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Metric cotype

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Abstract

We introduce the notion of *cotype of a metric space*, and prove that for Banach spaces it coincides with the classical notion of Rademacher cotype. This yields a concrete version of Ribe's theorem, settling a long standing open problem in the nonlinear theory of Banach spaces. We apply our results to several problems in metric geometry. Namely, we use metric cotype in the study of uniform and coarse embeddings, settling in particular the problem of classifying when L_p coarsely or uniformly embeds into L_q. We also prove a nonlinear analog of the Maurey-Pisier theorem, and use it to answer a question posed by Arora, Lovász, Newman, Rabani, Rabinovich and Vempala, and to obtain quantitative bounds in a metric Ramsey theorem due to Matoušek.

1. Introduction

In 1976 Ribe [62] (see also [63], [27], [9], [6]) proved that if X and Y are uniformly homeomorphic Banach spaces then X is finitely representable in Y, and vice versa (X is said to be finitely representable in Y if there exists a constant $K > 0$ such that any finite dimensional subspace of X is K-isomorphic to a subspace of Y). This theorem suggests that "local properties" of Banach spaces, i.e. properties whose definition involves statements about finitely many vectors, have a purely metric characterization. Finding explicit manifestations of this phenomenon for specific local properties of Banach spaces (such as type, cotype and super-reflexivity), has long been a major driving force in the bi-Lipschitz theory of metric spaces (see Bourgain's paper [8] for a discussion of this research program). Indeed, as will become clear below, the search for concrete versions of Ribe's theorem has fueled some of the field's most important achievements.

The notions of type and cotype of Banach spaces are the basis of a deep and rich theory which encompasses diverse aspects of the local theory of Banach spaces. We refer to [50], [59], [58], [68], [60], [36], [15], [71], [45] and the references therein for background on these topics. A Banach space X is said to have (Rademacher) type $p > 0$ if there exists a constant $T < \infty$ such that for every n and every $x_1, \dots, x_n \in X$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^p \le T^p \sum_{j=1}^n \|x_j\|_X^p. \tag{1}$$

where the expectation $\mathbb{E}_\varepsilon$ is with respect to a uniform choice of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$. X is said to have (Rademacher) cotype $q > 0$ if there exists a constant $C < \infty$ such that for every n and every $x_1, \dots, x_n \in X$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_X^q \ge \frac{1}{C^q} \sum_{j=1}^n \|x_j\|_X^q. \tag{2}$$

These notions are clearly *linear* notions, since their definition involves addition and multiplication by scalars. Ribe's theorem implies that these notions are preserved under uniform homeomorphisms of Banach spaces, and therefore it would be desirable to reformulate them using only distances between points in the given Banach space. Once this is achieved, one could define the notion of type and cotype of a metric space, and then hopefully transfer some of the deep theory of type and cotype to the context of arbitrary metric spaces. The need for such a theory has recently received renewed impetus due to the discovery of striking applications of metric geometry to theoretical computer science (see [44], [28], [41] and the references therein for part of the recent developments in this direction).

Enflo's pioneering work [18], [19], [20], [21] resulted in the formulation of a nonlinear notion of type, known today as *Enflo type*. The basic idea is that given a Banach space X and $x_1, \dots, x_n \in X$, one can consider the *linear* function $f : \{-1,1\}^n \to X$ given by $f(\varepsilon) = \sum_{j=1}^n \varepsilon_j x_j$. Then (1) becomes

$$\begin{aligned}\mathbb{E}_\varepsilon \|f(\varepsilon) - f(-\varepsilon)\|_X^p \le T^p \sum_{j=1}^n \mathbb{E}_\varepsilon \Big\| f(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n) \\ - f(\varepsilon_1, \dots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n)\Big\|_X^p.\end{aligned} \tag{3}$$

One can thus say that a metric space $(\mathcal{M}, d_\mathcal{M})$ has Enflo type p if there exists a constant T such that for every $n \in \mathbb{N}$ and *every* $f : \{-1,1\}^n \to \mathcal{M}$,

$$\begin{aligned}\mathbb{E}_\varepsilon\, d_\mathcal{M}(f(\varepsilon), f(-\varepsilon))^p \le T^p \sum_{j=1}^n \mathbb{E}_\varepsilon\, d_\mathcal{M}\Big(f(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n), \\ f(\varepsilon_1, \dots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n)\Big)^p.\end{aligned} \tag{4}$$

There are two natural concerns about this definition. First of all, while in the category of Banach spaces (4) is clearly a strengthening of (3) (as we are not restricting only to linear functions f), it isn't clear whether (4) follows from (3). Indeed, this problem was posed by Enflo in [21], and in full generality it remains open. Secondly, we do not know if (4) is a useful notion, in the sense that it yields metric variants of certain theorems from the linear theory of type (it should be remarked here that Enflo found striking applications of his notion of type to Hilbert's fifth problem in infinite dimensions [19], [20], [21], and to the uniform classification of L_p spaces [18]). As we will presently see, in a certain sense both of these issues turned out not to be problematic. Variants of Enflo type were studied by Gromov [24] and Bourgain, Milman and Wolfson [11]. Following [11] we shall say that a metric space $(\mathcal{M}, d_{\mathcal{M}})$ has BMW type $p > 0$ if there exists a constant $K < \infty$ such that for every $n \in \mathbb{N}$ and every $f : \{-1, 1\}^n \to \mathcal{M}$,

$$\begin{aligned}\mathbb{E}_\varepsilon\, d_{\mathcal{M}}(f(\varepsilon), f(-\varepsilon))^2 \le K^2 n^{\frac{2}{p}-1} \sum_{j=1}^n \mathbb{E}_\varepsilon\, d_{\mathcal{M}}\Big(&f(\varepsilon_1, \ldots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n),\\ &f(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n)\Big)^2.\end{aligned} \tag{5}$$

Bourgain, Milman and Wolfson proved in [11] that if a Banach space has BMW type $p > 0$ then it also has Rademacher type p' for all $0 < p' < p$. They also obtained a nonlinear version of the Maurey-Pisier theorem for type [55], [46], yielding a characterization of metric spaces which contain bi-Lipschitz copies of the Hamming cube. In [59] Pisier proved that for Banach spaces, Rademacher type p implies Enflo type p' for every $0 < p' < p$. Variants of these problems were studied by Naor and Schechtman in [53]. A stronger notion of nonlinear type, known as Markov type, was introduced by Ball [4] in his study of the *Lipschitz extension problem*. This important notion has since found applications to various fundamental problems in metric geometry [51], [42], [5], [52], [48]

Despite the vast amount of research on nonlinear type, a nonlinear notion of cotype remained elusive. Indeed, the problem of finding a notion of cotype which makes sense for arbitrary metric spaces, and which coincides (or almost coincides) with the notion of Rademacher type when restricted to Banach spaces, became a central open problem in the field.

There are several difficulties involved in defining nonlinear cotype. First of all, one cannot simply reverse inequalities (4) and (5), since the resulting condition fails to hold true even for Hilbert space (with $p = 2$). Secondly, if Hilbert space satisfies an inequality such as (4), then it must satisfy the same inequality where the distances are raised to any power $0 < r < p$. This is because Hilbert space, equipped with the metric $\|x - y\|^{r/p}$, is isometric to a subset of Hilbert space (see [65], [70]). In the context of nonlinear type, this observation makes perfect sense, since if a Banach space has type p then it also has type r for every $0 < r < p$. But, this is no longer true for cotype (in particular, no Banach space has cotype less than 2). One viable definition of cotype of a metric space X that was suggested in the early 1980s is the following: Let $\mathcal{M}$ be a metric space, and denote by $\mathrm{Lip}(\mathcal{M})$ the Banach space of all real-valued Lipschitz functions on $\mathcal{M}$, equipped with the Lipschitz norm. One can then define the nonlinear cotype of $\mathcal{M}$ as the (Rademacher) cotype of the (linear) dual $\mathrm{Lip}(\mathcal{M})^*$. This is a natural definition when $\mathcal{M}$ is a Banach space, since we can view $\mathrm{Lip}(\mathcal{M})$ as a nonlinear substitute for the dual space $\mathcal{M}^*$ (note that in [37] it is shown that there is a norm 1 projection from $\mathrm{Lip}(\mathcal{M})$ onto $\mathcal{M}^*$). With this point of view, the above definition of cotype is natural due to the principle of local reflexivity [39], [30]. Unfortunately, Bourgain [8] has shown that under this definition subsets of L_1 need not have finite nonlinear cotype (while L_1 has cotype 2). Additionally, the space $\mathrm{Lip}(M)^*$ is very hard to compute: for example it is an intriguing open problem whether even the unit square $[0,1]^2$ has nonlinear cotype 2 under the above definition.

In this paper we introduce a notion of cotype of metric spaces, and show that it coincides with Rademacher cotype when restricted to the category of Banach spaces. Namely, we introduce the following concept:

Definition 1.1 (Metric cotype). Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space and $q > 0$. The space $(\mathcal{M}, d_{\mathcal{M}})$ is said to have *metric cotype q with constant* Γ if for every integer $n \in \mathbb{N}$, there exists an even integer m, such that for every $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$\sum_{j=1}^{n} \mathbb{E}_x \left[d_{\mathcal{M}} \left(f\left(x + \frac{m}{2} e_j\right), f(x) \right)^q \right] \le \Gamma^q m^q \, \mathbb{E}_{\varepsilon,x} \left[d_{\mathcal{M}}(f(x+\varepsilon), f(x))^q \right], \tag{6}$$

where the expectations above are taken with respect to uniformly chosen $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,0,1\}^n$ (here, and in what follows we denote by $\{e_j\}_{j=1}^n$ the standard basis of $\mathbb{R}^n$). The smallest constant Γ with which inequality (6) holds true is denoted $\Gamma_q(\mathcal{M})$.

Several remarks on Definition 1.1 are in order. First of all, in the case of Banach spaces, if we apply inequality (6) to linear functions $f(x) = \sum_{j=1}^n x_j v_j$, then by homogeneity m would cancel, and the resulting inequality will simply become the Rademacher cotype q condition (this statement is not precise due to the fact that addition on $\mathbb{Z}_m^n$ is performed modulo m — see Section 5.1 for the full argument). Secondly, it is easy to see that in any metric space which contains at least two points, inequality (6) forces the scaling factor m to be large (see Lemma 2.3) — this is an essential difference between Enflo type and metric cotype. Finally, the averaging over $\varepsilon \in \{-1,0,1\}^n$ is natural here, since this forces the right-hand side of (6) to be a uniform average over all pairs in $\mathbb{Z}_m^n$ whose distance is at most 1 in the ℓ_∞ metric.

The following theorem is the main result of this paper:

THEOREM 1.2. *Let X be a Banach space, and $q \in [2, \infty)$. Then X has metric cotype q if and only if X has Rademacher cotype q. Moreover,*

$$\frac{1}{2\pi} C_q(X) \leq \Gamma_q(X) \leq 90 C_q(X).$$

Apart from settling the nonlinear cotype problem described above, this notion has various applications. Thus, in the remainder of this paper we proceed to study metric cotype and some of its applications, which we describe below. We believe that additional applications of this notion and its variants will be discovered in the future. In particular, it seems worthwhile to study the interaction between metric type and metric cotype (such as in Kwapien's theorem [35]), the possible "Markov" variants of metric cotype (à la Ball [4]) and their relation to the Lipschitz extension problem, and the relation between metric cotype and the nonlinear Dvoretzky theorem (see [10], [5] for information about the nonlinear Dvoretzky theorem, and [22] for the connection between cotype and Dvoretzky's theorem).

1.1. *Some applications of metric cotype.*

1) *A nonlinear version of the Maurey-Pisier theorem.* Given two metric spaces $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$, and an injective mapping $f : \mathcal{M} \hookrightarrow \mathcal{N}$, we denote the *distortion* of f by

$$\operatorname{dist}(f) := \|f\|_{\mathrm{Lip}} \cdot \|f^{-1}\|_{\mathrm{Lip}} = \sup_{\substack{x,y \in \mathcal{M} \\ x \neq y}} \frac{d_{\mathcal{N}}(f(x), f(y))}{d_{\mathcal{M}}(x,y)} \cdot \sup_{\substack{x,y \in \mathcal{M} \\ x \neq y}} \frac{d_{\mathcal{M}}(x,y)}{d_{\mathcal{N}}(f(x), f(y))}.$$

The smallest distortion with which $\mathcal{M}$ can be embedded into $\mathcal{N}$ is denoted $c_{\mathcal{N}}(\mathcal{M})$; i.e.,

$$c_{\mathcal{N}}(\mathcal{M}) := \inf\{\operatorname{dist}(f) : \ f : \mathcal{M} \hookrightarrow \mathcal{N}\}.$$

If $c_{\mathcal{N}}(\mathcal{M}) \leq \alpha$ then we sometimes use the notation $\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{N}$. When $\mathcal{N} = L_p$ for some $p \geq 1$, we write $c_{\mathcal{N}}(\cdot) = c_p(\cdot)$.

For a Banach space X write

$$p_X = \sup\{p \geq 1 : \ T_p(X) < \infty\} \quad \text{and} \quad q_X = \inf\{q \geq 2 : \ C_q(X) < \infty\}.$$

X is said to have nontrivial type if $p_X > 1$, and X is said to have nontrivial cotype if $q_X < \infty$.

In [55] Pisier proved that X has no nontrivial type if and only if for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, $\ell_1^n \overset{1+\varepsilon}{\hookrightarrow} X$. A nonlinear analog of this result was proved by Bourgain, Milman and Wolfson [11] (see also Pisier's exposition in [59]). They showed that a metric space $\mathcal{M}$ does not have BMW type larger than 1 if and only if for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, $(\{0,1\}^n, \|\cdot\|_1) \overset{1+\varepsilon}{\hookrightarrow} \mathcal{M}$. In [46] Maurey and Pisier proved that a Banach space X has no nontrivial cotype if and only for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, $\ell_\infty^n \overset{1+\varepsilon}{\hookrightarrow} X$. To obtain a nonlinear analog of this theorem we need to introduce a variant of metric cotype (which is analogous to the variant of Enflo type that was used in [11].

Definition 1.3 (Variants of metric cotype à la Bourgain, Milman and Wolfson). Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space and $1 \le p \le q$. We denote by $\Gamma_q^{(p)}(\mathcal{M})$ the least constant Γ such that for every integer $n \in \mathbb{N}$ there exists an even integer m, such that for every $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$(7) \quad \sum_{j=1}^{n} \mathbb{E}_x \left[d_{\mathcal{M}} \left(f\left(x + \frac{m}{2} e_j \right), f(x) \right)^p \right] \\ \le \Gamma^p m^p n^{1-\frac{p}{q}} \, \mathbb{E}_{\varepsilon, x} \left[d_{\mathcal{M}}(f(x+\varepsilon), f(x))^p \right],$$

where the expectations above are taken with respect to uniformly chosen $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1, 0, 1\}^n$. Note that $\Gamma_q^{(q)}(\mathcal{M}) = \Gamma_q(\mathcal{M})$. When $1 \le p < q$ we shall refer to (7) as a weak metric cotype q inequality with exponent p and constant Γ.

The following theorem is analogous to Theorem 1.2.

THEOREM 1.4. *Let X be a Banach space, and assume that for some $1 \le p < q$, $\Gamma_q^{(p)}(X) < \infty$. Then X has cotype q' for every $q' > q$. If $q = 2$ then X has cotype 2. On the other hand,*

$$\Gamma_q^{(p)}(X) \le c_{pq} C_q(X),$$

where c_{pq} is a universal constant depending only on p and q.

