{\cdot,\cdot}), (Y^{\widetilde{\ell}}_{(r)}(\cdot))_{\widetilde{\ell} \ge 0})$ has the same law under $\widetilde{\mathbb{Q}}$ as $(\overline{\omega}_{(r)}, (\overline{X}^{(r)}_{\cdot,\cdot}), (\overline{Y}^{\widetilde{\ell}}_{(r)}(\cdot))_{\widetilde{\ell} \ge 0})$ under Q, where

$$\begin{aligned} \overline{\omega}_{(r)} &= \mathcal{N}\Big(\sum_{\mathcal{I}_K} \delta_{\pi_i\ \exp\{\psi(\overline{Y}_i(x_M))\}}\Big), \quad \text{and for } \widetilde{\ell},\widetilde{\ell}' \ge 0, \\ \overline{X}^{(r)}_{\widetilde{\ell},\widetilde{\ell}'} &= \sup\{x_k;\ k \in [1,K] \ \text{ such that } \ [\tau(\widetilde{\ell})]_k = [\tau(\widetilde{\ell}')]_k\}, \\ \overline{Y}^{\widetilde{\ell}}_{(r)}(\cdot) &= \overline{Y}_{\tau(\widetilde{\ell})}(\cdot). \end{aligned} \tag{3.26}$$

The key observation is that we can write for $i \in \mathcal{I}_K$,

$$\begin{aligned} &\pi_i\ \exp\{\psi(\overline{Y}_i(x_M))\} = \eta^1_{[i]_1} \dots \eta^K_{[i]_K}\ \exp\{\psi_K(\overline{Y}_i(x_M))\} = \\ &e^{\psi_0(0)}\, \eta^1_{[i]_1}\ e^{\psi_1(z^1_{[i]_1}(\frac{q_M}{2^n})) - \psi_0(0)} \dots \eta^k_{[i]_k}\ e^{\psi_k(\overline{Y}_i(x_{k-1}) + z^k_{[i]_k}(\frac{q_M}{2^n})) - \psi_{k-1}(\overline{Y}_i(x_{k-1}))} \\ &\eta^K_{[i]_K}\ e^{\psi_K(\overline{Y}_i(x_{K-1}) + z^K_i(\frac{q_M}{2^n})) - \psi_{K-1}(\overline{Y}_i(x_{K-1}))}, \end{aligned} \tag{3.27}$$

and with the help of (A.7), and the definition (3.19), conditional to the σ-algebras:

$$\mathcal{G}_k \overset{\text{def}}{=} \sigma(\eta_i^{k'},\ z_i^{k'},\ i \in \mathcal{I}_{k'},\ k' < k),\ k \in [1, K],$$

the collection of marked point processes on $(0, \infty) \times C_0(\mathbb{R}_+, \mathbb{R})$

$$\sum_{j \geq 0} \delta_{(\eta^k_{i\cdot j}\, e^{\{\psi_k(\overline{Y}_i(x_{k-1})+z^k_{i\cdot j}(\frac{q_M}{2^n}))-\psi_{k-1}(\overline{Y}_i(x_{k-1}))\}},\, z^k_{i\cdot j}(\cdot))},\ i \in \mathcal{I}_{k-1}, \tag{3.28}$$

are independent Poisson with respective intensities:

$$x_k\ \eta^{-x_k - 1}\, d\eta \otimes \overline{\mu}_i^k(dz), \tag{3.29}$$

where $\overline{\mu}_i^k$ is the probability on $C_0(\mathbb{R}_+, \mathbb{R})$ defined by:

$$\overline{\mu}_i^k(dz) = e^{x_k\{\psi_k(\overline{Y}_i(x_{k-1})+z(\frac{q_M}{2^n}))-\psi_{k-1}(\overline{Y}_i(x_{k-1}))\}}\, W(dz), \tag{3.30}$$

where W stands for the Wiener measure.

As a result of (3.28), we can find variables $\widetilde{\eta}_i^k$, $\widetilde{z}_i^k$, $i \in \mathcal{I}_k$, $k \in [1, K]$, obtained by successive relabellings of the variables $\eta_i^k\ \exp\{\psi_k(\overline{Y}_i(x_k)) - \psi_{k-1}(\overline{Y}_{[i]_{k-1}}(x_{k-1}))\}$, $z_i^k(\cdot),\ i \in \mathcal{I}_k$, $k \in [1, K]$, such that:

$$\widetilde{\eta}_i^k,\ i \in \mathcal{I}_k, k \in [1, K], \text{ has same distributions as } \eta_i^k, i \in \mathcal{I}_k, k \in [1, K], \tag{3.31}$$

$$\text{the } \widetilde{z}_i^k \text{ variables are independent of the } \widetilde{\eta}_i^k \text{ variables}, \tag{3.32}$$

$$\text{conditional to } \widetilde{\mathcal{G}}_k \overset{\text{def}}{=} \sigma(\widetilde{\eta}_{i'}^{k'},\ \widetilde{z}_{i'}^{k'}, i' \in \mathcal{I}_{k'}, k' < k), \tag{3.33}$$

the $\widetilde{z}_{i\cdot j}^k(\cdot)$, $i \in \mathcal{I}_{k-1}$, $j \geq 0$, are independent, respectively $\widetilde{\mu}_i^k$-distributed, with

$$\widetilde{\mu}_i^k(dz) = e^{x_k\{\psi_k(\widetilde{Y}_i(x_{k-1})+z(\frac{q_M}{2^n}))-\psi_{k-1}(\widetilde{Y}_i(x_{k-1}))\}}\, W(dz), \tag{3.34}$$

and the $\widetilde{Y}_i(\cdot)$ are defined like the $\overline{Y}_i(\cdot)$ in (3.14), with the $\widetilde{z}$ variables in place of the z variables. Taking into account the fact that $\psi_0(0)$ in (3.27) is constant, and plays no role after normalization, we find that

$$\begin{aligned}
\overline{\omega}_{(r)} &= \mathcal{N}\Big(\sum_{\mathcal{I}_K} \delta_{\widetilde{\pi}_i}\Big), \quad \text{(with obvious notations), and for } \widetilde{\ell}, \widetilde{\ell}' \geq 0: \\
\overline{X}^{(r)}_{\widetilde{\ell},\widetilde{\ell}'} &= \sup\{x_k;\ k \in [1, K],\ [\widetilde{\tau}(\widetilde{\ell})]_k = [\widetilde{\tau}(\widetilde{\ell}')]_k\}, \\
\overline{Y}^{\widetilde{\ell}}_{(r)}(\cdot) &= \widetilde{Y}_{\widetilde{\tau}(\widetilde{\ell})}(\cdot),
\end{aligned} \tag{3.35}$$

where $\widetilde{\tau}(\cdot)$ is the Q-a.s. defined measurable bijection between $\mathbb{N}$ and $\mathcal{I}_K$, such that:

$$\widetilde{\pi}_{\widetilde{\tau}(\ell)} \text{ has rank } \ell \text{ among the collection } \widetilde{\pi}_i, i \in \mathcal{I}_K. \tag{3.36}$$