In what follows, for $m, n \in \mathbb{N}$ and $p \in [1, \infty]$ we let $[m]_p^n$ denote the set $\{0, 1, \dots, m\}^n$, equipped with the metric induced by ℓ_p^n. The following theorem is a metric version of the Maurey-Pisier theorem (for cotype):

THEOREM 1.5. *Let $\mathcal{M}$ be a metric space such that $\Gamma_q^{(2)}(\mathcal{M}) = \infty$ for all $q < \infty$. Then for every $m, n \in \mathbb{N}$ and every $\varepsilon > 0$,*

$$[m]_\infty^n \overset{1+\varepsilon}{\hookrightarrow} \mathcal{M}.$$

We remark that in [46] Maurey and Pisier prove a stronger result, namely that for a Banach space X, for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, $\ell_{p_X}^n \overset{1+\varepsilon}{\hookrightarrow} X$ and $\ell_{q_X}^n \overset{1+\varepsilon}{\hookrightarrow} X$. Even in the case of nonlinear type, the results of Bourgain, Milman and Wolfson yield an incomplete analog of this result in the case of BMW type greater than 1. The same phenomenon seems to occur when one tries to obtain a nonlinear analog of the full Maurey-Pisier theorem for cotype. We believe that this issue deserves more attention in future research.

2) *Solution of a problem posed by Arora, Lovász, Newman, Rabani, Rabinovich and Vempala.* The following question appears in [3, Conj. 5.1]:

> Let $\mathcal{F}$ be a *baseline* metric class which does not contain all finite metrics with distortion arbitrarily close to 1. Does this imply that there exists $\alpha > 0$ and arbitrarily large n-point metric spaces $\mathcal{M}_n$ such that for every $\mathcal{N} \in \mathcal{F}$, $c_{\mathcal{N}}(\mathcal{M}_n) \geq (\log n)^\alpha$?

We refer to [3, §2] for the definition of baseline metrics, since we will not use this notion in what follows. We also refer to [3] for background and motivation from combinatorial optimization for this problem, where several partial results in this direction are obtained. An extended abstract of the current paper [49] also contains more information on the connection to Computer Science. Here we apply metric cotype to settle this conjecture positively, without any restriction on the class $\mathcal{F}$.

To state our result we first introduce some notation. If $\mathcal{F}$ is a family of metric spaces we write

$$c_{\mathcal{F}}(\mathcal{N}) = \inf \left\{ c_{\mathcal{M}}(\mathcal{N}) : \mathcal{M} \in \mathcal{F} \right\}.$$

For an integer $n \geq 1$ we define

$$\mathcal{D}_n(\mathcal{F}) = \sup\{c_{\mathcal{F}}(\mathcal{N}) :\ \mathcal{N} \text{ is a metric space, } |\mathcal{N}| \leq n\}.$$

Observe that if, for example, $\mathcal{F}$ consists of all the subsets of Hilbert space (or L_1), then Bourgain's embedding theorem [7] implies that $\mathcal{D}_n(\mathcal{F}) = O(\log n)$.

For $K > 0$ we define the K-cotype (with exponent 2) of a family of metric spaces $\mathcal{F}$ as

$$q_{\mathcal{F}}^{(2)}(K) = \sup_{\mathcal{M} \in \mathcal{F}} \inf \left\{ q \in (0,\infty] :\ \Gamma_q^{(2)}(\mathcal{M}) \leq K \right\}.$$

Finally we let

$$q_{\mathcal{F}}^{(2)} = \inf_{\infty > K > 0} q_{\mathcal{F}}^{(2)}(K).$$

The following theorem settles positively the problem stated above:

Theorem 1.6. *Let $\mathcal{F}$ be a family of metric spaces. Then the following conditions are equivalent*:

1. *There exists a finite metric space $\mathcal{M}$ for which $c_{\mathcal{F}}(\mathcal{M}) > 1$.*
2. $q_{\mathcal{F}}^{(2)} < \infty$.
3. *There exists $0 < \alpha < \infty$ such that $\mathcal{D}_n(\mathcal{F}) = \Omega\left((\log n)^\alpha\right)$.*

3) *A quantitative version of Matoušek's* BD *Ramsey theorem.* In [43] Matoušek proved the following result, which he calls the Bounded Distortion (BD) Ramsey theorem. We refer to [43] for motivation and background on these types of results.

THEOREM 1.7 (Matoušek's BD Ramsey theorem). *Let X be a finite metric space and $\varepsilon > 0$, $\gamma > 1$. Then there exists a metric space $Y = Y(X, \varepsilon, \gamma)$, such that for every metric space Z,*

$$c_Z(Y) < \gamma \implies c_Z(X) < 1 + \varepsilon.$$

We obtain a new proof of Theorem 1.7, which is quantitative and concrete:

THEOREM 1.8 (Quantitative version of Matoušek's BD Ramsey theorem). *There exists a universal constant C with the following properties. Let X be an n-point metric space and $\varepsilon \in (0,1)$, $\gamma > 1$. Then for every integer $N \geq (C\gamma)^{2^{5A}}$, where*

$$A = \max\left\{\frac{4\operatorname{diam}(X)}{\varepsilon \cdot \min_{x\neq y} d_X(x,y)}, n\right\},$$

if a metric space Z satisfies $c_Z(X) > 1 + \varepsilon$ then, $c_Z\left(\left[N^5\right]_\infty^N\right) > \gamma$.

We note that Matoušek's argument in [43] uses Ramsey theory, and is nonconstructive (at best it can yield tower-type bounds on the size of Z, which are much worse than what the cotype-based approach gives).

4) *Uniform embeddings and Smirnov's problem.* Let $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$ be metric spaces. A mapping $f : \mathcal{M} \to \mathcal{N}$ is called a *uniform embedding* if f is injective, and both f and f^{-1} are uniformly continuous. There is a large body of work on the uniform classification of metric spaces — we refer to the survey article [38], the book [6], and the references therein for background on this topic. In spite of this, several fundamental questions remain open. For example, it was not known for which values of $0 < p,\ q < \infty$, L_p embeds uniformly into L_q. As we will presently see, our results yield a complete characterization of these values of p, q.

In the late 1950's Smirnov asked whether every separable metric space embeds uniformly into L_2 (see [23]). Smirnov's problem was settled negatively by Enflo in [17]. Following Enflo, we shall say that a metric space $\mathcal{M}$ is a *universal uniform embedding space* if every separable metric space embeds uniformly into $\mathcal{M}$. Since every separable metric space is isometric to a subset of $C[0,1]$, this is equivalent to asking whether $C[0,1]$ is uniformly homeomorphic to a subset of $\mathcal{M}$ (the space $C[0,1]$ can be replaced here by c_0 due to Aharoni's theorem [1]). Enflo proved that c_0 does not uniformly embed into Hilbert space. In [2], Aharoni, Maurey and Mityagin systematically studied metric spaces which are uniformly homeomorphic to a subset of Hilbert space, and obtained an elegant characterization of Banach spaces which are uniformly homeomorphic to a subset of L_2. In particular, the results of [2] imply that for $p > 2$, L_p is not uniformly homeomorphic to a subset of L_2.

Here we prove that in the class of Banach spaces with nontrivial type, if Y embeds uniformly into X, then Y inherits the cotype of X. More precisely:

THEOREM 1.9. *Let X be a Banach space with nontrivial type. Assume that Y is a Banach space which uniformly embeds into X. Then $q_Y \le q_X$.*

As a corollary, we complete the characterization of the values of $0 < p$, $q < \infty$ for which L_p embeds uniformly into L_q:

THEOREM 1.10. *For $p, q > 0$, L_p embeds uniformly into L_q if and only if $p \le q$ or $q \le p \le 2$.*

We believe that the assumption that X has nontrivial type in Theorem 1.9 can be removed — in Section 8 we present a concrete problem which would imply this fact. If true, this would imply that cotype is preserved under uniform embeddings of Banach spaces. In particular, it would follow that a universal uniform embedding space cannot have nontrivial cotype, and thus by the Maurey-Pisier theorem [46] it must contain ℓ_∞^n's with distortion uniformly bounded in n.

5) *Coarse embeddings.* Let $(\mathcal{M}, d_\mathcal{M})$ and $(\mathcal{N}, d_\mathcal{N})$ be metric spaces. A mapping $f : \mathcal{M} \to \mathcal{N}$ is called a *coarse embedding* if there exists two nondecreasing functions $\alpha, \beta : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} \alpha(t) = \infty$, and for every $x, y \in \mathcal{M}$,

$$\alpha(d_\mathcal{M}(x, y)) \le d_\mathcal{N}(f(x), f(y)) \le \beta(d_\mathcal{M}(x, y)).$$

This (seemingly weak) notion of embedding was introduced by Gromov (see [25]), and has several important geometric applications. In particular, Yu [72] obtained a striking connection between the Novikov and Baum-Connes conjectures and coarse embeddings into Hilbert spaces. In [33] Kasparov and Yu generalized this to coarse embeddings into arbitrary uniformly convex Banach spaces. It was unclear, however, whether this is indeed a strict generalization, i.e. whether or not the existence of a coarse embedding into a uniformly convex Banach space implies the existence of a coarse embedding into a Hilbert space. This was resolved by Johnson and Randrianarivony in [29], who proved that for $p > 2$, L_p does not coarsely embed into L_2. In [61], Randrianarivony proceeded to obtain a characterization of Banach spaces which embed coarsely into L_2, in the spirit of the result of Aharoni, Maurey and Mityagin [2]. There are very few known methods of proving coarse nonembeddability results. Apart from the papers [29], [61] quoted above, we refer to [26], [16], [54] for results of this type. Here we use metric cotype to prove the following coarse variants of Theorem 1.9 and Theorem 1.10, which generalize, in particular, the theorem of Johnson and Randrianarivony.

THEOREM 1.11. *Let X be a Banach space with nontrivial type. Assume that Y is a Banach space which coarsely embeds into X. Then $q_Y \le q_X$. In* *particular, for $p, q > 0$, L_p embeds coarsely into L_q if and only if $p \le q$ or $q \le p \le 2$.*

6) *Bi-Lipschitz embeddings of the integer lattice.* Bi-Lipschitz embeddings of the integer lattice $[m]_p^n$ were investigated by Bourgain in [9] and by the present authors in [48] where it was shown that if $2 \le p < \infty$ and Y is a Banach space which admits an equivalent norm whose modulus of uniform convexity has power type 2, then

$$c_Y\left([m]_p^n\right) = \Theta\left(\min\left\{n^{\frac{1}{2}-\frac{1}{p}}, m^{1-\frac{2}{p}}\right\}\right). \tag{8}$$

The implied constants in the above asymptotic equivalence depend on p and on the 2-convexity constant of Y. Moreover, it was shown in [48] that

$$c_Y([m]_\infty^n) = \Omega\left(\min\left\{\sqrt{\frac{n}{\log n}}, \frac{m}{\sqrt{\log m}}\right\}\right).$$

It was conjectured in [48] that the logarithmic terms above are unnecessary. Using our results on metric cotype we settle this conjecture positively, by proving the following general theorem:

Theorem 1.12. *Let Y be a Banach space with nontrivial type which has cotype q. Then*

$$c_Y([m]_\infty^n) = \Omega\left(\min\left\{n^{1/q}, m\right\}\right).$$

Similarly, our methods imply that (8) holds true for any Banach space Y with nontrivial type and cotype 2 (note that these conditions are strictly weaker than being 2-convex, as shown e.g. in [40]). Moreover, it is possible to generalize the lower bound in (8) to Banach spaces with nontrivial type, and cotype $2 \le q \le p$, in which case the lower bound becomes $\min\left\{n^{\frac{1}{q}-\frac{1}{p}}, m^{1-\frac{q}{p}}\right\}$.

7) *Quadratic inequalities on the cut-cone.* An intriguing aspect of Theorem 1.2 is that L_1 has metric cotype 2. Thus, we obtain a nontrivial inequality on L_1 which involves distances *squared*. To the best of our knowledge, all the known nonembeddability results for L_1 are based on Poincaré type inequalities in which distances are raised to the power 1. Clearly, any such inequality reduces to an inequality on the real line. Equivalently, by the cut-cone representation of L_1 metrics (see [14]) it is enough to prove any such inequality for *cut metrics*, which are particularly simple. Theorem 1.2 seems to be the first truly "infinite dimensional" metric inequality in L_1, in the sense that its nonlinearity does not allow a straightforward reduction to the one-dimensional case. We believe that understanding such inequalities on L_1 deserves further scrutiny, especially as they hint at certain nontrivial (and nonlinear) interactions between cuts.

2. Preliminaries and notation

We start by setting notation and conventions. Consider the standard ℓ_∞ Cayley graph on $\mathbb{Z}_m^n$, namely $x, y \in \mathbb{Z}_m^n$ are joined by an edge if and only if they are distinct and $x - y \in \{-1,0,1\}^n$. This induces a shortest-path metric on $\mathbb{Z}_m^n$ which we denote by $d_{\mathbb{Z}_m^n}(\cdot,\cdot)$. Equivalently, the metric space $(\mathbb{Z}_m^n, d_{\mathbb{Z}_m^n})$ is precisely the quotient $(\mathbb{Z}^n, \|\cdot\|_\infty)/(m\mathbb{Z})^n$ (for background on quotient metrics see [13], [25]). The ball of radius r around $x \in \mathbb{Z}_m^n$ will be denoted $B_{\mathbb{Z}_m^n}(x,r)$. We denote by μ the normalized counting measure on $\mathbb{Z}_m^n$ (which is clearly the Haar measure on this group). We also denote by σ the normalized counting measure on $\{-1,0,1\}^n$. In what follows, whenever we average over uniformly chosen signs $\varepsilon \in \{-1,1\}^n$ we use the probabilistic notation $\mathbb{E}_\varepsilon$ (in this sense we break from the notation used in the introduction, for the sake of clarity of the ensuing arguments).

In what follows all Banach spaces are assumed to be over the complex numbers $\mathbb{C}$. All of our results hold for real Banach spaces as well, by a straightforward complexification argument.

Given a Banach space X and $p, q \in [1,\infty)$ we denote by $C_q^{(p)}(X)$ the infimum over all constants $C > 0$ such that for every integer $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in X$,

$$\left(\mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j x_j\right\|_X^p\right)^{1/p} \ge \frac{1}{C}\left(\sum_{j=1}^n \|x_j\|_X^q\right)^{1/q}. \tag{9}$$

Thus, by our previous notation, $C_q^{(q)}(X) = C_q(X)$. Kahane's inequality [31] says that for $1 \le p, q < \infty$ there exists a constant $1 \le A_{pq} < \infty$ such that for every Banach space X, every integer $n \in \mathbb{N}$, and every $x_1, \ldots, x_n \in X$,

$$\left(\mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j x_j\right\|_X^p\right)^{1/p} \le A_{pq}\left(\mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j x_j\right\|_X^q\right)^{1/q}. \tag{10}$$

Where clearly $A_{pq} = 1$ if $p \le q$, and for every $1 \le q < p < \infty$, $A_{pq} = O\left(\sqrt{p}\right)$ (see [66]). It follows in particular from (10) that if X has cotype q then for every $p \in [1,\infty)$, $C_q^{(p)}(X) = O_{p,q}(C_q(X))$, where the implied constant may depend on p and q.