As a result, we see that $(\overline{\omega}_{(r)}, (\overline{X}^{(r)}_{\cdot,\cdot}), \overline{Y}^{\ell}_{(r)}(\cdot))_{0\le\ell\le N})$ has same law under Q as $(\overline{\omega}, (\overline{X}_{\cdot,\cdot}), (\overline{Y}_{i(\ell)}(\cdot))_{0\le\ell\le N})$ under $\overline{H}_N\, Q$, where

$$\begin{aligned}\overline{H}_0 \quad &= \exp\Big\{\sum_1^K x_k[\psi_k(\overline{Y}_{i(0)}(x_k)) - \psi_{k-1}(\overline{Y}_{i(0)}(x_{k-1}))]\Big\}, \ \text{ and for } \ N \ge 0,\\ \overline{H}_{N+1} &= \overline{H}_N \ \exp\Big\{\sum_{\overline{X}_{N+1,L}<k\le K} x_k[\psi_k(\overline{Y}_{i(N+1)}(x_k)) - \psi_{k-1}(\overline{Y}_{i(N+1)}(x_{k-1}))]\Big\},\end{aligned} \tag{3.37}$$

with $L \in [0, N]$ any number such that $\overline{X}_{N+1,L} = \sup\{\overline{X}_{N+1,\ell}, 0 \le \ell \le N\}$. Thus with the help of Lemma 3.1, and the identity in law above (3.26), Proposition 3.2 follows. □

4. Reshuffling

The object of this section is to study the true reshuffling operation. The description of the effect of this operation on the random weights, the clustering process, and the $y^\ell(\cdot)$ components will exhibit several quantities which arise in the prediction of the "limit picture" for the Sherrington–Kirkpatrick model, see Mézard–Parisi–Virasoro [6], p. 45, and [5]. In the light of these references, the operation of reshuffling, we introduce in (4.4) and (4.24) below, can be seen as a kind of abstract cavity method. Supposedly in the context of the Sherrington–Kirkpatrick model, the cavity method yields a description of the disordered averaged SK-measure on N and $(N + 1)$ sites, when N is large.

We keep the notations of Sect. 3, and assume from now on that

$$\psi(\cdot) \ \text{ belongs to } \ C_b^4(\mathbb{R}). \tag{4.1}$$

We define on $(\widetilde{\Sigma}, \widetilde{\mathcal{B}}, \widetilde{\mathbb{Q}})$ the sequence:

$$\mu_\ell = \eta_\ell(m) \ \exp\{\psi(Y^\ell(x_M))\}\,, \ \ell \ge 0. \tag{4.2}$$

By the same argument as in (3.22), (3.23), we can introduce on a set of full $\widetilde{\mathbb{Q}}$-probability a measurable permutation $\widetilde{\sigma}(\cdot)$ of $\mathbb{N}$, with inverse $\sigma(\cdot)$ such that:

$$\widetilde{\ell} = \widetilde{\sigma}(\ell) \text{ is the rank of } \mu_\ell \text{ among the sequence } \mu_{\ell'},\, \ell' \ge 0. \tag{4.3}$$

As in (3.24), we can then define the quantities:

$$m_{(R)} = \mathcal{N}\Big(\sum_{\ell\ge 0} \delta_{\mu_\ell}\Big),\ X^{(R)}_{\widetilde{\ell},\widetilde{\ell}'} = X_{\sigma(\widetilde{\ell}),\sigma(\widetilde{\ell}')},\ Y^{\widetilde{\ell}}_{(R)}(\cdot) = Y^{\sigma(\widetilde{\ell})}(\cdot),\ \widetilde{\ell}, \widetilde{\ell}' \ge 0, \tag{4.4}$$

which describe the reshuffling operation. Our main objective is to find the law of the random vector (4.4). To this end we introduce for $n \ge 0$, $q \in [0, q_M]$, $y \in \mathbb{R}$:

$$f_n(q, y) = \frac{1}{x_{k,n}}\ \log\{P_{\frac{k}{2^n}\, q_M - q}(\exp\{x_{k,n}\, \psi_k\})(y)\},\ \text{when}\ \frac{k-1}{2^n}\, q_M \le q \le \frac{k}{2^n}\, q_M\,, \tag{4.5}$$

with the notations of (3.19).

Lemma 4.1. *As n tends to infinity, f_n converges uniformly on compact sets of $[0,q_M]\times\mathbb{R}$, to the unique solution $f(q,y)$ in $C_b^{1,2}([0,q_M]\times\mathbb{R})$ of*

$$\begin{cases} \partial_q f + \frac{1}{2}\,\partial_y^2 f + \dfrac{x(q)}{2}\,(\partial_y f)^2 = 0, & \text{on } (0,q_M)\times\mathbb{R}, \\ f(q_M,y) = \psi(y). \end{cases} \tag{4.6}$$

Proof. It follows from (4.5) that on $[\frac{k-1}{2^n}\,q_M,\ \frac{k}{2^n}\,q_M]\times\mathbb{R}$:

$$e^{x_{k,n}\,f_n} = P_{\frac{k}{2^n}\,q_M-q}(e^{x_{k,n}\psi_k}). \tag{4.7}$$

Differentiating twice in the y variable, we find

$$\partial_y\, f_n = P_{\frac{k}{2^n}\,q_M-q}(\partial_y\,\psi_k\, e^{x_{k,n}\psi_k})\,/\,P_{\frac{k}{2^n}\,q_M-q}(e^{x_{k,n}\,\psi_k}), \quad \text{and} \tag{4.8}$$

$$\partial_y^2\, f_n + x_{k,n}(\partial_y\, f_n)^2 = P_{\frac{k}{2^n}\,q_M-q}((\partial_y^2\,\psi_k + x_{k,n}(\partial_y\,\psi_k)^2)\, e^{x_{k,n}\,\psi_k})/P_{\frac{k}{2^n}\,q_M-q}(e^{x_{k,n}\,\psi_k}). \tag{4.9}$$

From (4.8), we see recursively on k that:

$$\sup_{n,q,y}\,|\partial_y\, f_n| \le \|\partial_y\,\psi\|_\infty,$$

and then choosing $q = \frac{k-1}{2^n}\,q_M$ in (4.9), we see that:

$$\begin{aligned} &\|\partial_y^2\,\psi_{k-1} + x_{k-1,n}(\partial_y\,\psi_{k-1})^2\|_\infty \le \|\partial_y^2\,\psi_k + x_{k,n}(\partial_y\,\psi_k)^2\|_\infty \\ &+ (x_{k,n} - x_{k-1,n})\,\|\partial_y\,\psi\|_\infty^2, \end{aligned} \tag{4.10}$$

which together with (4.9) easily implies that:

$$\sup_{n,q,y}\,|\partial_y^2\, f_n| < \infty.$$

Moreover, differentiating (4.7) in the q variable and recalling that P_t is the Brownian semigroup, we find that:

$$\begin{cases} \partial_q\, f_n + \frac{1}{2}\,\partial_y^2\, f_n + \dfrac{x_n(q)}{2}\,(\partial_y\, f_n)^2 = 0, & \text{for } q \ne \dfrac{k}{2^n}\,q_M,\ k\in[1,2^n] \\ f_n(q_M,y) = \psi(y), \end{cases} \tag{4.11}$$

with the notation:

$$x_n(q) = \sum_1^{2^n}\, x_{k,n}\; 1\Big\{\frac{k-1}{2^n}\,q_M < q \le \frac{k}{2^n}\,q_M\Big\}. \tag{4.12}$$

If we write the relations analogous to (4.8) and (4.9) obtained for the third and fourth derivative of f_n in the y variable, and derive analogous controls to (4.10), we easily deduce that:

$$\sup_{n\ge 0,\ 0\le j\le 4}\ \sup_{q,y}\ |\partial_y^j\, f_n(q,y)| < \infty. \tag{4.13}$$

Taking first and second derivatives of (4.11) in the y variable, we see that

$$\sup_{n\ge 0,\ 0\le j\le 2}\ \|\partial_q\,\partial_y^j\, f_n\|_{L^\infty([0,q_M]\times\mathbb{R})} < \infty. \tag{4.14}$$

It then follows from (4.13), (4.14) and $f_n(q_M, \cdot) = \psi(\cdot)$, that $\partial_y^j f_n$, $0 \le j \le 2$, are relatively compact sequences for the topology of uniform convergence on compact sets of $[0, q_M] \times \mathbb{R}$. From any subsequence of f_n, we can extract a subsequence along which f_n, $\partial_y f_n$, $\partial_y^2 f_n$ converge uniformly on compact subsets of $[0, q_M] \times \mathbb{R}$ respectively to the bounded continuous function f_∞, $\partial_y f_\infty$, $\partial_y^2 f_\infty$. Coming back to (4.11) in integral form, we see that $f_\infty \in C_b^{1,2}([0, q_M] \times \mathbb{R})$, and f_∞ satisfies (4.6).

Observe now that (4.6) has a unique solution. Indeed, if f and f' are two solutions, $w = f - f' \in C_b^{1,2}\,([0, q_M] \times \mathbb{R})$ satisfies

$$\begin{cases} \partial_q\, w + \frac{1}{2}\, \partial_y^2\, w + \dfrac{x(q)}{2}\, (\partial_y\, f + \partial_y\, f')\, \partial_y\, w = 0 \\ w(q_M, y) = 0, \end{cases} \tag{4.15}$$

and the maximum principle, (see Theorem 8.1.4 of Krylov [3]), implies that $w = 0$, (one could also give an argument based on an S.D.E. representation of w). This shows that f_∞ is uniquely determined, and thus f_n converges uniformly on compact sets of $[0, q_M] \times \mathbb{R}$ to the solution of (4.6). □

We shall use the notation

$$m(q, y) = \partial_y\, f(q, y),\ \ (q, y) \in [0, q_M] \times \mathbb{R}, \tag{4.16}$$

where f is the unique solution of (4.6). We can now introduce a sequence of random variables I_N, $N \geq 0$, on $\widetilde{\Sigma}$:

$$\begin{aligned} I_0 &= \exp\Big\{x_M\, \psi(Y^0(x_M)) - \int_0^{x_M} f(q(x), Y^0(x))dx\Big\}, \quad \text{and for } \ N \geq 0, \\ I_{N+1} &= I_N\ \exp\Big\{x_M\, \psi(Y^{N+1}(x_M)) - X_{N+1,L}\, \psi(Y^{N+1}(X_{N+1,L})) \\ &\quad - \int_{X_{N+1,L}}^{x_M} f(q(x), Y^{N+1}(x))dx\Big\}, \end{aligned} \tag{4.17}$$

where $L \in [0, N]$, and $X_{N+1,L}$ are as in (3.6). With the help of (3.7), this unambiguously defines I_N, $N \geq 0$. We can give an alternative expression for I_N, if we notice that for $x_1 \in [0, x_M]$, $\ell \geq 0$, Ito's formula implies

$$\begin{aligned} & x_1\, f(q(x_1), Y^\ell(x_1)) - \int_0^{x_1} f(q(x), Y^\ell(x))dx = \int_0^{x_1} x\, m(q(x), Y^\ell(x))\, dY^\ell(x) \\ & + \int_0^{x_1} x(\partial_q + \frac{1}{2}\, \partial_y^2)\, f(q(x), Y^\ell(x))\, dq(x) \overset{(4.6)}{=} \\ & \int_0^{x_1} x\, m(q(x), Y^\ell(x))dY^\ell(x) - \frac{1}{2} \int_0^{x_1} x^2\, m^2(q(x), Y^\ell(x))\, dq(x). \end{aligned} \tag{4.18}$$