Given $A \subseteq \{1, \ldots, n\}$, we consider the Walsh functions $W_A : \{-1,1\}^n \to \mathbb{C}$, defined as

$$W_A(\varepsilon_1, \ldots, \varepsilon_m) = \prod_{j\in A} \varepsilon_j.$$

Every $f : \{-1,1\}^n \to X$ can be written as

$$f(\varepsilon_1, \ldots, \varepsilon_n) = \sum_{A\subseteq\{1,\ldots,n\}} \widehat{f}(A) W_A(\varepsilon_1, \ldots, \varepsilon_n),$$

where $\widehat{f}(A) \in X$ are given by

$$\widehat{f}(A) = \mathbb{E}_\varepsilon\Big(f(\varepsilon)W_A(\varepsilon)\Big).$$

The *Rademacher projection* of f is defined by

$$\mathbf{Rad}(f) = \sum_{j=1}^{n} \widehat{f}(A)W_{\{j\}}.$$

The K-convexity constant of X, denoted $K(X)$, is the smallest constant K such that for every n and every $f : \{-1,1\}^n \to X$,

$$\mathbb{E}_\varepsilon \|\mathbf{Rad}(f)(\varepsilon)\|_X^2 \le K^2\, \mathbb{E}_\varepsilon \|f(\varepsilon)\|_X^2.$$

In other words,

$$K(X) = \sup_{n\in\mathbb{N}} \|\mathbf{Rad}\|_{L_2(\{-1,1\}^n,X)\to L_2(\{-1,1\}^n,X)}.$$

X is said to be K-convex if $K(X) < \infty$. More generally, for $p \ge 1$ we define

$$K_p(X) = \sup_{n\in\mathbb{N}} \|\mathbf{Rad}\|_{L_p(\{-1,1\}^n,X)\to L_p(\{-1,1\}^n,X)}.$$

It is a well known consequence of Kahane's inequality and duality that for every $p > 1$,

$$K_p(X) \le O\left(\frac{p}{\sqrt{p-1}}\right)\cdot K(X).$$

The following deep theorem was proved by Pisier in [57]:

Theorem 2.1 (Pisier's K-convexity theorem [57]). *Let X be a Banach space. Then*

$$q_X > 1 \iff K(X) < \infty.$$

Next, we recall some facts concerning Fourier analysis on the group $\mathbb{Z}_m^n$. Given $k = (k_1, \dots, k_n) \in \mathbb{Z}_m^n$ we consider the Walsh function $W_k : \mathbb{Z}_m^n \to \mathbb{C}$:

$$W_k(x) = \exp\left(\frac{2\pi i}{m}\sum_{j=1}^{m} k_j x_j\right).$$

Then, for any Banach space X, any $f : Z_m^n \to X$ can be decomposed as follows:

$$f(x) = \sum_{k\in\mathbb{Z}_m^n} W_k(x)\widehat{f}(k),$$

where

$$\widehat{f}(k) = \int_{\mathbb{Z}_m^n} f(y)\overline{W_k(y)}d\mu(y) \in X.$$

If X is a Hilbert space then Parseval's identity becomes:

$$\int_{Z_m^n} \|f(x)\|_X^2 d\mu(x) = \sum_{k\in\mathbb{Z}_m^n} \left\|\widehat{f}(k)\right\|_X^2.$$

2.1. *Definitions and basic facts related to metric cotype.*

Definition 2.2. Given $1 \le p \le q$, an integer n and an even integer m, let $\Gamma_q^{(p)}(\mathcal{M};n,m)$ be the infimum over all $\Gamma > 0$ such that for every $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right), f(x)\right)^p d\mu(x) \le \Gamma^p m^p n^{1-\frac{p}{q}} \int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\varepsilon\right), f(x)\right)^p d\mu(x)d\sigma(\varepsilon). \tag{11}$$

When $p = q$ we write $\Gamma_q(\mathcal{M};n,m) := \Gamma_q^{(q)}(\mathcal{M};n,m)$. With this notation,

$$\Gamma_q^{(p)}(\mathcal{M}) = \sup_{n\in\mathbb{N}} \inf_{m\in 2\mathbb{N}} \Gamma_q^{(p)}(\mathcal{M};n,m).$$

We also denote by $m_q^{(p)}(\mathcal{M};n,\Gamma)$ the smallest even integer m for which (11) holds. As usual, when $p = q$ we write $m_q(\mathcal{M};n,\Gamma) := m_q^{(q)}(\mathcal{M};n,\Gamma)$.

The following lemma shows that for nontrivial metric spaces $\mathcal{M}$, $m_q(\mathcal{M};n,\Gamma)$ must be large.

Lemma 2.3. *Let $(\mathcal{M},d_{\mathcal{M}})$ be a metric space which contains at least two points. Then for every integer n, every $\Gamma > 0$, and every $p,q > 0$,*

$$m_q^{(p)}(\mathcal{M};n,\Gamma) \ge \frac{n^{1/q}}{\Gamma}.$$

Proof. Fix $u,v \in \mathcal{M}$, $u \ne v$, and without loss of generality normalize the metric so that $d_{\mathcal{M}}(u,v) = 1$. Denote $m = m_q^{(p)}(\mathcal{M};n,\Gamma)$. Let $f : \mathbb{Z}_m^n \to \mathcal{M}$ be the random mapping such that for every $x \in \mathbb{Z}_m^n$, $\Pr[f(x) = u] = \Pr[f(x) = v] = \frac{1}{2}$, and $\{f(x)\}_{x\in\mathbb{Z}_m^n}$ are independent random variables. Then for every distinct $x,y \in \mathbb{Z}_m^n$, $\mathbb{E}\left[d_{\mathcal{M}}(f(x),f(y))^p\right] = \frac{1}{2}$. Thus, the required result follows by applying (11) to f and taking expectation. □

Lemma 2.4. *For every two integers n,k, and every even integer m,*

$$\Gamma_q^{(p)}(\mathcal{M};n,km) \le \Gamma_q^{(p)}(\mathcal{M};n,m).$$

Proof. Fix $f : \mathbb{Z}_{km}^n \to \mathcal{M}$. For every $y \in \mathbb{Z}_k^n$ define $f_y : \mathbb{Z}_m^n \to \mathcal{M}$ by

$$f_y(x) = f(kx+y).$$

Fix $\Gamma > \Gamma_q^{(p)}(\mathcal{M};n,m)$. Applying the definition of $\Gamma_q^{(p)}(\mathcal{M};n,m)$ to f_y, we get that

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(kx+\frac{km}{2}e_j+y\right), f(kx+y)\right)^p d\mu_{\mathbb{Z}_m^n}(x)$$
$$\le \Gamma^p m^p n^{1-\frac{p}{q}} \int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(kx+k\varepsilon+y\right), f(kx+y)\right)^p d\mu_{\mathbb{Z}_m^n}(x)d\sigma(\varepsilon).$$

Integrating this inequality with respect to $y \in \mathbb{Z}_k^n$ we see that

$$\sum_{j=1}^n \int_{\mathbb{Z}_{km}^n} d_{\mathcal{M}}\left(f\left(z+\frac{km}{2}e_j\right), f(z)\right)^p d\mu_{\mathbb{Z}_{km}^n}(z)$$
$$=\sum_{j=1}^n \int_{\mathbb{Z}_k^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(kx+\frac{km}{2}e_j+y\right), f(kx+y)\right)^p d\mu_{\mathbb{Z}_m^n}(x)d\mu_{\mathbb{Z}_k^n}(y)$$
$$\le \Gamma^p m^p n^{1-\frac{p}{q}}\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_k^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(kx+k\varepsilon+y), f(kx+y))^p d\mu_{\mathbb{Z}_m^n}(x)d\mu_{\mathbb{Z}_k^n}(y)d\sigma(\varepsilon)$$
$$=\Gamma^p m^p n^{1-\frac{p}{q}}\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_{km}^n} d_{\mathcal{M}}\left(f\left(z+k\varepsilon\right), f(z)\right)^p d\mu_{\mathbb{Z}_{km}^n}(z)d\sigma(\varepsilon)$$
$$\le \Gamma^p m^p n^{1-\frac{p}{q}}\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_{km}^n} k^{p-1}\sum_{s=1}^k d_{\mathcal{M}}\left(f\left(z+s\varepsilon\right), f(z+(s-1)\varepsilon)\right)^p d\mu_{\mathbb{Z}_{km}^n}(z)d\sigma(\varepsilon)$$
$$=\Gamma^p (km)^p n^{1-\frac{p}{q}}\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_{km}^n} d_{\mathcal{M}}\left(f\left(z+\varepsilon\right), f(z)\right)^p d\mu_{\mathbb{Z}_{km}^n}(z)d\sigma(\varepsilon). \qquad \square$$

LEMMA 2.5. *Let k,n be integers such that $k \le n$, and let m be an even integer. Then*

$$\Gamma_q^{(p)}(\mathcal{M};k,m) \le \left(\frac{n}{k}\right)^{1-\frac{p}{q}} \cdot \Gamma_q^{(p)}(\mathcal{M};n,m).$$

Proof. Given an $f:\mathbb{Z}_m^k \to \mathcal{M}$, we define an $\mathcal{M}$-valued function on $\mathbb{Z}_m^n \cong \mathbb{Z}_m^k \times \mathbb{Z}_m^{n-k}$ by $g(x,y)=f(x)$. Applying the definition $\Gamma_q^{(p)}(\mathcal{M};n,m)$ to g yields the required inequality. $\square$

We end this section by recording some general inequalities which will be used in the ensuing arguments. In what follows $(\mathcal{M}, d_{\mathcal{M}})$ is an arbitrary metric space.

LEMMA 2.6. *For every $f:\mathbb{Z}_m^n \to \mathcal{M}$,*

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+e_j), f(x)\right)^p d\mu(x)$$
$$\le 3\cdot 2^{p-1} n \cdot \int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+\varepsilon), f(x)\right)^p d\mu(x)d\sigma(\varepsilon).$$

Proof. For every $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,0,1\}^n$,

$$d_{\mathcal{M}}(f(x+e_j), f(x))^p \le 2^{p-1} d_{\mathcal{M}}(f(x+e_j), f(x+\varepsilon))^p + 2^{p-1} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^p.$$

Thus

$$\begin{aligned}
&\frac{2}{3}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+e_j), f(x)\right)^p d\mu(x) \\
&\quad = \sigma(\{\varepsilon \in \{-1,0,1\}^n :\ \varepsilon_j \neq -1\}) \cdot \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+e_j), f(x)\right)^p d\mu(x) \\
&\quad \le 2^{p-1} \int_{\{\varepsilon\in\{-1,0,1\}^n:\ \varepsilon_j\neq -1\}} \int_{\mathbb{Z}_m^n} \Big(d_{\mathcal{M}}\left(f(x+e_j), f(x+\varepsilon)\right)^p \\
&\qquad + d_{\mathcal{M}}(f(x+\varepsilon), f(x))^p \Big) d\mu(x) d\sigma(\varepsilon) \\
&\quad = 2^{p-1} \int_{\{\varepsilon\in\{-1,0,1\}^n:\ \varepsilon_j\neq 1\}} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(y+\varepsilon), f(y))^p d\mu(y) d\sigma(\varepsilon) \\
&\qquad + 2^{p-1} \int_{\{\varepsilon\in\{-1,0,1\}^n:\ \varepsilon_j\neq -1\}} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^p d\mu(x) d\sigma(\varepsilon) \\
&\quad \le 2^p \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^p d\mu(x) d\sigma(\varepsilon).
\end{aligned}$$

Summing over $j = 1, \ldots, n$ yields the required result. $\square$

LEMMA 2.7. *Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space. Assume that for an integer n and an even integer m we have for every integer $\ell \le n$ and every $f : \mathbb{Z}_m^\ell \to \mathcal{M}$,*

$$\begin{aligned}
&\sum_{j=1}^{\ell} \int_{\mathbb{Z}_m^\ell} d_{\mathcal{M}}\left(f\left(x + \frac{m}{2}e_j\right), f(x)\right)^p d\mu(x) \\
&\qquad \le C^p m^p n^{1-\frac{p}{q}} \Bigg(\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^\ell} d_{\mathcal{M}}\left(f(x+\varepsilon), f(x)\right)^p d\mu(x) \\
&\qquad\qquad + \frac{1}{\ell}\sum_{j=1}^{\ell} \int_{\mathbb{Z}_m^\ell} d_{\mathcal{M}}\left(f(x+e_j), f(x)\right)^p d\mu(x) \Bigg).
\end{aligned}$$

Then

$$\Gamma_q^{(p)}(\mathcal{M}; n, m) \le 5C.$$

Proof. Fix $f : \mathbb{Z}_m^n \to \mathcal{M}$ and $\emptyset \neq A \subseteq \{1, \ldots, n\}$. Our assumption implies that

$$\sum_{j\in A}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right),f(x)\right)^p d\mu(x)$$
$$\le C^p m^p n^{1-\frac{p}{q}}\left(\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\sum_{j\in A}\varepsilon_j e_j\right),f(x)\right)^p d\mu(x)\right.$$
$$\left.+\frac{1}{|A|}\sum_{j\in A}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+e_j),f(x)\right)^p d\mu(x)\right).$$

Multiplying this inequality by $\frac{2^{|A|}}{3^n}$, and summing over all $\emptyset\neq A\subseteq\{1,\ldots,n\}$, we see that

(12)
$$\frac{2}{3}\sum_{j=1}^n\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right),f(x)\right)^p d\mu(x)$$
$$=\sum_{\emptyset\neq A\subseteq\{1,\ldots,n\}}\frac{2^{|A|}}{3^n}\sum_{j\in A}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right),f(x)\right)^p d\mu(x)$$
$$\le C^p m^p n^{1-\frac{p}{q}}\left(\sum_{\emptyset\neq A\subseteq\{1,\ldots,n\}}\frac{2^{|A|}}{3^n}\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\sum_{j\in A}\varepsilon_j e_j\right),f(x)\right)^p d\mu(x)\right.$$
$$\left.+\sum_{\emptyset\neq A\subseteq\{1,\ldots,n\}}\frac{2^{|A|}}{|A|3^n}\sum_{j\in A}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+e_j),f(x)\right)^p d\mu(x)\right)$$

(13)
$$\le C^p m^p n^{1-\frac{p}{q}}\left(\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+\delta),f(x)\right)^p d\mu(x)d\sigma(\delta)\right.$$
$$\left.+\frac{1}{n}\sum_{j=1}^n\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+e_j),f(x)\right)^p d\mu(x)\right)$$
$$\le C^p m^p n^{1-\frac{p}{q}}\left(3^p+1\right)\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f(x+\delta),f(x)\right)^p d\mu(x)d\sigma(\delta),$$

where we used the fact that in (12), the coefficient of $d_{\mathcal{M}}(f(x+e_j),f(x))^p$ equals $\sum_{k=1}^n\frac{2^k}{k3^n}\binom{n-1}{k-1}\le\frac{1}{n}$, and in (13) we used Lemma 2.6. □

3. Warmup: the case of Hilbert space

The fact that Hilbert spaces have metric cotype 2 is particularly simple to prove. This is contained in the following proposition.