As a result, we can write

$$\begin{aligned} I_0 &= \exp\Big\{ \int_0^{x_M} xm(q(x),\, Y^0(x))\, dY^0(x) \\ &\quad - \frac{1}{2} \int_0^{x_M} x^2\, m^2(q(x),\, Y^0(x))\, dq(x) \Big\}, \\ I_{N+1} &= I_N \,\exp\Big\{ \int_{X_{N+1,L}}^{x_M} xm(q(x),\, Y^{N+1}(x))\, dY^{N+1}(x) \\ &\quad - \frac{1}{2} \int_{X_{N+1,L}}^{x_M} x^2\, m^2(q(x),\, Y^{N+1}(x))\, dq(x) \Big\}, \end{aligned} \tag{4.19}$$

for $N \geq 0$. This gives a more transparent interpretation for the I_N-variables if one keeps in mind the Girsanov formula. On the other hand (4.17) is better suited to the approximation scheme.

Theorem 4.2. *For $N \geq 0$, $(m_{(R)}$, $(X^{(R)}_{\cdot,\cdot})$, $(Y^{\widetilde{\ell}}_{(R)}(\cdot))_{0\leq\widetilde{\ell}\leq N})$ has the same law under $\widetilde{\mathbb{Q}}$ as $(m$, $(X_{\cdot,\cdot})$, $(Y^{\ell}(\cdot))_{0\leq\ell\leq N})$ under $I_N \cdot \widetilde{\mathbb{Q}}$.*

Proof. We reintroduce the n-dependence in the notation $m_{n,(r)}$, $X^{n,(r)}_{\cdot,\cdot}$, $Y^{\cdot}_{n,(r)}(\cdot)$, for the quantities defined in (3.24). As a result of (3.12), (3.13), on a set of full $\widetilde{\mathbb{Q}}$-probability, as $n \to \infty$,

$$m_{n,(r)} \to m_{(R)}, \quad \text{vaguely on } (0,\infty), \tag{4.20}$$

$\sigma_n(\cdot)$ and $\widetilde{\sigma}_n(\cdot)$ converge simply on $\mathbb{N}$, respectively to $\sigma(\cdot)$ and $\widetilde{\sigma}(\cdot)$. Therefore, when $n \to \infty$:

$$\begin{aligned} &X^{n,(r)}_{\widetilde{\ell},\widetilde{\ell}'} = X^n_{\sigma_n(\widetilde{\ell}),\sigma_n(\widetilde{\ell}')} \to X^{(R)}_{\widetilde{\ell},\widetilde{\ell}'} = X_{\sigma(\widetilde{\ell}),\sigma(\widetilde{\ell}')}, \text{ for } \widetilde{\ell},\widetilde{\ell}' \geq 0, \\ &Y^{\widetilde{\ell}}_{n,(r)}(\cdot) = Y^{\sigma_n(\widetilde{\ell})}_n(\cdot) \to Y^{\widetilde{\ell}}_{(R)}(\cdot) = Y^{\sigma(\widetilde{\ell})}(\cdot) \text{ uniformly on } [0,x_M], \text{ for } \widetilde{\ell} \geq 0. \end{aligned}$$

If we endow $M_p \times [0,x_M]^{\mathbb{N}\times\mathbb{N}} \times C([0,x_M],\mathbb{R})^N$, with the canonical product topology, and denote by F a continuous bounded function in this space, it follows from these convergences, Proposition 3.2 and Lemma 4.1, that:

$$\begin{aligned} &E^{\widetilde{\mathbb{Q}}}[F(m_{(R)},\, (X^{(R)}_{\cdot,\cdot}),\, (Y^{\widetilde{\ell}}_{(R)}(\cdot))_{0\leq\widetilde{\ell}\leq N})] \\ &= \lim_n E^{\widetilde{\mathbb{Q}}}[F(m_{n,(r)}, (X^{n,(r)}_{\cdot,\cdot}),\, (Y^{\widetilde{\ell}}_{n,(r)})_{0\leq\ell\leq N})] \\ &= \lim_{n\to\infty} E^{\widetilde{\mathbb{Q}}}[F(m, (X^n_{\cdot,\cdot}),\, (Y^{\ell}_n(\cdot))_{0\leq\ell\leq N})\, H_N] \\ &= E^{\widetilde{\mathbb{Q}}}[F(m, (X_{\cdot,\cdot}),\, (Y^{\ell}(\cdot))_{0\leq\ell\leq N})\, I_N]. \end{aligned} \tag{4.21}$$

Since N and F are arbitrary, this proves our claim. □

We can give a slightly different formulation of Theorem 4.2 by considering:

$$\Sigma = M \times T \times C([0,x_M],\mathbb{R})^{\mathbb{N}}, \tag{4.22}$$

endowed with the natural product σ-algebra $\mathcal{B}$ and with the probability $\mathbb{Q}$ for which the canonical coordinates m and $(\Gamma_u)_{u\geq 0}$ are independent, respectively $\overline{\mathbb{P}}_{x_M}$ and P

distributed, and conditional to $m, (\Gamma_u)_{u\ge 0}$, the $y^\ell(\cdot)$, $\ell \ge 0$, are centered Gaussian processes with

$$E^{\mathbb{Q}}[y^\ell(x)\, y^{\ell'}(x') \,|\, m, (\Gamma_u)_{u\ge 0}] = q(x\wedge x' \wedge X_{\ell,\ell'}),\ x, x' \in [0, x_m],\ \ell, \ell' \ge 0. \quad (4.23)$$

Then we can define the reshuffling operation as the measurable $\Phi:\ \Sigma_0 \to \Sigma$, where Σ_0 has full $\mathbb{Q}$ probability, and Φ is such that:

$$\Phi(m, (\Gamma_u)_{u\ge 0},\ (y^\ell(\cdot))_{\ell\ge 0}) = \Big(\mathcal{N}\Big(\sum_{\ell\ge 0} \delta_{\eta_\ell e^{\psi(y^\ell(x_M))}}\Big), ((\widetilde{\sigma}\otimes\widetilde{\sigma})(\Gamma_u))_{u\ge 0}, (y^{\sigma(\widetilde{\ell})}(\cdot))_{\widetilde{\ell}\ge 0}\Big), \quad (4.24)$$

with $\widetilde{\sigma}(\cdot)$ and $\sigma(\cdot)$, as in (4.3), with Y^ℓ replaced by y^ℓ. Furthermore, we can introduce on Σ the $\mathbb{Q}$-densities:

$$\begin{aligned} J_N = \exp\Big\{ &\sum_{\ell\ge 0} \int_0^{x_M} 1_{\{\ell\in L_N(x)\}}\, xm(q(x),\ y^\ell(x))\, dy^\ell(x) \\ &- \frac{1}{2} \int_0^{x_M} 1_{\{\ell\in L_N(x)\}}\, x^2\, m^2(q(x), y^\ell(m))\, dq(x)\Big\}, \end{aligned} \quad (4.25)$$

for $N \ge 0$, where L_N is any map of the form $S(r_{[0,N]}\circ \Gamma_{\log(\frac{x_M}{x})})$: composition of the restriction to $[0,N]$ of $\Gamma_{\log(\frac{x_M}{x})}$ (a piecewise constant map), with S which to $\Gamma \in E_{[0,N]}$ associates a selection $S(\Gamma) \subseteq [0,N]$ of representatives of the Γ-equivalence classes in $[0,N]$. As a result of (4.23), J_N is unambiguously defined up to null equivalence. The key Theorem 4.2 can be reformulated as:

Theorem 4.3. *If $\mathcal{H}_N$ is the σ-algebra generated by $(m, (\Gamma_u)_{u\ge 0},\ (y^\ell(\cdot)_{0\le \ell\le N})$, for $N \ge 0$, then*

$$\Phi\circ\mathbb{Q} = J_N\cdot\mathbb{Q} \ \text{ on }\ \mathcal{H}_N. \quad (4.26)$$

Proof. We only need to notice that I_N in (4.19) can be rewritten as

$$\begin{aligned} I_N = \exp\Big\{ &\sum_{\ell\ge 0} \int_0^{x_M} 1_{\{\ell\in L_N(x)\}}\, xm(q(x),\ Y^\ell(x))\, dY^\ell(x) \\ &- \frac{1}{2} \int_0^{x_M} 1_{\{\ell\in L_N(x)\}}\, x^2\, m^2(q(x), Y^\ell(m))\, dq(x)\Big\}, \end{aligned}$$

for a suitable $L_N(x)$, as after (4.25). □

Corollary 4.4. *Let τ be a permutation of $[0,N]$, $N \ge 0$, then under $\Phi\circ\mathbb{Q}$,*

$$\begin{aligned} G\ &= ((X_{\ell,\ell'})_{0\le \ell,\ell'\le N},\ (y^\ell(\cdot))_{0\le\ell\le N}) \ \text{ and} \\ G_\tau &= ((X_{\tau(\ell),\tau(\ell')}))_{0\le \ell,\ell'\le N},\ (y^{\tau(\ell)}(\cdot))_{0\le\ell\le N}) \ \textit{ have the same law}. \end{aligned}$$

Proof. It is obvious that $(X_{\ell,\ell'})_{0\le\ell,\ell'\le N}$ and $(X_{\tau(\ell),\tau(\ell')})_{0\le\ell,\ell'\le N}$ have the same law under P. Together with (4.23), this shows that G and G_τ have the same law under $\mathbb{Q}$. Expressing J_N as a measurable function of G_τ, we find that G_τ under $J_N\cdot\mathbb{Q}$ has the same law as G under $\widetilde{J}_N\cdot\mathbb{Q}$, with

$$\begin{aligned} \widetilde{J}_N = \exp\Big\{ &\sum_{\ell\ge 0} \int_0^{x_M} 1_{\{\ell\in \widetilde{L}_N(x)\}}\, xm(q(x),\ y^\ell(x))\, dy^\ell(x) \\ &- \frac{1}{2} \int_0^{x_M} 1_{\{\ell\in \widetilde{L}_N(x)\}}\, x^2\, m^2(q(x), y^\ell(x))\, dq(x)\Big\}, \end{aligned}$$

with $\widetilde{L}_N(x) = \widetilde{S}(r_{[0,N]} \circ \Gamma_{\log(\frac{x_M}{x})})$, where $\widetilde{S} : E_{[0,N]} \to \mathcal{P}([0,N])$ is defined by:

$$\widetilde{S}(\Gamma) = \tau^{-1}\Big(S((\tau \otimes \tau)(\Gamma))\Big),$$

and associates to $\Gamma \in E_{[0,N]}$ a selection of representatives of the Γ-equivalence classes. As a result $J_N = \widetilde{J}_N$, $\mathbb{Q}$-a.s., and this proves our claim. □

5. Single and Double Replicas Calculations

We want to apply the results of the previous section to investigate the "single replica distribution":

$$E^{\Phi\circ\,\mathbb{Q}}\Big[\sum_{\ell\geq 0} \eta_\ell\, \delta_{y^\ell(x_M)}\Big], \tag{5.1}$$

which is a probability on $\mathbb{R}$, as well as the "double replicas distribution":

$$E^{\Phi\circ\,\mathbb{Q}}\Big[\sum_{\ell,\ell'\geq 0} \eta_\ell\, \eta_{\ell'}\; 1_{\{X_{\ell,\ell'}\,\in\,\cdot\}}\, \delta_{y^\ell(x_M)} \otimes \delta_{y^{\ell'}(x_M)}\Big], \tag{5.2}$$

which is a probability on $[0, x_M] \times \mathbb{R} \times \mathbb{R}$.

As a result when ψ is symmetric non-constant, we shall define a transformation of the function $q(\cdot)$. The equation for fixed points of this transformation in the context of the SK-model is the so-called "consistency equation", see [6], p. 45.