Proposition 3.1. *Let H be a Hilbert space. Then for every integer n, and every integer $m \ge \frac{2}{3}\pi\sqrt{n}$ which is divisible by 4,*

$$\Gamma_2(H;n,m) \le \frac{\sqrt{6}}{\pi}.$$

Proof. Fix $f : Z_m^n \to H$ and decompose it into Fourier coefficients:

$$f(x) = \sum_{k\in\mathbb{Z}_m^n} W_k(x)\widehat{f}(k).$$

For every $j = 1,2,\dots,n$ we have that

$$f\left(x+\frac{m}{2}e_j\right) - f(x) = \sum_{k\in\mathbb{Z}_m^n} W_k(x)\left(e^{\pi i k_j}-1\right)\widehat{f}(k).$$

Thus

$$\begin{aligned}&\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|f\left(x+\frac{m}{2}e_j\right)-f(x)\right\|_H^2 d\mu(x)\\ &= \sum_{k\in\mathbb{Z}_m^n}\left(\sum_{j=1}^n \left|e^{\pi i k_j}-1\right|^2\right)\left\|\widehat{f}(k)\right\|_H^2 = 4\sum_{k\in\mathbb{Z}_m^n} |\{j : k_j \equiv 1 \bmod 2\}|\cdot\left\|\widehat{f}(k)\right\|_H^2.\end{aligned}$$

Additionally, for every $\varepsilon \in \{-1,0,1\}^n$,

$$f(x+\varepsilon)-f(x) = \sum_{k\in\mathbb{Z}_m^n} W_k(x)(W_k(\varepsilon)-1)\widehat{f}(k).$$

Thus

$$\begin{aligned}&\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} \|f(x+\varepsilon)-f(x)\|_H^2 d\mu(x)d\sigma(\varepsilon)\\ &\qquad= \sum_{k\in\mathbb{Z}_m^n}\left(\int_{\{-1,0,1\}^n} |W_k(\varepsilon)-1|^2\, d\sigma(\varepsilon)\right)\left\|\widehat{f}(k)\right\|_H^2.\end{aligned}$$

Observe that

$$\begin{aligned}\int_{\{-1,0,1\}^n} |W_k(\varepsilon)-1|^2\, d\sigma(\varepsilon) &= \int_{\{-1,0,1\}^n}\left|\exp\left(\frac{2\pi i}{m}\sum_{j=1}^m k_j\varepsilon_j\right)-1\right|^2 d\sigma(\varepsilon)\\ &= 2-2\prod_{j=1}^n\int_{\{-1,0,1\}^n}\exp\left(\frac{2\pi i}{m}k_j\varepsilon_j\right)d\sigma(\varepsilon)\\ &= 2-2\prod_{j=1}^n\frac{1+2\cos\left(\frac{2\pi}{m}k_j\right)}{3}\\ &\ge 2-2\prod_{j:\ k_j\equiv 1 \mod 2}\frac{1+2\left|\cos\left(\frac{2\pi}{m}k_j\right)\right|}{3}.\end{aligned}$$

Note that if m is divisible by 4 and $\ell \in \{0, \dots, m-1\}$ is an odd integer, then

$$\left|\cos\left(\frac{2\pi\ell}{m}\right)\right| \le \left|\cos\left(\frac{2\pi}{m}\right)\right| \le 1 - \frac{\pi^2}{m^2}.$$

Hence

$$\begin{aligned}\int_{\{-1,0,1\}^n} |W_k(\varepsilon) - 1|^2 \, d\sigma(\varepsilon) &\ge 2\left(1 - \left(1 - \frac{2\pi^2}{3m^2}\right)^{|\{j:\ k_j \equiv 1 \mod 2\}|}\right)\\ &\ge 2\left(1 - e^{-\frac{2|\{j:\ k_j\equiv 1 \mod 2\}|\pi^2}{3m^2}}\right)\\ &\ge |\{j:\ k_j \equiv 1 \mod 2\}| \cdot \frac{2\pi^2}{3m^2},\end{aligned}$$

provided that $m \ge \frac{2}{3}\pi\sqrt{n}$. □

4. K-convex spaces

In this section we prove the "hard direction" of Theorem 1.2 and Theorem 1.4 when X is a K-convex Banach space; namely, we show that in this case Rademacher cotype q implies metric cotype q. There are two reasons why we single out this case before passing to the proofs of these theorems in full generality. First of all, the proof for K-convex spaces is different and simpler than the general case. More importantly, in the case of K-convex spaces we are able to obtain optimal bounds on the value of m in Definition 1.1 and Definition 1.3. Namely, we show that if X is a K-convex Banach space of cotype q, then for every $1 \le p \le q$, $m_q^{(p)}(X; n, \Gamma) = O(n^{1/q})$, for some $\Gamma = \Gamma(X)$. This is best possible due to Lemma 2.3. In the case of general Banach spaces we obtain worse bounds, and this is why we have the restriction that X is K-convex in Theorem 1.9 and Theorem 1.11. This issue is taken up again in Section 8.

THEOREM 4.1. *Let X be a K-convex Banach space with cotype q. Then for every integer n and every integer m which is divisible by* 4,

$$m \ge \frac{2n^{1/q}}{C_q^{(p)}(X) K_p(X)} \implies \Gamma_q^{(p)}(X; n, m) \le 15 C_q^{(p)}(X) K_p(X).$$

Proof. For $f : \mathbb{Z}_m^n \to X$ we define the following operators:

$$\begin{aligned}\widetilde{\partial}_j f(x) &= f(x + e_j) - f(x - e_j),\\ \mathcal{E}_j f(x) &= \mathbb{E}_\varepsilon\, f\left(x + \sum_{\ell \ne j} \varepsilon_\ell e_\ell\right),\end{aligned}$$

and for $\varepsilon \in \{-1, 0, 1\}^n$,

$$\partial_\varepsilon f(x) = f(x + \varepsilon) - f(x).$$

These operators operate diagonally on the Walsh basis $\{W_k\}_{k\in\mathbb{Z}_m^n}$ as follows:

$$\widetilde{\partial}_j W_k = \left(W_k(e_j) - W_k(-e_j)\right) W_k = 2\sin\left(\frac{2\pi i k_j}{m}\right)\cdot W_k, \tag{14}$$

$$\mathcal{E}_j W_k = \left(\mathbb{E}_\varepsilon \prod_{\ell\neq j} e^{\frac{2\pi i \varepsilon_\ell k_\ell}{m}}\right) W_k = \left(\prod_{\ell\neq j}\cos\left(\frac{2\pi k_\ell}{m}\right)\right) W_k, \tag{15}$$

and for $\varepsilon\in\{-1,1\}^n$,

$$\begin{aligned}\partial_\varepsilon W_k &= \left(W(\varepsilon)-1\right) W_k \\ &= \left(\prod_{j=1}^n e^{\frac{2\pi i \varepsilon_j k_j}{m}} - 1\right) W_k \\ &= \left(\prod_{j=1}^n \left(\cos\left(\frac{2\pi\varepsilon_j k_j}{m}\right) + i\sin\left(\frac{2\pi\varepsilon_j k_j}{m}\right)\right) - 1\right) W_k \\ &= \left(\prod_{j=1}^n \left(\cos\left(\frac{2\pi k_j}{m}\right) + i\varepsilon_j\sin\left(\frac{2\pi k_j}{m}\right)\right) - 1\right) W_k.\end{aligned} \tag{16}$$

The last step was a crucial observation, using the fact that $\varepsilon_j \in \{-1,1\}$. Thinking of $\partial_\varepsilon W_k$ as a function of $\varepsilon\in\{-1,1\}^n$, equations (14), (15) and (16) imply that

$$\begin{aligned}\mathbf{Rad}(\partial_\varepsilon W_k) &= i\left(\sum_{j=1}^n \varepsilon_j \sin\left(\frac{2\pi k_j}{m}\right)\cdot\prod_{\ell\neq j}\cos\left(\frac{2\pi k_\ell}{m}\right)\right) W_k \\ &= \frac{i}{2}\left(\sum_{j=1}^n \varepsilon_j \widetilde{\partial}_j \mathcal{E}_j\right) W_k.\end{aligned}$$

Thus for every $x\in\mathbb{Z}_m^n$ and $f:\mathbb{Z}_m^n\to X$,

$$\mathbf{Rad}(\partial_\varepsilon f(x)) = \frac{i}{2}\left(\sum_{j=1}^n \varepsilon_j \widetilde{\partial}_j \mathcal{E}_j\right) f(x).$$

It follows that

$$\begin{aligned}&\int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j\left[\mathcal{E}_j f(x+e_j) - \mathcal{E}_j f(x-e_j)\right]\right\|_X^p d\mu(x) \\ &\qquad = \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j \widetilde{\partial}_j \mathcal{E}_j f(x)\right\|_X^p d\mu(x) \\ &\qquad = \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\|\mathbf{Rad}(\partial_\varepsilon f(x))\right\|_X^p d\mu(x) \\ &\qquad \le K_p(X)^p \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\|\partial_\varepsilon f(x)\right\|_X^p d\mu(x).\end{aligned} \tag{17}$$

By (17) and the definition of $C_q^{(p)}(X)$, for every $C > C_q^{(p)}(X)$ we have that

$$\begin{aligned}
(18) \qquad & [K_p(X)C]^p\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon)-f(x)\|_X^p d\mu(x) \\
& \ge C^p \cdot \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j [\mathcal{E}_j f(x+e_j) - \mathcal{E}_j f(x-e_j)] \right\|_X^p d\mu(x) \\
& \ge \int_{\mathbb{Z}_m^n} \left(\sum_{j=1}^n \|\mathcal{E}_j f(x+e_j) - \mathcal{E}_j f(x-e_j)\|_X^q \right)^{p/q} d\mu(x) \\
& \ge \frac{1}{n^{1-p/q}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|\mathcal{E}_j f(x+e_j) - \mathcal{E}_j f(x-e_j)\|_X^p \, d\mu(x).
\end{aligned}$$

Now, for $j \in \{1,\dots,n\}$,

$$\begin{aligned}
(19) \quad & \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j f\left(x + \frac{m}{2}e_j\right) - \mathcal{E}_j f(x) \right\|_X^p d\mu(x) \\
& \le \left(\frac{m}{4}\right)^{p-1} \sum_{s=1}^{m/4} \int_{\mathbb{Z}_m^n} \|\mathcal{E}_j f(x+2se_j) - \mathcal{E}_j f(x+2(s-1)e_j)\|_X^p \, d\mu(x) \\
& = \left(\frac{m}{4}\right)^p \int_{\mathbb{Z}_m^n} \|\mathcal{E}_j f(x+e_j) - \mathcal{E}_j f(x-e_j)\|_X^p \, d\mu(x).
\end{aligned}$$

Plugging (19) into (18) we get

$$\begin{aligned}
& \left(\frac{m}{4}\right)^p n^{1-\frac{p}{q}} [K_p(X)C]^p\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon)-f(x)\|_X^p d\mu(x) \\
& \ge \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j f\left(x+\frac{m}{2}e_j\right) - \mathcal{E}_j f(x) \right\|_X^p d\mu(x) \\
& \ge \frac{1}{3^{p-1}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| f\left(x+\frac{m}{2}e_j\right) - f(x) \right\|_X^p d\mu(x) \\
& \quad -2 \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|\mathcal{E}_j f(x) - f(x)\|_X^p \, d\mu(x) \\
& = \frac{1}{3^{p-1}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| f\left(x+\frac{m}{2}e_j\right) - f(x) \right\|_X^p d\mu(x) \\
& \quad -2 \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| \mathbb{E}_\varepsilon \left(f\left(x + \sum_{\ell\neq j} \varepsilon_\ell e_\ell \right) - f(x) \right) \right\|_X^p d\mu(x)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{3^{p-1}}\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|f\left(x+\frac{m}{2}e_j\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
&\quad -2\sum_{j=1}^n \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|f\left(x+\sum_{\ell\neq j}\varepsilon_\ell e_\ell\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
&\geq \frac{1}{3^{p-1}}\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|f\left(x+\frac{m}{2}e_j\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
&\quad -2^p n\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|f\left(x+\varepsilon\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
&\quad -2^p\sum_{j=1}^n \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|f\left(x+\varepsilon_j e_j\right)-f\left(x\right)\right\|_X^p d\mu(x).
\end{aligned}$$

Thus, the required result follows from Lemma 2.7. □

The above argument actually gives the following generalization of Theorem 4.1, which holds for products of arbitrary compact Abelian groups.

THEOREM 4.2. *Let $G_1,\ldots,G_n$ be compact Abelian groups, $(g_1,\ldots,g_n)\in G_1\times\cdots\times G_n$, and let X be a K-convex Banach space. Then for every integer k and every $f: G_1\times\cdots\times G_n\to X$,*

$$\begin{aligned}
&\sum_{j=1}^n \int_{G_1\times\cdots\times G_n} \|f(x+2kg_je_j)-f(x)\|_X^p d(\mu_{G_1}\otimes\cdots\otimes\mu_{G_n})(x)\\
&\leq C^p \int_{\{-1,0,1\}^n}\int_{G_1\times\cdots\times G_n} \left\|f\left(x+\sum_{j=1}^n \varepsilon_j g_j e_j\right)-f(x)\right\|_X^p d(\mu_{G_1}\otimes\cdots\otimes\mu_{G_n})(x)d\sigma(\varepsilon),
\end{aligned}$$

where

$$C\leq 5\max\left\{C_q^{(p)}(X)K_p(X)kn^{\frac{1}{p}-\frac{1}{q}},n^{\frac{1}{p}}\right\}.$$

Here μ_G denotes the normalized Haar measure on a compact Abelian group G. We refer the interested reader to the book [64], which contains the necessary background required to generalize the proof of Theorem 4.1 to this setting.

5. The equivalence of Rademacher cotype and metric cotype

We start by establishing the easy direction in Theorem 1.2 and Theorem 1.4, i.e. that metric cotype implies Rademacher cotype.

5.1. *Metric cotype implies Rademacher cotype.* Let X be a Banach space and assume that $\Gamma_q^{(p)}(X)<\infty$ for some $1\leq p\leq q$. Fix $\Gamma>\Gamma_q^{(p)}(X)$,

$v_1, \ldots, v_n \in X$, and let m be an even integer. Define $f : \mathbb{Z}_m^n \to X$ by

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} e^{\frac{2\pi i x_j}{m}} v_j.$$

Then

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} \left\| f\left(x + \frac{m}{2} e_j\right) - f(x) \right\|_X^p d\mu(x) = 2^p \sum_{j=1}^{n} \|v_j\|_X^p, \tag{20}$$

and

$$\begin{aligned} &\int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \|f(x+\delta) - f(x)\|_X^p \, d\mu(x) d\sigma(\delta) \\ &\qquad = \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^{n} e^{\frac{2\pi i x_j}{m}} \left(e^{\frac{2\pi i \delta_j}{m}} - 1 \right) v_j \right\|_X^p d\mu(x) d\sigma(\delta). \end{aligned} \tag{21}$$

We recall the *contraction principle* (see [36]), which states that for every $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j a_j v_j \right\|_X^p \le \left(\max_{1 \le j \le n} |a_j| \right)^p \cdot \mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j v_j \right\|_X^p.$$

Observe that for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$,

$$\begin{aligned} &\int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^{n} e^{\frac{2\pi i x_j}{m}} \left(e^{\frac{2\pi i \delta_j}{m}} - 1 \right) v_j \right\|_X^p d\mu(x) d\sigma(\delta) \\ &\quad = \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^{n} e^{\frac{2\pi i}{m}\left(x_j + \frac{m(1-\varepsilon_j)}{4}\right)} \left(e^{\frac{2\pi i \delta_j}{m}} - 1 \right) v_j \right\|_X^p d\mu(x) d\sigma(\delta) \\ &\quad = \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^{n} \varepsilon_j e^{\frac{2\pi i x_j}{m}} \left(e^{\frac{2\pi i \delta_j}{m}} - 1 \right) v_j \right\|_X^p d\mu(x) d\sigma(\delta). \end{aligned}$$

Taking expectation with respect to ε, and using the contraction principle, we see that

$$\begin{aligned} &\int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^{n} e^{\frac{2\pi i x_j}{m}} \left(e^{\frac{2\pi i \delta_j}{m}} - 1 \right) v_j \right\|_X^p d\mu(x) d\sigma(\delta) \\ &\quad = \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j e^{\frac{2\pi i x_j}{m}} \left(e^{\frac{2\pi i \delta_j}{m}} - 1 \right) v_j \right\|_X^p d\mu(x) d\sigma(\delta) \\ &\quad \le \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} 2^p \left(\max_{1 \le j \le n} \left| e^{\frac{2\pi i \delta_j}{m}} - 1 \right| \right)^p \mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j v_j \right\|_X^p d\mu(x) d\sigma(\delta) \\ &\quad \le \left(\frac{4\pi}{m} \right)^p \mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j v_j \right\|_X^p, \end{aligned} \tag{22}$$

where in the last inequality above we used the fact that for $\theta \in [0,\pi]$, $|e^{i\theta}-1| \le \theta$.