It is useful to introduce the time inhomogeneous transition probability $(R_{x_0,x_1})_{0\leq x_0\leq x_1\leq x_M}$ of the solution of the SDE

$$dy(x) = dM(x) + xm(q(x),\, y(x))\, dq(x) \tag{5.3}$$

with $M(x) = w(q(x))$, $0 \leq x \leq x_M$, the time changed of the standard Brownian motion $w(\cdot)$. More precisely, we denote by $P^y_{x_0,x_1}$ the law of $\Big(y + w(q(x) - q(x_0))\Big)_{x_0\leq x\leq x_1}$, for $y \in \mathbb{R}$, $x_0 \leq x_1$ in $[0, x_M]$, and introduce

$$\begin{aligned}
&R_{x_0,x_1}\, h(y) = \int\; h(y(x_1))\;\exp\Big\{\int_{x_0}^{x_1} xm(q(x),\, y(x))\, dy(x)\\
&-\frac{1}{2}\int_{x_0}^{x_1} x^2\, m(q(x),\, y(x))\, dq(x)\Big\}\, dP^y_{x_0,x_1}(y(\cdot))\\
&= \int h(y(x_1))\;\exp\Big\{x_1\, f(q(x_1),\, y(x_1)) - x_0\, f(q(x_0),\, y(x_0))\\
&-\int_{x_0}^{x_1} f(q(x),\, y(x))\, dx\Big\}\, dP^y_{x_0,x_1}(y(\cdot)),
\end{aligned} \tag{5.4}$$

where the second equality follows from Ito's formula and (4.6), as in (4.18). It is immediate from the second line of (5.4) to check the composition rule:

$$R_{x_0,x_1}\; R_{x_1,x_2} = R_{x_0,x_2}, \quad \text{for}\;\; 0 \leq x_0 \leq x_1 \leq x_2 \leq x_M.$$

Lemma 5.1.

$$R_{x,x_M}(\partial_y\, \psi)(\cdot) = \partial_y\, f(q(x),\, \cdot) = m(q(x),\, \cdot), \quad \text{for}\;\; x \in [0, x_M]. \tag{5.5}$$

Proof. We introduce a regularization by convolution $f_\epsilon = f * \psi_\epsilon$, and $m_\epsilon = \partial_y(f * \psi_\epsilon) = m * \psi_\epsilon$, where $\psi_\epsilon = \epsilon^{-2}\,\psi(\frac{\cdot}{\epsilon})$, with $\psi(q, y) \geq 0$, smooth, compactly supported and $\int \psi\, dqdy = 1$. It follows from (4.6) that when $I = [q_0, q_1] \subset (0, q_M)$, for small ϵ:

$$\begin{aligned}&\partial_1\, m_\epsilon + \frac{1}{2}\,\partial_y^2\, m_\epsilon + H_\epsilon = 0 \quad \text{in } I \times \mathbb{R}, \text{ with}\\ &H_\epsilon = \partial_y\left(\Big[\frac{x(\cdot)}{2}\,(\partial_y\, f)^2\Big] * \psi_\epsilon\right) = [x(\cdot)\, m\, \partial_y\, m] * \psi_\epsilon.\end{aligned} \tag{5.6}$$

Let $x_0 = x(q_0)$, $x_1 = x(q_1)$ and Z_x, $x \in [x_0, x_M]$, stand for the exponential martingale (under $P^y_{x_0,x_M}$):

$$Z_x = \exp\left\{\int_{x_0}^x u\, m(q(u),\, y(u))\, dy(u) - \frac{1}{2}\int_{x_0}^x u^2\, m^2(q(u),\, y(u))\, dq(u)\right\}.$$

Observe that by a similar calculation as in the second line of (5.4), $Z_.$ is bounded. It follows from Ito's formula and (5.6) that

$$\begin{aligned}m_\epsilon(q(x_1),\, y(x_1)) = m_\epsilon(q(x_0),\, y(x_0)) &+ \int_{x_0}^{x_1} \partial_y\, m_\epsilon(q(x),\, y(x))\, dy(x)\\ &- \int_{x_0}^{x_1} H_\epsilon(q(x),\, y(x))\, dq(x), \quad \text{when } \epsilon \text{ is small.}\end{aligned}$$

Letting ϵ tend to 0, we find

$$\begin{aligned}&m(q(x_1),\, y(x_1)) = m(q(x_0,\, y(x_0)) + \int_{x_0}^{x_1} \partial_y\, m(q(x),\, y(x))\, dN_x, \quad \text{where}\\ &N_x = y(x) - \int_{x_0}^x u\, m(q(u),\, y(u))\, dq(u),\ \ x \in [x_0, x_1] \ \text{ is a martingale}\\ &\text{under } Z_{x_1}\cdot P^y_{x_0,x_M}.\end{aligned} \tag{5.7}$$

If we take expectations of (5.7) with respect to the above probability and let x_1 tend to x_M and x_0 vary in $(0, x_M]$, we find our claim (5.5). □

Theorem 5.2. *For h bounded measurable and $x_0 \in [0, x_M]$,*

$$E^{\Phi\circ\mathbb{Q}}\Big[\sum_{\ell\geq 0} \eta_\ell\, h(y^\ell(x_M))\Big] = R_{0,x_M}\, h(0), \tag{5.8}$$

$$\begin{aligned}&E^{\Phi\circ\mathbb{Q}}\Big[\sum_{\ell,\ell'\geq 0} \eta_\ell\, \eta_{\ell'}\ 1(X_{\ell,\ell'} \geq x_0)\ h(y^\ell(x_M))\, h(y^{\ell'}(x_M))\Big]\\ &= \int_{x_0}^{x_M} R_{0,x}(R_{x,x_M}\, h)^2(0)\, dx + (1 - x_M)\, R_{0,x_M}(h^2)(0).\end{aligned} \tag{5.9}$$

Proof. As a result of Theorem 4.3, $(\eta_\ell)_{\ell\geq 0}$ and $(y^\ell(\cdot))_{\ell\geq 0}$, are independent under $\Phi\circ\mathbb{Q}$, therefore the left member of (5.8) equals:

$$\sum_{\ell\geq 0} E^{\Phi\circ\mathbb{Q}}[\eta_\ell]\; E^{\Phi\circ\mathbb{Q}}\,[h(y^\ell(x_M))],$$

and using Corollary 4.4 and Lemma 5.1, this equals:

$$E^{\mathbb{Q}}[h(y^0(x_M))\,J_0] = R_{0,x_M}\,h(0).$$

This proves (5.8). By similar considerations, the left-hand member of (5.9) equals:

$$\begin{aligned}&\sum_{\ell\neq\ell'} E^{\overline{\mathbb{P}}_{x_M}}[\eta_\ell\,\eta_{\ell'}]\;E^{\mathbb{Q}}[1_{\{X_{0,1}\geq x_0\}}\,h(y^0(x_M))\,h(y^1(x_M))\,J_1]\\ &+\sum_{\ell\geq 0} E^{\overline{\mathbb{P}}_{x_M}}[\eta_\ell^2]\;E^{\mathbb{Q}}[h^2(y^0(x_M))J_0].\end{aligned}\tag{5.10}$$

In view of (A.4) and (5.4), the last term of (5.10) equals $(1-x_M)\,R_{0,x_M}\,h^2(0)$.