Combining (7), (20), (21), and (22), we get that

$$2^p \sum_{j=1}^n \|v_j\|_X^p \le \Gamma^p m^p \left(\frac{4\pi}{m}\right)^p n^{1-\frac{p}{q}}\, \mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j v_j\right\|_X^p = (4\pi\Gamma)^p\, n^{1-\frac{p}{q}}\, \mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j v_j\right\|_X^p .$$

If $p = q$ we see that $C_q(X) \le 2\pi\Gamma_q(X)$. If $p < q$ then when $\|v_1\|_X = \cdots = \|v_n\|_X = 1$ we get that

$$\left(\mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j v_j\right\|_X^q\right)^{1/q} \ge \left(\mathbb{E}_\varepsilon \left\|\sum_{j=1}^n \varepsilon_j v_j\right\|_X^p\right)^{1/p} = \Omega\left(\frac{n^{1/q}}{\Gamma}\right).$$

This means that X has "equal norm cotype q", implying that X has cotype q' for every $q' > q$ (see [69], [34], [68] for quantitative versions of this statement). When $q = 2$ this implies that X has cotype 2 (see [69] and the references therein).

5.2. *Proof of Theorem* 1.2 *and Theorem* 1.4. The proof of Theorem 1.2 and Theorem 1.4 is based on several lemmas. Fix an odd integer $k \in \mathbb{N}$, with $k < \frac{m}{2}$, and assume that $1 \le p \le q$. Given $j \in \{1,\ldots,n\}$, define $S(j,k) \subseteq \mathbb{Z}_m^n$ by

$$S(j,k) := \{y \in [-k,k]^n \subseteq \mathbb{Z}_m^n : \ y_j \equiv 0 \mod 2 \text{ and } \forall\, \ell \neq j,\ y_\ell \equiv 1 \mod 2\}.$$

For $f : \mathbb{Z}_m^n \to X$ we define

$$\mathcal{E}_j^{(k)} f(x) = \left(f * \frac{\mathbf{1}_{S(j,k)}}{\mu(S(j,k))}\right)(x) = \frac{1}{\mu(S(j,k))}\int_{S(j,k)} f(x+y)\,d\mu(y). \tag{23}$$

LEMMA 5.1. *For every $p \ge 1$, every $j \in \{1,\ldots,n\}$, and every $f : \mathbb{Z}_m^n \to X$,*

$$\int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j^{(k)} f(x) - f(x)\right\|_X^p d\mu(x) \le 2^p k^p\, \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x)\|_X^p d\mu(x) + 2^{p-1}\int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_X^p d\mu(x).$$

Proof. By convexity, for every $x \in \mathbb{Z}_m^n$,

$$\begin{aligned} \left\|\mathcal{E}_j^{(k)} f(x) - f(x)\right\|_X^p &= \left\|\frac{1}{\mu(S(j,k))}\int_{S(j,k)} [f(x+y) - f(x)]\,d\mu(y)\right\|_X^p \\ &\le \frac{1}{\mu(S(j,k))}\int_{S(j,k)} \|f(x) - f(x+y)\|_X^p d\mu(y). \end{aligned} \tag{24}$$

Let $x \in \{0,\ldots,k\}^n$ be such that for all $j \in \{1,\ldots,n\}$, x_j is a positive odd integer. Observe that there exists a geodesic $\gamma : \{0,1,\ldots,\|x\|_\infty\} \to \mathbb{Z}_m^n$ such that $\gamma(0) = 0$, $\gamma(\|x\|_\infty) = x$ and for every $t \in \{1,\ldots,\|x\|_\infty\}$,

$\gamma(t)-\gamma(t-1)\in\{-1,1\}^n$. Indeed, we define $\gamma(t)$ inductively as follows: $\gamma(0)=0$, $\gamma(1)=(1,1,\dots,1)$, and if $t\ge 2$ is odd then

$$\gamma(t)=\gamma(t-1)+\sum_{s=1}^n e_s \quad\text{and}\quad \gamma(t+1)=\gamma(t-1)+2\sum_{\substack{s\in\{1,\dots,n\}\\ \gamma(t-1)_s<x_s}} e_s.$$

Since all the coordinates of x are odd, $\gamma(\|x\|_\infty)=x$. In what follows we fix an arbitrary geodesic $\gamma_x:\{0,1,\dots,\|x\|_\infty\}\to\mathbb{Z}_m^n$ as above. For $x\in(\mathbb{Z}_m\setminus\{0\})^n$ we denote $|x|=(|x_1|,\dots,|x_n|)$ and $\operatorname{sign}(x)=(\operatorname{sign}(x_1),\dots,\operatorname{sign}(x_n))$. If $x\in[-k,k]^n$ is such that all of its coordinates are odd, then we define $\gamma_x=\operatorname{sign}(x)\cdot\gamma_{|x|}$ (where the multiplication is coordinate-wise).

If $y\in S(j,k)$ then all the coordinates of $y\pm e_j$ are odd. We can thus define two geodesic paths

$$\gamma_{x,y}^{+1}=x+e_j+\gamma_{y-e_j}\quad\text{and}\quad \gamma_{x,y}^{-1}=x-e_j+\gamma_{y+e_j},$$

where the addition is point-wise.

For $z\in\mathbb{Z}_m^n$ and $\varepsilon\in\{-1,1\}^n$ define

$$F^{+1}(z,\varepsilon)=\Big\{(x,y)\in\mathbb{Z}_m^n\times S(j,k):\ \exists t\in\{1,\dots,\|y-e_j\|_\infty\},\\ \gamma_{x,y}^{+1}(t-1)=z,\ \gamma_{x,y}^{+1}(t)=z+\varepsilon\Big\},$$

and

$$F^{-1}(z,\varepsilon)=\Big\{(x,y)\in\mathbb{Z}_m^n\times S(j,k):\ \exists t\in\{1,\dots,\|y+e_j\|_\infty\},\\ \gamma_{x,y}^{-1}(t-1)=z,\ \gamma_{x,y}^{-1}(t)=z+\varepsilon\Big\}.$$

CLAIM 5.2. *For every $z,w\in\mathbb{Z}_m^n$ and $\varepsilon,\delta\in\{-1,1\}^n$,*

$$|F^{+1}(z,\varepsilon)|+|F^{-1}(z,\varepsilon)|=|F^{+1}(w,\delta)|+|F^{-1}(w,\delta)|.$$

Proof. Define $\psi:\mathbb{Z}_m^n\times S(j,k)\to\mathbb{Z}_m^n\times S(j,k)$ by

$$\psi(x,y)=(w-\varepsilon\delta z+\varepsilon\delta x,\varepsilon\delta y).$$

We claim that ψ is a bijection between $F^{+1}(z,\varepsilon)$ and $F^{\varepsilon_j\delta_j}(w,\delta)$, and also ψ is a bijection between $F^{-1}(z,\varepsilon)$ and $F^{-\varepsilon_j\delta_j}(w,\delta)$. Indeed, if $(x,y)\in F^{+1}(z,\varepsilon)$ then there exists $t\in\{1,\dots,\|y-e_j\|_\infty\}$ such that $\gamma_{x,y}^{+1}(t-1)=z$ and $\gamma_{x,y}^{+1}(t)=z+\varepsilon$. The path $w-\varepsilon\delta z+\varepsilon\delta\gamma_{x,y}^{+1}$ equals the path $\gamma_{\psi(x,y)}^{\varepsilon_j\delta_j}$, which by definition goes through w at time $t-1$ and $w+\delta$ at time t. Since these transformations are clearly invertible, we obtain the required result for $F^{+1}(z,\varepsilon)$. The proof for $F^{-1}(z,\varepsilon)$ is analogous. □

CLAIM 5.3. *Denote* $N = |F^{+1}(z,\varepsilon)| + |F^{-1}(z,\varepsilon)|$, *which is independent of* $z \in \mathbb{Z}_m^n$ *and* $\varepsilon \in \{-1,1\}^n$, *by Claim* 5.2. *Then*

$$N \le \frac{k \cdot |S(j,k)|}{2^{n-1}}.$$

Proof. We have that

$$\begin{aligned}
N \cdot m^n \cdot 2^n &= \sum_{(z,\varepsilon)\in \mathbb{Z}_m^n \times \{-1,1\}^n} \left(|F^{+1}(z,\varepsilon)| + |F^{-1}(z,\varepsilon)|\right)\\
&= \sum_{(z,\varepsilon)\in \mathbb{Z}_m^n \times \{-1,1\}^n} \left(\sum_{(x,y)\in \mathbb{Z}_m^n \times S(j,k)} \sum_{t=1}^{\|y-e_j\|_\infty} \mathbf{1}_{\{\gamma_{x,y}^{+1}(t-1)=z \ \wedge\ \gamma_{x,y}^{+1}(t)=z+\varepsilon\}} \right)\\
&\quad + \sum_{(z,\varepsilon)\in \mathbb{Z}_m^n \times \{-1,1\}^n} \left(\sum_{(x,y)\in \mathbb{Z}_m^n \times S(j,k)} \sum_{t=1}^{\|y+e_j\|_\infty} \mathbf{1}_{\{\gamma_{x,y}^{-1}(t-1)=z \ \wedge\ \gamma_{x,y}^{-1}(t)=z+\varepsilon\}} \right)\\
&= \sum_{(x,y)\in \mathbb{Z}_m^n \times S(j,k)} \|y-e_j\|_\infty + \sum_{(x,y)\in \mathbb{Z}_m^n \times S(j,k)} \|y+e_j\|_\infty\\
&\le 2k \cdot m^n \cdot |S(j,k)|.
\end{aligned}$$

□

We now conclude the proof of Lemma 5.1. Observe that for $x \in \mathbb{Z}_m^n$ and $y \in S(j,k)$,

$$\begin{aligned}
\frac{\|f(x)-f(x+y)\|_X^p}{2^{p-1}} &\le \|f(x)-f(x+e_j)\|_X^p\\
&\quad + \|y-e_j\|_\infty^{p-1} \sum_{t=1}^{\|y-e_j\|_\infty} \|f(\gamma_{x,y}^{+1}(t)) - f(\gamma_{x,y}^{+1}(t-1))\|_X^p\\
&\le \|f(x)-f(x+e_j)\|_X^p\\
&\quad + k^{p-1} \sum_{t=1}^{\|y-e_j\|_\infty} \|f(\gamma_{x,y}^{+1}(t)) - f(\gamma_{x,y}^{+1}(t-1))\|_X^p,
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\frac{\|f(x)-f(x+y)\|_X^p}{2^{p-1}} &\le \|f(x)-f(x-e_j)\|_X^p\\
&\quad + \|y+e_j\|_\infty^{p-1} \sum_{t=1}^{\|y+e_j\|_\infty} \|f(\gamma_{x,y}^{-1}(t)) - f(\gamma_{x,y}^{-1}(t-1))\|_X^p\\
&\le \|f(x)-f(x-e_j)\|_X^p\\
&\quad + k^{p-1} \sum_{t=1}^{\|y+e_j\|_\infty} \|f(\gamma_{x,y}^{-1}(t)) - f(\gamma_{x,y}^{-1}(t-1))\|_X^p.
\end{aligned} \tag{26}$$

Averaging inequalities (25) and (26), and integrating, we get that

$$\begin{aligned}
(27)\qquad & \frac{1}{\mu(S(j,k))}\int_{\mathbb{Z}_m^n}\int_{S(j,k)}\|f(x)-f(x+y)\|_X^p d\mu(y)d\mu(x)\\
&\le 2^{p-1}\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x)\\
&\quad +(2k)^{p-1}\frac{N\cdot 2^n}{|S(j,k)|}\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\|f(z+\varepsilon)-f(z)\|_X^p d\mu(z)\\
(28)\qquad &\le 2^{p-1}\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x)\\
&\quad +(2k)^p\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\|f(z+\varepsilon)-f(z)\|_X^p d\mu(z),
\end{aligned}$$

where in (27) we used Claim 5.2 and in (28) we used Claim 5.3. By (24), this completes the proof of Lemma 5.1. □

Lemma 5.4 below is the heart of our proof. It contains the cancellation of terms which is key to the validity of Theorem 1.2 and Theorem 1.4.

LEMMA 5.4. *For every* $f:\mathbb{Z}_m^n\to X$, *every integer* n, *every even integer* m, *every* $\varepsilon\in\{-1,1\}^n$, *every odd integer* $k<m/2$, *and every* $p\ge 1$,

$$\begin{aligned}
&\int_{\mathbb{Z}_m^n}\left\|\sum_{j=1}^n \varepsilon_j\left[\mathcal{E}_j^{(k)}f(x+e_j)-\mathcal{E}_j^{(k)}f(x-e_j)\right]\right\|_X^p d\mu(x)\\
&\qquad\le 3^{p-1}\int_{\mathbb{Z}_m^n}\|f(x+\varepsilon)-f(x-\varepsilon)\|_X^p d\mu(x)\\
&\qquad\quad+\frac{\cdot 24^p n^{2p-1}}{k^p}\sum_{j=1}^n\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x).
\end{aligned}$$

We postpone the proof of Lemma 5.4 to Section 5.3, and proceed to prove Theorem 1.2 and Theorem 1.4 assuming its validity.

Proof of Theorem 1.2 *and Theorem* 1.4. Taking expectations with respect to $\varepsilon\in\{-1,1\}^n$ in Lemma 5.4 we get that

(29)

$$\begin{aligned}
&\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\left\|\sum_{j=1}^n \varepsilon_j\left[\mathcal{E}_j^{(k)}f(x+e_j)-\mathcal{E}_j^{(k)}f(x-e_j)\right]\right\|_X^p d\mu(x)\\
&\le 3^{p-1}\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}2^{p-1}\left(\|f(x+\varepsilon)-f(x)\|_X^p+\|f(x)-f(x-\varepsilon)\|_X^p\right)d\mu(x)\\
&\quad+\frac{24^p n^{2p-1}}{k^p}\sum_{j=1}^n\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x)
\end{aligned}$$

$$\leq \frac{6^p}{3}\,\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon)-f(x)\|_X^p d\mu(x)$$
$$+\frac{24^p n^{2p-1}}{k^p}\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_X^p d\mu(x).$$

Fix $x \in \mathbb{Z}_m^n$ and let m be an integer which is divisible by 4 such that $m \geq 6n^{2+1/q}$. Fixing $C > C_q^{(p)}(X)$, and applying the definition of $C_q^{(p)}(X)$ to the vectors $\left\{\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right\}_{j=1}^n$, we get

$$\text{(30)}\quad \mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right]\right\|_X^p$$
$$\geq \frac{1}{C^p \cdot n^{1-p/q}} \sum_{j=1}^n \left\|\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right\|_X^p.$$

Now, for every $j \in \{1,\ldots,n\}$,

$$\text{(31)}\quad \sum_{s=1}^{m/4} \left\|\mathcal{E}_j^{(k)} f\left(x+2se_j\right) - \mathcal{E}_j^{(k)} f\left(x+2(s-1)e_j\right)\right\|_X^p$$
$$\geq \left(\frac{4}{m}\right)^{p-1} \left\|\mathcal{E}_j^{(k)} f\left(x+\frac{m}{2}e_j\right) - \mathcal{E}_j^{(k)} f\left(x\right)\right\|_X^p.$$

Averaging (31) over $x \in \mathbb{Z}_m^n$ we get that

$$\text{(32)}\quad \int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right\|_X^p d\mu(x)$$
$$\geq \left(\frac{4}{m}\right)^{p} \int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j^{(k)} f\left(x+\frac{m}{2}e_j\right) - \mathcal{E}_j^{(k)} f\left(x\right)\right\|_X^p d\mu(x).$$

Combining (30) and (32) we get the inequality

$$\text{(33)}\quad \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right]\right\|_X^p d\mu(x)$$
$$\geq \frac{1}{C^p \cdot n^{1-p/q}} \cdot \left(\frac{4}{m}\right)^{p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j^{(k)} f\left(x+\frac{m}{2}e_j\right) - \mathcal{E}_j^{(k)} f\left(x\right)\right\|_X^p d\mu(x).$$

Now, for every $j \in \{1,\ldots,n\}$,

$$\text{(34)}\quad \int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j^{(k)} f\left(x+\frac{m}{2}e_j\right) - \mathcal{E}_j^{(k)} f\left(x\right)\right\|_X^p d\mu(x)$$
$$\geq \frac{1}{3^{p-1}} \int_{\mathbb{Z}_m^n} \left\| f\left(x+\frac{m}{2}e_j\right) - f\left(x\right)\right\|_X^p d\mu(x)$$

$$
\begin{aligned}
&-\int_{\mathbb{Z}_m^n}\left\|\mathcal{E}_j^{(k)}f\left(x+\frac{m}{2}e_j\right)-f\left(x+\frac{m}{2}e_j\right)\right\|_X^p d\mu(x)\\
&-\int_{\mathbb{Z}_m^n}\left\|\mathcal{E}_j^{(k)}f\left(x\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
=&\ \frac{1}{3^{p-1}}\int_{\mathbb{Z}_m^n}\left\|f\left(x+\frac{m}{2}e_j\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
&-2\int_{\mathbb{Z}_m^n}\left\|\mathcal{E}_j^{(k)}f\left(x\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
\ge&\ \frac{1}{3^{p-1}}\int_{\mathbb{Z}_m^n}\left\|f\left(x+\frac{m}{2}e_j\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
&-2^{p+1}k^p\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\|f(x+\varepsilon)-f(x)\|_X^p d\mu(x)\\
&-2^p\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x),
\end{aligned}
$$

where we used Lemma 5.1.

Combining (34) with (33), we see that

(35)
$$
\begin{aligned}
&\sum_{j=1}^n\int_{\mathbb{Z}_m^n}\left\|f\left(x+\frac{m}{2}e_j\right)-f\left(x\right)\right\|_X^p d\mu(x)\\
&\le\frac{(3Cm)^p n^{1-\frac{p}{q}}}{3\cdot 4^p}\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\left\|\sum_{j=1}^n\varepsilon_j\left[\mathcal{E}_j^{(k)}f(x+e_j)-\mathcal{E}_j^{(k)}f(x-e_j)\right]\right\|_X^p d\mu(x)\\
&\quad+6^pk^pn\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\|f(x+\varepsilon)-f(x)\|_X^p d\mu(x)\\
&\quad+6^p\sum_{j=1}^n\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x)\\
&\le\left(\frac{(18Cm)^pn^{1-\frac{p}{q}}}{4^p}+6^pk^pn\right)\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\|f(x+\varepsilon)-f(x)\|_X^p d\mu(x)\\
&\quad+\left(\frac{(3Cm)^pn^{1-\frac{p}{q}}}{4^p}\cdot\frac{24^pn^{2p-1}}{k^p}+6^p\right)\sum_{j=1}^n\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x)
\end{aligned}
$$

(36)
$$
\begin{aligned}
&\le(18Cm)^pn^{1-\frac{p}{q}}\left(\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n}\|f(x+\varepsilon)-f(x)\|_X^p d\mu(x)d\sigma(\varepsilon)\right.\\
&\quad\left.+\frac{1}{n}\sum_{j=1}^n\int_{\mathbb{Z}_m^n}\|f(x+e_j)-f(x)\|_X^p d\mu(x)\right),
\end{aligned}
$$

where in (35) we used (29), and (36) holds true when we choose $4n^2 \le k \le \frac{3m}{4n^{1/q}}$ (which is possible if we assume that $m \ge 6n^{2+1/q}$). By Lemma 2.7, this completes the proof of Theorem 1.4. $\square$

5.3. *Proof of Lemma* 5.4. Fix $\varepsilon \in \{-1,1\}^n$, and $x \in \mathbb{Z}_m^n$. Consider the following two sums:

$$A_f(x,\varepsilon) = \sum_{j=1}^{n} \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right] = \frac{1}{k(k+1)^{n-1}} \sum_{y\in\mathbb{Z}_m^n} a_y(x,\varepsilon) f(y), \tag{37}$$

and

$$B_f(x,\varepsilon) = \frac{1}{k(k+1)^{n-1}} \sum_{z-x\in(-k,k)^n\cap(2\mathbb{Z})^n} [f(z+\varepsilon)-f(z-\varepsilon)] = \frac{1}{k(k+1)^{n-1}} \sum_{y\in\mathbb{Z}_m^n} b_y(x,\varepsilon) f(y), \tag{38}$$

where $a_y(x,\varepsilon), b_y(x,\varepsilon) \in \mathbb{Z}$ are appropriately chosen coefficients, which are independent of f.

For $x \in \mathbb{Z}_m^n$ define $S(x) \subset \mathbb{Z}_m^n$,

$$S(x) = \Big\{ y \in x + (2\mathbb{Z}+1)^n :\ d_{\mathbb{Z}_m^n}(y,x) = k, \quad \text{and } |\{j :\ |y_j - x_j| \equiv k \mod m\}| \ge 2 \Big\}.$$

Claim 5.5. *For $x \in \mathbb{Z}_m^n$ and $y \notin S(x)$, $a_y(x,\varepsilon) = b_y(x,\varepsilon)$.*

Proof. If there exists a coordinate $j \in \{1,\dots,n\}$ such that $x_j - y_j$ is even, then it follows from our definitions that $a_y(x,\varepsilon) = b_y(x,\varepsilon) = 0$. Similarly, if $d_{\mathbb{Z}_m^n}(x,y) > k$ then $a_y(x,\varepsilon) = b_y(x,\varepsilon) = 0$ (because k is odd). Assume that $x - y \in (2\mathbb{Z}+1)^n$. If $d_{\mathbb{Z}_m^n}(y,x) < k$ then for each j the term $f(y)$ cancels in $\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)}(x-e_j)$, implying that $a_y(x,\varepsilon) = 0$. Similarly, in the sum defining $B_f(x,\varepsilon)$ the term $f(y)$ appears twice, with opposite signs, so that $b_y(x,\varepsilon) = 0$.

It remains to deal with the case $|\{j :\ |y_j - x_j| \equiv k \mod m\}| = 1$. We may assume without loss of generality that

$$|y_1 - x_1| \equiv k \mod m \quad \text{ and for } j \ge 2, \quad y_j - x_j \in (-k,k) \mod m.$$

If $y_1 - x_1 \equiv k \mod m$ then $a_y(x,\varepsilon) = \varepsilon_1$, since in the terms corresponding to $j \ge 2$ in the definition of $A_f(x,\varepsilon)$ the summand $f(y)$ cancels out. We also claim that in this case $b_y(x,\varepsilon) = \varepsilon_1$. Indeed, if $\varepsilon_1 = 1$ then $f(y)$ appears in the sum defining $B_f(x,\varepsilon)$ only in the term corresponding to $z = y - \varepsilon$, while if $\varepsilon_1 = -1$ then $f(y)$ appears in this sum only in the term corresponding to $z = y+\varepsilon$, in which case its coefficient is -1. In the case $y_1 - x_1 \equiv -k \mod m$ the same reasoning shows that $a_y(x,\varepsilon) = b_y(x,\varepsilon) = -\varepsilon_1$. □

By Claim 5.5 we have

$$A_f(x,\varepsilon) - B_f(x,\varepsilon) = \frac{1}{k(k+1)^{n-1}} \sum_{y\in S(x)} [a_y(x,\varepsilon) - b_y(x,\varepsilon)] f(y). \tag{39}$$

Thus,

$$\begin{aligned}\int_{\mathbb{Z}_m^n} \|A_f(x,\varepsilon)\|_X^p \, d\mu(x) \le 3^{p-1} &\int_{\mathbb{Z}_m^n} \|B_f(x,\varepsilon)\|_X^p \, d\mu(x) \\ &+3^{p-1} \int_{\mathbb{Z}_m^n} \left\| \frac{1}{k(k+1)^{n-1}} \sum_{y\in S(x)} a_y(x,\varepsilon) f(y) \right\|_X^p d\mu(x) \\ &+3^{p-1} \int_{\mathbb{Z}_m^n} \left\| \frac{1}{k(k+1)^{n-1}} \sum_{y\in S(x)} b_y(x,\varepsilon) f(y) \right\|_X^p d\mu(x).\end{aligned}$$

Thus Lemma 5.4 will be proved once we establish the following inequalities

$$\int_{\mathbb{Z}_m^n} \|B_f(x,\varepsilon)\|_X^p \, d\mu(x) \le \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x-\varepsilon)\|_X^p d\mu(x), \tag{40}$$

$$\begin{aligned}\int_{\mathbb{Z}_m^n} &\left\| \frac{1}{k(k+1)^{n-1}} \sum_{y\in S(x)} a_y(x,\varepsilon) f(y) \right\|_X^p d\mu(x) \\ &\le \frac{8^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_X^p ,\end{aligned} \tag{41}$$

and

$$\begin{aligned}\int_{\mathbb{Z}_m^n} &\left\| \frac{1}{k(k+1)^{n-1}} \sum_{y\in S(x)} b_y(x,\varepsilon) f(y) \right\|_X^p d\mu(x) \\ &\le \frac{8^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_X^p .\end{aligned} \tag{42}$$

Inequality (40) follows directly from the definition of $B_f(x,\varepsilon)$, by convexity. Thus, we pass to the proof of (41) and (42).

For $j = 1, 2, \ldots, n$ define for $y \in S(x)$,

$$\tau_j^x(y) = \begin{cases} y - 2ke_j & y_j - x_j \equiv k \mod m, \\ y & \text{otherwise,} \end{cases}$$

and set $\tau_j^x(y) = y$ when $y \notin S(x)$. Observe that the following identity holds true:

$$\tau_j^x(y) = \tau_j^0(y-x) + x. \tag{43}$$

Claim 5.6. *Assume that for every $j \in \{1,2,\dots,n\}$, $x,y \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,1\}^n$, we are given a real number $\eta_j(x,y,\varepsilon) \in [-1,1]$. Then*

$$\int_{\mathbb{Z}_m^n} \left\| \frac{1}{k(k+1)^{n-1}} \sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \eta_j(x,y,\varepsilon)\left[f(y)-f(\tau_j^x(y))\right]\right\|_X^p d\mu(x) \le \frac{8^p n^{2p-1}}{2k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_X^p\, d\mu(x).$$

Proof. Denote by $N(x,\varepsilon)$ the number of nonzero summands in

$$\sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \eta_j(x,y,\varepsilon)\left[f(y)-f(\tau_j^x(y))\right].$$

For every $\ell \ge 2$ let $S^\ell(x)$ be the set of all $y \in S(x)$ for which the number of coordinates j such that $y_j - x_j \in \{k,-k\} \mod m$ equals ℓ. Then $|S^\ell(x)| = \binom{n}{\ell}2^\ell(k-1)^{n-\ell}$. Moreover, for $y \in S^\ell(x)$ we have that $y \neq \tau_j^x(y)$ for at most ℓ values of j. Hence

$$\begin{aligned} N(x,\varepsilon) &\le \sum_{\ell=2}^n |S^\ell(x)|\ell = \sum_{\ell=2}^n \binom{n}{\ell} 2^\ell (k-1)^{n-\ell}\ell \\ &= 2n\left[(k+1)^{n-1}-(k-1)^{n-1}\right] \le \frac{4n^2}{k^2}k(k+1)^{n-1}. \end{aligned}$$

Now, using (43), we get

$$\begin{aligned} &\int_{\mathbb{Z}_m^n} \left\| \frac{1}{k(k+1)^{n-1}} \sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \eta_j(x,y,\varepsilon)\left[f(y)-f(\tau_j^x(y))\right]\right\|_X^p d\mu(x) \\ &= \int_{\mathbb{Z}_m^n} \left(\frac{N(x,\varepsilon)}{k(k+1)^{n-1}}\right)^p \left\| \frac{1}{N(x,\varepsilon)} \sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \eta_j(x,y,\varepsilon)[f(y)-f(\tau_j^x(y))]\right\|_X^p d\mu(x) \\ &\le \int_{\mathbb{Z}_m^n} \frac{N(x,\varepsilon)^{p-1}}{k^p(k+1)^{(n-1)p}} \sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \left\|f(y)-f(\tau_j^x(y))\right\|_X^p d\mu(x) \\ &\le \frac{4^{p-1}n^{2p-2}}{k^{2p-1}(k+1)^{n-1}} \sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \int_{\mathbb{Z}_m^n} \left\|f(y)-f(\tau_j^x(y))\right\|_X^p d\mu(x) \\ &= \frac{4^{p-1}n^{2p-2}}{k^{2p-1}(k+1)^{n-1}} \sum_{j=1}^n \sum_{z\in\mathbb{Z}_m^n} \int_{\mathbb{Z}_m^n} \left\|f(z+x)-f(\tau_j^0(z)+x)\right\|_X^p d\mu(x). \end{aligned} \tag{44}$$

Consider the following set:

$$E_j = \{z \in \mathbb{Z}_m^n :\ \tau_j^0(z) = z - 2ke_j\}.$$

Observe that that for every j,

$$|E_j| = \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} 2^\ell (k-1)^{n-1-\ell} \tag{45}$$
$$\le (k+1)^{n-1} - (k-1)^{n-1} \le \frac{2n}{k}(k+1)^{n-1}.$$

Using the translation invariance of the Haar measure on $\mathbb{Z}_m^n$ we get that

$$\sum_{j=1}^n \sum_{z\in\mathbb{Z}_m^n} \int_{\mathbb{Z}_m^n} \left\|f(z+x) - f(\tau_j^0(z)+x)\right\|_X^p d\mu(x) \tag{46}$$
$$= \sum_{j=1}^n \sum_{z\in E_j} \int_{\mathbb{Z}_m^n} \|f(z+x) - f(z+x-2ke_j)\|_X^p d\mu(x)$$
$$= \sum_{j=1}^n |E_j| \int_{\mathbb{Z}_m^n} \|f(w) - f(w-2ke_j)\|_X^p d\mu(w)$$
$$\le \frac{2n}{k}(k+1)^{n-1} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(w) - f(w-2ke_j)\|_X^p d\mu(w)$$
$$\le \frac{2n}{k}(k+1)^{n-1} \tag{47}$$
$$\times \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left((2k)^{p-1} \sum_{t=1}^{2k} \|f(w-(t-1)e_j) - f(w-te_j)\|_X^p \right) d\mu(w)$$
$$\le 2^{p+1} n k^{p-1} (k+1)^{n-1} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(z+e_j) - f(z)\|_X^p d\mu(z),$$

where in (46) we used (45). Combining (44) and (47) completes the proof of Claim 5.6. □

By Claim 5.6, inequalities (41) and (42), and hence also Lemma 5.4, will be proved once we establish the following identities:

$$\sum_{y\in S(x)} a_y(x,\varepsilon) f(y) = \sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \varepsilon_j \left[f(y) - f(\tau_j^x(y))\right]. \tag{48}$$

and

$$\sum_{y\in S(x)} b_y(x,\varepsilon) f(y) = \sum_{j=1}^n \sum_{y\in\mathbb{Z}_m^n} \delta_j(x,y,\varepsilon) \left[f(y) - f(\tau_j^x(y))\right], \tag{49}$$

for some $\delta_j(x,y,\varepsilon) \in \{-1,0,1\}$.