As for the first term of (5.10), note that $X_{0,1}$ is uniformly distributed on $(0,x_M)$ under $\mathbb{Q}$, see (1.30), (3.31), and thus:

$$\begin{aligned}&E^{\mathbb{Q}}[1_{\{X_{0,1}\geq x_0\}}\,h(y^0(x_M))\,h(y^1(x_M))\,J_1]\\ &=\frac{1}{x_M}\int_{x_0}^{x_M} dx\,E^{\mathbb{Q}}[h(y^0(x_M))\,h(y^1(x_M))\,J_1\mid X_{0,1}=x]\\ &=\frac{1}{x_M}\int_{x_0}^{x_1} R_{0,x}(R_{x,x_M}\,h)^2(0)dx,\end{aligned}\tag{5.11}$$

by the definition of J_1 and (5.4). Since $E^{\overline{\mathbb{P}}_{x_M}}[\sum_{\ell\neq\ell'}\eta_\ell\,\eta_{\ell'}]=x_M$, this concludes the proof of (5.9). $\square$

Remark 5.3. In the special case $h=\partial_y\,\psi$, whose special virtue is explained below, Lemma 5.1 and Theorem 5.2 show that:

$$E^{\Phi\circ\mathbb{Q}}\Big[\sum_{\ell\geq 0}\eta_\ell\,\partial_y\,\psi(y^\ell(x_M))\Big]=m(0,0),\quad\text{and}\tag{5.12}$$

$$\begin{aligned}&E^{\Phi\circ\mathbb{Q}}\Big[\sum_{\ell,\ell'\geq 0}\eta_\ell\,\eta_{\ell'}\;1_{\{X_{\ell,\ell'}\geq x_0\}}\,\partial_y\,\psi(y^\ell(x_M))\,\partial_y\,\psi(y^{\ell'}(x_M))\Big]\\ &=\int_{x_0}^{x_M} R_{0,x}(m(q(x),\,\cdot)^2)(0)\,dx+(1-x_M)\,R_{0,x_M}((\partial_y\,\psi)^2)(0).\end{aligned}\tag{5.13}$$

We now specialise to the case where ψ is symmetric and non-constant. As a result $f(q,\cdot)$ and $m(q,\cdot)$ are respectively symmetric and antisymmetric functions, so that $m(q,0)=0$. In the context of the SK measure, a (very) non-rigorous cavity calculation in the spirit of Chapter 5 of [6] leads to an "approximate identity":

$$\begin{aligned}&\int_{x_0}^{1} q^{(n+1)}(x)\,dx\\ &\text{"}\approx\text{"}\;\mathbb{E}\Big[\sum_{\alpha,\alpha'}\eta_\alpha^{(n+1)}\,\eta_{\alpha'}^{(n+1)}\,1\{q_{\alpha,\alpha'}^{(n+1)}\geq q^{(n+1)}(x_0)\}\,\tanh\big(\beta y_\alpha^{(n)}\big)\,\tanh\big(\beta y_{\alpha'}\big)^{(n)}\Big],\end{aligned}$$

where $q^{(n+1)}(\cdot)$ stands for the overlap function on $(n+1)$ spins, $\eta_\alpha^{(n+1)}$ are the weights of the respective "states" in the decomposition of the SK measure on $(n+1)$ spins, $q_{\alpha,\alpha'}^{(n+1)}$ are the mutual overlaps, $y_\alpha^{(n)}$ are the respective cavity fields, $\mathbb{E}$ denotes the disorder expectation, and $q^{(n+1)}(x_0)$ is smaller than the maximum value of $q^{(n+1)}(\cdot)$. Recall that in this situation $\psi(x)$ should be viewed as log(cosh βx).

In our abstract set up, this suggests defining an "inverse temperature"

$$\beta = \|\partial_y \psi\|_\infty \in (0, \infty), \tag{5.14}$$

and a reshuffled function $q^{(R)}(\cdot)$ via:

$$\begin{aligned} q^{(R)}(x) \;&=\; -\frac{1}{\beta^2} \;\frac{d}{dx_0}\; E^{\Phi \circ \mathbb{Q}} \Big[\sum_{\ell, \ell' \geq 0} \eta_\ell \, \eta_{\ell'} \, 1\{X_{\ell,\ell'} \geq x_0\} \\ &\qquad \partial_y \, \psi(y^\ell(x_M)) \, \partial_y \, \psi(y^{\ell'}(x_M)) \Big] \Big|_{x_0 = x \wedge x_M} \\ &\overset{(5.13)}{=} \beta^{-2} \, R_{0, x \wedge x_M}(m(q(x \wedge x_M), \cdot)^2)(0), \;\; x \in [0,1]. \end{aligned} \tag{5.15}$$

As we shall now see, $q^{(R)}(\cdot)$ fulfills analogous properties to the function $q(\cdot)$.

Theorem 5.4. *The function $q^{(R)}(\cdot)$ is continuous increasing, with values in $[0,1]$, constant on $[x_M, 1]$,*

$$q^{(R)}(0) = 0, \;\; q^{(R)}(x_M) = \beta^{-2} \, R_{0,x_M}((\partial_y \, \psi)^2)(0), \quad \text{and} \tag{5.16}$$

$$q^{(R)'}(x) = \beta^{-2} \, R_{0,x} \left((\partial_y^2 \, f(q(x), \, \cdot))^2 \right) (0) \cdot q'(x), \quad \text{for} \;\; x \in [0, x_M]. \tag{5.17}$$