Identity (48) follows directly from the fact that (37) implies that for every $y \in S(x)$,

$$a_y(x,\varepsilon) = \sum_{j:\ y_j - x_j \equiv k \mod m} \varepsilon_j - \sum_{j:\ y_j - x_j \equiv -k \mod m} \varepsilon_j.$$

It is enough to prove identity (49) for $x = 0$, since $b_y(x,\varepsilon) = b_{y-x}(0,\varepsilon)$. To this end we note that it follows directly from (38) that for every $y \in S(0)$

$$b_y(0,\varepsilon) = \begin{cases} 1 & \exists j\ y_j \equiv \varepsilon_j k \mod m \text{ and } \forall \ell\ y_\ell \not\equiv -\varepsilon_j k \mod m \\ -1 & \exists j\ y_j \equiv -\varepsilon_j k \mod m \text{ and } \forall \ell\ y_\ell \not\equiv \varepsilon_j k \mod m \\ 0 & \text{otherwise.} \end{cases}$$

For $y \in S(0)$ define

$$y_j^{\ominus} = \begin{cases} -y_j & y_j \in \{k,-k\} \mod m \\ y_j & \text{otherwise.} \end{cases}$$

Since $b_y(0,\varepsilon) = -b_{y^\ominus}(0,\varepsilon)$ we get that

$$\sum_{y\in S(0)} b_y(0,\varepsilon) f(y) = \frac{1}{2}\sum_{y\in S(0)} b_y(0,\varepsilon)\left[f(y) - f(y^\ominus)\right]. \tag{50}$$

Define for $\ell \in \{1,\ldots,n+1\}$ a vector $y^{\ominus_\ell} \in \mathbb{Z}_m^n$ by

$$y_j^{\ominus_\ell} = \begin{cases} -y_j & j < \ell \text{ and } y_j \in \{k,-k\} \mod m \\ y_j & \text{otherwise.} \end{cases}$$

Then $y^{\ominus_{n+1}} = y^\ominus$, $y^{\ominus_1} = y$ and by (50)

$$\sum_{y\in S(0)} b_y(0,\varepsilon) f(y) = \frac{1}{2}\sum_{\ell=1}^n \sum_{y\in S(0)} b_y(0,\varepsilon)\left[f(y^{\ominus_\ell}) - f(y^{\ominus_{\ell+1}})\right].$$

Since whenever $y^{\ominus_\ell} \neq y^{\ominus_{\ell+1}}$, each of these vectors is obtained from the other by flipping the sign of the ℓ-th coordinate, which is in $\{k,-k\} \mod m$, this implies the representation (49). The proof of Lemma 5.4 is complete. □

6. A nonlinear version of the Maurey-Pisier theorem

In what follows we denote by $\mathbf{diag}(\mathbb{Z}_m^n)$ the graph on $\mathbb{Z}_m^n$ in which $x,y \in \mathbb{Z}_m^n$ are adjacent if for every $i \in \{1,\ldots,n\}$, $x_i - y_i \in \{\pm 1 \mod m\}$.

For technical reasons that will become clear presently, given $\ell, n \in \mathbb{N}$ we denote by $\mathcal{B}(\mathcal{M};n,\ell)$ the infimum over $\mathcal{B} > 0$ such that for every even $m \in \mathbb{N}$ and for every $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\ell e_j\right), f(x)\right)^2 d\mu(x) \\ \le \mathcal{B}^2 \ell^2 n\, \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^2 d\mu(x).$$

Lemma 6.1. *For every metric space $(\mathcal{M}, d_{\mathcal{M}})$, every $n, a \in \mathbb{N}$, every even $m, r \in \mathbb{N}$ with $0 \le r < m$, and every $f : \mathbb{Z}_m^n \to \mathcal{M}$,*

$$\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+(am+r)e_j\right),f(x)\right)^2 d\mu(x) \le \min\left\{r^2,(m-r)^2\right\}\cdot n\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2 d\mu(x). \tag{51}$$

In particular, $\mathcal{B}(\mathcal{M};n,\ell)\le 1$ *for every* $n\in\mathbb{N}$ *and every even* $\ell\in\mathbb{N}$.

Proof. The left-hand side of (51) depends only on r, and remains unchanged if we replace r by $m-r$. We may thus assume that $a=0$ and $r\le m-r$. Fix $x\in\mathbb{Z}_m^n$ and $j\in\{1,\dots n\}$. Observe that

$$\left\{x+\frac{1-(-1)^k}{2}\sum_{r\neq j}e_r+ke_j\right\}_{k=0}^{r}$$

is a path of length r joining x and $x+re_j$ in the graph $\mathbf{diag}(\mathbb{Z}_m^n)$. Thus the distance between x and $x+re_j$ in the graph $\mathbf{diag}(\mathbb{Z}_m^n)$ equals r. If $(x=w_0,w_1,\dots,w_r=x+re_j)$ is a geodesic joining x and $x+re_j$ in $\mathbf{diag}(\mathbb{Z}_m^n)$, then by the triangle inequality

$$d_{\mathcal{M}}(f(x+re_j),f(x))^2\le r\sum_{k=1}^{r}d_{\mathcal{M}}(f(w_k),f(w_{k-1}))^2. \tag{52}$$

Observe that if we sum (52) over all geodesics joining x and $x+re_j$ in $\mathbf{diag}(\mathbb{Z}_m^n)$, and then over all $x\in\mathbb{Z}_m^n$, then in the resulting sum each edge in $\mathbf{diag}(\mathbb{Z}_m^n)$ appears the same number of times. Thus, averaging this inequality over $x\in\mathbb{Z}_m^n$ we get

$$\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+re_j),f(x))^2 d\mu(x)\le r^2\,\mathbb{E}_\varepsilon[d_{\mathcal{M}}(f(x+\varepsilon),f(x))]^2.$$

Summing over $j=1,\dots n$ we obtain the required result. □

LEMMA 6.2. *For every four integers* $\ell,k,s,t\in\mathbb{N}$,

$$\mathcal{B}\left(\mathcal{M};\ell k,st\right)\le\mathcal{B}\left(\mathcal{M};\ell,s\right)\cdot\mathcal{B}\left(\mathcal{M};k,t\right).$$

Proof. Let m be an even integer and take a function $f:\mathbb{Z}_m^{\ell k}\to\mathcal{M}$. Fix $x\in\mathbb{Z}_m^{\ell k}$ and $\varepsilon\in\{-1,1\}^{\ell k}$. Define $g:\mathbb{Z}_m^{\ell}\to\mathcal{M}$ by

$$g(y)=f\left(x+\sum_{r=1}^{k}\sum_{j=1}^{\ell}\varepsilon_{j+(r-1)\ell}\cdot y_j\cdot e_{j+(r-1)\ell}\right).$$

By the definition of $\mathcal{B}\left(\mathcal{M};\ell,s\right)$, applied to g, for every $\mathcal{B}_1>\mathcal{B}\left(\mathcal{M};\ell,s\right)$,

$$\sum_{a=1}^{\ell}\int_{\mathbb{Z}_m^\ell} d_{\mathcal{M}}\left(f\left(x+\sum_{r=1}^{k}\sum_{j=1}^{\ell}\varepsilon_{j+(r-1)\ell}\cdot y_j\cdot e_{j+(r-1)\ell}+s\sum_{r=1}^{k}\varepsilon_{a+(r-1)\ell}\cdot e_{a+(r-1)\ell}\right),\right.$$
$$\left.f\left(x+\sum_{r=1}^{k}\sum_{j=1}^{\ell}\varepsilon_{j+(r-1)\ell}\cdot y_j\cdot e_{j+(r-1)\ell}\right)\right)^2 d\mu_{\mathbb{Z}_m^\ell}(y)$$
$$\le \mathcal{B}_1^2 s^2\ell\cdot\mathbb{E}_\delta\int_{\mathbb{Z}_m^\ell} d_{\mathcal{M}}\left(f\left(x+\sum_{r=1}^{k}\sum_{j=1}^{\ell}\varepsilon_{j+(r-1)\ell}\cdot (y_j+\delta_j)\cdot e_{j+(r-1)\ell}\right),\right.$$
$$\left.f\left(x+\sum_{r=1}^{k}\sum_{j=1}^{\ell}\varepsilon_{j+(r-1)\ell}\cdot y_j\cdot e_{j+(r-1)\ell}\right)\right)^2 d\mu_{\mathbb{Z}_m^\ell}(y).$$

Averaging this inequality over $x\in\mathbb{Z}_m^{\ell k}$ and $\varepsilon\in\{-1,1\}^{\ell k}$, and using the translation invariance of the Haar measure, we get that

$$\mathbb{E}_\varepsilon\sum_{a=1}^{\ell}\int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+s\sum_{r=1}^{k}\varepsilon_{a+(r-1)\ell}\cdot e_{a+(r-1)\ell}\right),f(x)\right)^2 d\mu_{\mathbb{Z}_m^{\ell k}}(x)$$
$$\le \mathcal{B}_1^2 s^2\ell\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+\varepsilon\right),f\left(x\right)\right)^2 d\mu_{\mathbb{Z}_m^{\ell k}}(x). \tag{53}$$

Next we fix $x\in\mathbb{Z}_m^{\ell k}$, $u\in\{1,\ldots,\ell\}$, and define $h_u:\mathbb{Z}_m^k\to\mathcal{M}$ by

$$h_u(y)=f\left(x+s\sum_{r=1}^{k}y_r\cdot e_{u+(r-1)\ell}\right).$$

By the definition of $\mathcal{B}(\mathcal{M};k,t)$, applied to h_u, for every $\mathcal{B}_2>\mathcal{B}(\mathcal{M};k,t)$ we have

$$\sum_{j=1}^{k}\int_{\mathbb{Z}_m^k} d_{\mathcal{M}}\left(f\left(x+s\sum_{r=1}^{k}y_r\cdot e_{u+(r-1)\ell}+st\cdot e_{u+(j-1)\ell}\right),\right.$$
$$\left.f\left(x+s\sum_{r=1}^{k}y_r\cdot e_{u+(r-1)\ell}\right)\right)^2 d\mu_{\mathbb{Z}_m^k}(y)$$
$$=\sum_{j=1}^{k}\int_{\mathbb{Z}_m^k} d_{\mathcal{M}}\Big(h_u\big(y+te_j\big),h_u(y)\Big)^2 d\mu_{\mathbb{Z}_m^k}(y)$$
$$\le \mathcal{B}_2^2t^2k\cdot\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^k} d\left(h_u\left(y+\varepsilon\right),h_u(y)\right)^2 d\mu_{\mathbb{Z}_m^k}(y)$$
$$=\mathcal{B}_2^2t^2k\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^k} d_{\mathcal{M}}\left(f\left(x+s\sum_{r=1}^{k}(y_r+\varepsilon_{u+(r-1)\ell})\cdot e_{u+(r-1)\ell}\right),\right.$$
$$\left.f\left(x+s\sum_{r=1}^{k}y_r\cdot e_{u+(r-1)\ell}\right)\right)^2 d\mu_{\mathbb{Z}_m^k}(y).$$

Summing this inequality over $u \in \{1, \dots, \ell\}$ and averaging over $x \in \mathbb{Z}_m^{\ell k}$, we get, using (53), that

$$\begin{aligned}\sum_{a=1}^{\ell k} \int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+ste_a\right), f(x)\right)^2 d\mu(x) \\ \le \mathcal{B}_2^2 t^2 k\, \mathbb{E}_\varepsilon \sum_{u=1}^{\ell} \int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+s\sum_{r=1}^{k} \varepsilon_{u+(r-1)\ell} \cdot e_{u+(r-1)\ell}\right), f\left(x\right)\right)^2 d\mu(x) \\ \le \mathcal{B}_2^2 t^2 k \cdot \mathcal{B}_1^2 s^2 \ell\, \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^{\ell k}} d_{\mathcal{M}}\left(f\left(x+\varepsilon\right), f\left(x\right)\right)^2 d\mu(x).\end{aligned}$$

This implies the required result. □

LEMMA 6.3. *Assume that there exist integers $n_0, \ell_0 > 1$ such that $\mathcal{B}(\mathcal{M}; n_0, \ell_0) < 1$. Then there exists $0 < q < \infty$ such that for every integer n,*

$$m_q^{(2)}(\mathcal{M}; n, 3n_0) \le 2\ell_0 n^{\log_{n_0} \ell_0}.$$

In particular, $\Gamma_q^{(2)}(\mathcal{M}) < \infty$.

Proof. Let $q < \infty$ satisfy $\mathcal{B}(\mathcal{M}, n_0, \ell_0) < n_0^{-1/q}$. Iterating Lemma 6.2 we get that for every integer k, $\mathcal{B}(n_0^k, \ell_0^k) \le n_0^{-k/q}$. Denoting $n = n_0^k$ and $m = 2\ell_0^k$, this implies that for every $f : \mathbb{Z}_m^n \to \mathcal{M}$,

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right), f(x)\right)^2 d\mu(x) \le \frac{1}{4} m^2 n^{1-\frac{2}{q}}\, \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^2 d\mu(x).$$

For $f : \mathbb{Z}_m^{n'} \to \mathcal{M}$, where $n' \le n$, we define $g : \mathbb{Z}_m^{n'} \times \mathbb{Z}_m^{n-n'} \to \mathcal{M}$ by $g(x, y) = f(x)$. Applying the above inqeuality to g we obtain,

$$\sum_{j=1}^{n'} \int_{\mathbb{Z}_m^{n'}} d_{\mathcal{M}}\left(f\left(x+\tfrac{m}{2}e_j\right), f(x)\right)^2 s\mu(x) \le \frac{1}{4} m^2 n^{1-\frac{2}{q}}\, \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^{n'}} d\mathcal{M}(f(x+\varepsilon), f(x))^2 d\mu(x).$$

Hence, by Lemma 2.7 we deduce that $\Gamma_q^{(2)}(\mathcal{M}; n_0^k, 2\ell_0^k) \le 3$. For general n, let k be the minimal integer such that $n \le n_0^k$. By Lemma 2.5 we get that $\Gamma(\mathcal{M}; n, 2\ell_0^k) \le 3n_0^{1-2/q} \le 3n_0$. In other words,

$$m_q^{(2)}(\mathcal{M}; n, 3n_0) \le 2\ell_0^k \le 2\ell_0 n^{\log_{n_0} \ell_0}.$$ □

THEOREM 6.4. *Let $n > 1$ be an integer, m an even integer, and s an integer divisible by 4. Assume that $\eta \in (0,1)$ satisfies $8^{sn}\sqrt{\eta} < \frac{1}{2}$, and that there exists a mapping $f : \mathbb{Z}_m^n \to \mathcal{M}$ such that*

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x + se_j\right), f(x)\right)^2 d\mu(x) \\ > (1-\eta)s^2 n\, \mathbb{E}_{\varepsilon} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon), f(x))^2 d\mu(x). \tag{54}$$

Then

$$c_{\mathcal{M}}\left([s/4]_{\infty}^n\right) \le 1 + 8^{sn}\sqrt{\eta}.$$

In particular, if $\mathcal{B}(\mathcal{M}; n, s) = 1$ then $c_{\mathcal{M}}\left([s/4]_{\infty}^n\right) = 1$.