Proof. The formula (5.15) clearly defines a continuous increasing function, constant on $[x_M, 1]$, for which (5.16) holds. The calculation in (5.7) shows that

$$\begin{aligned} &m(q(x), \, y(x)) = \int_0^x \partial_y \, m(q(v), \, y(v)) \, dN_v, \quad \text{where} \\ &N_x = y(x) - \int_0^x v \, m(q(v), \, y(v)) \, dv, \;\; x \in [0, x_M], \end{aligned} \tag{5.18}$$

defines a martingale, with increasing process:

$$\int_0^x \partial_y \, m(q(v), \, y(v))^2 \, dq(v),$$

under the law $\widetilde{P} \overset{\text{def}}{=} Z_{x_M} \cdot P^{y=0}_{x_0=0, x_1=x_M}$. Therefore for $0 \leq x_0 < x_1 \leq x_M$,

$$\begin{aligned} &q^{(R)}(x_1) - q^{(R)}(x_0) = \beta^{-2} \, E^{\widetilde{P}}[m^2(q(x_1), \, y(x_1)) - m^2(q(x_0), \, y(x_0))] \\ &= \beta^{-2} \int_{x_0}^{x_1} E^{\widetilde{P}}[(\partial_y \, m)^2(q(x), \, y(x))] \, dq(x), \end{aligned} \tag{5.19}$$

and our claim (5.17) readily follows. □

Remark 5.5. The above theorem suggests looking at iterations of the transformation $q(\cdot) \to q^{(R)}(\cdot)$. The fixed point equation $q(\cdot) = q^{(R)}(\cdot)$ essentially corresponds to the selfconsistency equation (III.63), p. 45 of Mézard–Parisi–Virasoro [6], for the SK-model. We thus see once again that several quantities related to the physical prediction of the SK-model appear in the context of our abstract cavity method.

Appendix

We shall collect in this appendix some useful results on the laws $\mathbb{P}_x$ and $\overline{\mathbb{P}}_x$ which are defined above (0.2) and (1.3), and are used throughout this article. We recall that M_p is the set of simple pure point Radon measures on $(0,\infty)$, and it is endowed with the topology of vague convergence.

Proposition A.1.

For $x \in (0,1)$, $\mathbb{P}_x$ is the weak limit of the laws on M_p of $\sum_1^n \delta_{\exp\{\frac{1}{x}(X_i - \log n)\}}$ *where $X_1, \dots, X_n$ are standard i.i.d. exponential variables.* (A.1)

For $x_1 \in (0,1)$ and $x_1 x_2 \in (0,1)$, $\mathbb{P}_{x_1 x_2}$ is the image of $\mathbb{P}_{x_1}$ under the map: $m \to \eta^{\frac{1}{x_2}} \circ m$, (i.e. $m = \sum \delta_{\eta_\ell} \to \sum \delta_{(\eta_\ell)^{\frac{1}{x_2}}}$). (A.2)

$$E^{\mathbb{P}_{x_2}}[|m|^{x_1}] < \infty, \quad \text{if} \quad 0 < x_1 < x_2 < 1. \tag{A.3}$$

For $x \in (0,1)$, $k \geq 1$, $E^{\overline{\mathbb{P}}_x}[< m, \eta^k >] = \dfrac{\Gamma(k-x)}{\Gamma(1-x)\Gamma(k)} = \dfrac{(k-1-x)\dots(1-x)}{(k-1)!}$. (A.4)

Proof. For the proof of (A.1), observe that $\sum_1^n \delta_{(X_i - \log n)}$ converges in law to a Poisson point process with intensity $\exp\{-z\}dz$. Indeed for $f \in C_c(\mathbb{R})$,

$$\begin{aligned} E\Big[\exp\Big\{-\sum_{i=1}^n f(x_i - \log n)\Big\}\Big] &= \Big(\int_0^\infty e^{-f(z-\log n)-z} dz\Big)^n \\ &= \Big(1 - \int_{-\log n}^\infty (1-e^{-f(x)}) \frac{e^{-x}}{n}\, dx\Big)^n \overset{n\to\infty}{\longrightarrow} \exp\Big\{-\int_{\mathbb{R}} (1-e^{-f(x)})\, e^{-x}\, dx\Big\}. \end{aligned}$$

Our claim (A.1) now follows once we notice that the image of the Poisson law with intensity $e^{-z}dz$ on $\mathbb{R}$ under the continuous map: $m \in M_p(\mathbb{R}) \to \exp\{\frac{\cdot}{x}\} \circ m \in M_p$, is the Poisson law with intensity $\exp\{\frac{\cdot}{x}\} \circ (e^{-z}dz) = x\eta^{-x-1}d\eta$. As for (A.2), it is an immediate consequence of (A.1). Finally (A.3) and (A.4) can be found in Corollary 2.2 of Ruelle [10]. $\square$

We also want to look at the situation where $\mathbb{P}$ is the law of $m(d\eta\, dy)$, a Poisson point process on $(0,\infty)$ with marks in E a Polish space, with intensity $x\eta^{-x-1}d\eta \otimes d\mu$, where μ is a probability on E. We consider a positive measurable function g on E, such that

$$\int_E g(y)^x \, d\mu(y) < \infty. \tag{A.5}$$

We can then define a new random pure point measure on $(0,\infty) \times E$ through the formula

$$m(d\eta\, dy) \longrightarrow \widetilde{m}(d\eta\, dy) : \int_{(0,\infty)\times E} f\, d\widetilde{m} = \int_{(0,\infty)\times E} f(\eta g(y), y)\, dm, \tag{A.6}$$

for $f \geq 0$ measurable.

Proposition A.2.

$$\widetilde{m} \text{ is a Poisson point process on } (0,\infty) \times E \text{ with intensity } x\eta^{-x-1} d\eta \otimes g(y)^x \, d\mu(y). \tag{A.7}$$

Proof. For $f \geq 0$ measurable on $(0,\infty) \times E$:

$$E^{\mathbb{P}}[\exp\{- < \widetilde{m}, f >\}] = E^{\mathbb{P}}\Big[\exp\Big\{ - \int_{(0,\infty)\times E} f(\eta g(y), y) \, dm\Big\}\Big]$$
$$= \exp\Big\{ - \int_{(0,\infty)\times E} (1 - e^{-f(\eta g(y),y)}) x\eta^{-x-1} \, d\eta \, d\mu(y)\Big\}$$
$$= \exp\Big\{ - \int_{(0,\infty)\times E} (1 - e^{-f}) \, x\eta^{-x-1} \, d\eta \, g(y)^x \, d\mu(y)\Big\}.$$

This proves our claim. □

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