Proof. Observe first of all that (54) and Lemma 6.1 imply that $m \ge 2s\sqrt{1-\eta} > 2s - 1$, so that $m \ge 2s$. In what follows we will use the following numerical fact: If $a_1, \ldots, a_r \ge 0$ and $0 \le b \le \frac{1}{r}\sum_{j=1}^{r} a_j$, then

$$\sum_{j=1}^{r} (a_j - b)^2 \le \sum_{j=1}^{r} a_j^2 - rb^2. \tag{55}$$

For $x \in \mathbb{Z}_m^n$ let $\mathcal{G}_j^+(x)$ (resp. $\mathcal{G}_j^-(x)$) be the set of all geodesics joining x and $x + se_j$ (resp. $x - se_j$) in the graph $\mathbf{diag}(\mathbb{Z}_m^n)$. As we have seen in the proof of Lemma 6.1, since s is even, these sets are nonempty. Notice that if $m = 2s$ then $\mathcal{G}_j^+(x) = \mathcal{G}_j^-(x)$; otherwise $\mathcal{G}_j^+(x) \cap \mathcal{G}_j^-(x) = \emptyset$. Denote $\mathcal{G}_j^{\pm}(x) = \mathcal{G}_j^+(x) \cup \mathcal{G}_j^-(x)$, and for $\pi \in \mathcal{G}_j^{\pm}(x)$,

$$\operatorname{sgn}(\pi) = \begin{cases} +1 & \text{if } \pi \in \mathcal{G}_j^+(x) \\ -1 & \text{otherwise.} \end{cases}$$

Each geodesic in $\mathcal{G}_j^{\pm}(x)$ has length s. We write each $\pi \in \mathcal{G}_j^{\pm}(x)$ as a sequence of vertices $\pi = (\pi_0 = x, \pi_1, \ldots, \pi_s = x + \operatorname{sgn}(\pi)se_j)$. Using (55) with $a_j = d_{\mathcal{M}}(f(\pi_j), f(\pi_{j-1}))$ and $b = \frac{1}{s}d_{\mathcal{M}}\left(f\left(x + se_j\right), f(x)\right)$, which satisfy the conditions of (55) due to the triangle inequality, we get that for each $\pi \in \mathcal{G}_j^{\pm}(x)$,

$$\sum_{\ell=1}^{s} \left[d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1})) - \frac{1}{s}d_{\mathcal{M}}\left(f\left(x + \operatorname{sgn}(\pi)se_j\right), f(x)\right)\right]^2 \\ \le \sum_{k=1}^{s} d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1}))^2 - \frac{1}{s}d_{\mathcal{M}}\left(f\left(x + \operatorname{sgn}(\pi)se_j\right), f(x)\right)^2. \tag{56}$$

By symmetry $|\mathcal{G}_j^+(x)| = |\mathcal{G}_j^-(x)|$, and this value is independent of $x \in \mathbb{Z}_m^n$ and $j \in \{1, \ldots, n\}$. Denote $g = |\mathcal{G}_j^{\pm}(x)|$, and observe that $g \le 2 \cdot 2^{ns}$. Averaging (56) over all $x \in \mathbb{Z}_m^n$ and $\pi \in \mathcal{G}_j^{\pm}(x)$, and summing over $j \in \{1, \ldots, n\}$, we get that

$$\begin{aligned}(57)\qquad &\frac{1}{g}\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n}\sum_{\pi\in\mathcal{G}_j^{\pm}(x)}\sum_{\ell=1}^{s}\Bigg[d_{\mathcal{M}}(f(\pi_\ell),f(\pi_{\ell-1}))\\
&\qquad\qquad -\frac{1}{s}d_{\mathcal{M}}\left(f\left(x+\mathrm{sgn}(\pi)se_j\right),f(x)\right)\Bigg]^2 d\mu(x)\\
&\le sn\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2\,d\mu(x)\\
&\quad -\frac{1}{s}\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+se_j\right),f(x)\right)^2\,d\mu(x)\\
&< \eta sn\,\mathbb{E}_\varepsilon\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2\,d\mu(x).\end{aligned}$$

Define $\psi:\mathbb{Z}_m^n\to\mathbb{R}$ by

$$\begin{aligned}&\psi(x)=2\eta sn2^{sn}\,\mathbb{E}_\varepsilon[d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2]\\
&-\sum_{j=1}^{n}\sum_{\pi\in\mathcal{G}_j^{\pm}(x)}\sum_{\ell=1}^{s}\left[d_{\mathcal{M}}(f(\pi_\ell),f(\pi_{\ell-1}))-\frac{1}{s}d_{\mathcal{M}}\left(f\left(x+\mathrm{sgn}(\pi)se_j\right),f(x)\right)\right]^2.\end{aligned}$$

Inequality (57), together with the bound on g, implies that

$$0<\int_{\mathbb{Z}_m^n}\psi(x)d\mu(x)=\frac{1}{(2s-1)^n}\int_{\mathbb{Z}_m^n}\sum_{\substack{y\in\mathbb{Z}_m^n\\ d_{\mathbb{Z}_m^n}(x,y)<s}}\psi(y)d\mu(x).$$

It follows that there exists $x^0\in\mathbb{Z}_m^n$ such that

$$\begin{aligned}(58)\qquad &\sum_{\substack{y\in\mathbb{Z}_m^n\\ d_{\mathbb{Z}_m^n}(x^0,y)<s}}\sum_{j=1}^{n}\sum_{\pi\in\mathcal{G}_j^{+}(x)\bigcup\mathcal{G}_j^{-}(x)}\\
&\quad\sum_{\ell=1}^{s}\left[d_{\mathcal{M}}(f(\pi_\ell),f(\pi_{\ell-1}))-\frac{1}{s}d_{\mathcal{M}}\left(f\left(y+\mathrm{sgn}(\pi)se_j\right),f(y)\right)\right]^2\\
&<2\eta sn2^{sn}\sum_{\substack{y\in\mathbb{Z}_m^n\\ d_{\mathbb{Z}_m^n}(x^0,y)<s}}\mathbb{E}_\varepsilon\left[d_{\mathcal{M}}(f(y+\varepsilon),f(y))^2\right].\end{aligned}$$

By scaling the metric $d_{\mathcal{M}}$ we may assume without loss of generality that

$$(59)\qquad \frac{1}{(2s-1)^n}\sum_{\substack{y\in\mathbb{Z}_m^n\\ d_{\mathbb{Z}_m^n}(x^0,y)<s}}\mathbb{E}_\varepsilon\left[d_{\mathcal{M}}(f(y+\varepsilon),f(y))^2\right]=1.$$

It follows that there exists $y^0\in\mathbb{Z}_m^n$ satisfying $d_{\mathbb{Z}_m^n}(x^0,y^0)<s$ such that

$$(60)\qquad \mathbb{E}_\varepsilon\left[d_{\mathcal{M}}(f(y^0+\varepsilon),f(y^0))^2\right]\ge 1.$$

By translating the argument of f, and multiplying (coordinate-wise) by an appropriate sign vector in $\{-1,1\}^n$, we may assume that $y^0 = 0$ and all the coordinates of x^0 are nonnegative. Observe that this implies that every $y \in \{0,1,\dots,s-1\}^n$ satisfies $d_{\mathbb{Z}_m^n}(x^0,y) < s$. Thus (58), and (59) imply that for every $y \in \{0,1,\dots,s-1\}^n$, every $j \in \{1,\dots,n\}$, every $\pi \in \mathcal{G}_j^{\pm}(y)$, and every $\ell \in \{1,\dots,s\}$,

$$\tag{61} \left| d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1})) - \frac{1}{s} d_{\mathcal{M}}\left(f\left(y + \mathrm{sgn}(\pi) s e_j\right), f(y)\right) \right| \le \sqrt{2\eta(2s-1)^n sn 2^{sn}} \le 2^{2sn}\sqrt{\eta}.$$

Claim 6.5. *For every $\varepsilon, \delta \in \{-1,1\}^n$ and every $x \in \mathbb{Z}_m^n$, such that $x+\varepsilon \in \{0,1,\dots,s-1\}^n$,*

$$|d_{\mathcal{M}}(f(x+\varepsilon), f(x)) - d_{\mathcal{M}}(f(x+\delta), f(x))| \le 2\sqrt{\eta}\cdot 2^{2sn}.$$

Proof. If $\varepsilon = \delta$ then there is nothing to prove, so assume that $\varepsilon_\ell = -\delta_\ell$. Denote $S = \{j \in \{1,\dots,n\} : \ \varepsilon_j = -\delta_j\}$ and define $\theta, \tau \in \{-1,1\}^n$ by

$$\theta_j = \begin{cases} -\varepsilon_\ell & j = \ell \\ \varepsilon_j & j \in S \setminus \{\ell\} \\ 1 & j \notin S \end{cases} \qquad \text{and} \qquad \tau_j = \begin{cases} -\varepsilon_\ell & j = \ell \\ \varepsilon_j & j \in S \setminus \{\ell\} \\ -1 & j \notin S. \end{cases}$$

Consider the following path π in $\mathbf{diag}(\mathbb{Z}_m^n)$: Start at $x+\varepsilon \in \{0,1,\dots,s-1\}^n$, go in direction $-\varepsilon$ (i.e. pass to x), then go in direction δ (i.e. pass to $x+\delta$), then go in direction θ (i.e. pass to $x+\delta+\theta$), then go in direction τ (i.e. pass to $x+\delta+\theta+\tau$), and repeat this process $s/4$ times. It is clear from the construction that $\pi \in \mathcal{G}_\ell^{\varepsilon_\ell}(x+\varepsilon)$. Thus, by (61) we get that

$$\begin{aligned} &|d_{\mathcal{M}}(f(x+\varepsilon), f(x)) - d_{\mathcal{M}}(f(x+\delta), f(x))| \\ &\qquad = |d_{\mathcal{M}}(f(\pi_1), f(\pi_0)) - d_{\mathcal{M}}(f(\pi_2), f(\pi_1))| \le 2\sqrt{\eta}\cdot 2^{2sn}. \end{aligned}$$

□

Corollary 6.6. *There exists a number $A \ge 1$ such that for every $\varepsilon \in \{-1,1\}^n$,*

$$\left(1 - 4\sqrt{\eta}\cdot 2^{2sn}\right) A \le d_{\mathcal{M}}(f(\varepsilon), f(0)) \le \left(1 + 4\sqrt{\eta}\cdot 2^{2sn}\right) A.$$

Proof. Denote $e = \sum_{j=1}^n e_j = (1,1,\dots,1)$ and take

$$A = \left(\mathbb{E}_\delta\left[d_{\mathcal{M}}(f(\delta), f(0))^2\right]\right)^{1/2}.$$

By (60), $A \ge 1$. By Claim 6.5 we know that for every $\varepsilon, \delta \in \{-1,1\}^{2^s}$,

$$d_{\mathcal{M}}(f(\varepsilon), f(0)) \le d_{\mathcal{M}}(f(e), f(0)) + 2\sqrt{\eta}\cdot 2^{2sn} \le d_{\mathcal{M}}(f(\delta), f(0)) + 4\sqrt{\eta}\cdot 2^{2sn}.$$

Averaging over δ, and using the Cauchy-Schwartz inequality, we get that

$$\begin{aligned} d_{\mathcal{M}}(f(\varepsilon),f(0)) &\le \left(\mathbb{E}_\delta\left[d_{\mathcal{M}}(f(\delta),f(0))^2\right]\right)^{1/2}+4\sqrt{\eta}\cdot 2^{2sn}\\ &= A+4\sqrt{\eta}\cdot 2^{2sn}\le \left(1+4\sqrt{\eta}\cdot 2^{2sn}\right)A. \end{aligned}$$

In the reverse direction we also know that

$$A^2=\mathbb{E}_\delta[d_{\mathcal{M}}(f(\delta),f(0))^2]\le\left[d_{\mathcal{M}}(f(\varepsilon),f(0))+4\sqrt{\eta}\cdot 2^{2sn}\right]^2,$$

which implies the required result since $A\ge 1$. $\square$

CLAIM 6.7. *Denote*

$$V=\left\{x\in\mathbb{Z}_m^n:\ \forall j\ 0\le x_j\le\frac{s}{2}\text{ and }x_j\text{ is even}\right\}. \tag{62}$$

Then the following assertions hold true:

1. *For every $x,y\in V$ there is some $z\in\{x,y\}$, $j\in\{1,\dots,n\}$, and a path $\pi\in\mathcal{G}_j^+(z)$ of length s which goes through x and y. Moreover, we can ensure that if $\pi=(\pi_0,\dots,\pi_s)$ then for all $\ell\in\{1,\dots,s\}$, $\{\pi_\ell,\pi_{\ell-1}\}\cap\{0,\dots,s-1\}^n\neq\emptyset$.*

2. *For every $x,y\in V$, $d_{\mathbf{diag}(\mathbb{Z}_m^n)}(x,y)=d_{\mathbb{Z}_m^n}(x,y)=\|x-y\|_\infty$.*

Proof. Let $j\in\{1,\dots,n\}$ be such that $|y_j-x_j|=\|x-y\|_\infty:=t$. Without loss of generality $y_j\ge x_j$. We will construct a path of length s in $\mathcal{G}_j^+(x)$ which goes through y. To begin with, we define $\varepsilon^\ell,\delta^\ell\in\{-1,1\}^n$ inductively on ℓ as follows:

$$\varepsilon_r^\ell=\begin{cases}1 & x_r+2\sum_{k=1}^{\ell-1}(\varepsilon_r^k+\delta_r^k)<y_r\\ -1 & x_r+2\sum_{k=1}^{\ell-1}(\varepsilon_r^k+\delta_r^k)>y_r\\ 1 & x_r+2\sum_{k=1}^{\ell-1}(\varepsilon_r^k+\delta_r^k)=y_r\end{cases}$$

and

$$\delta_r^\ell=\begin{cases}1 & x_r+2\sum_{k=1}^{\ell-1}(\varepsilon_r^k+\delta_r^k)<y_r\\ -1 & x_r+2\sum_{k=1}^{\ell-1}(\varepsilon_r^k+\delta_r^k)>y_r\\ -1 & x_r+2\sum_{k=1}^{\ell-1}(\varepsilon_r^k+\delta_r^k)=y_r.\end{cases}$$

If we define $a_\ell=x+\sum_{k=1}^{\ell}\varepsilon^k+\sum_{k=1}^{\ell-1}\delta^k$ and $b_\ell=x+\sum_{k=1}^{\ell}\varepsilon^k+\sum_{k=1}^{\ell}\delta^k$ then the sequence

$$(x,a_1,b_1,a_2,b_2,\dots,a_{t/2-1},b_{t/2}=y)$$

is a path of length t in $\mathbf{diag}(\mathbb{Z}_m^n)$ joining x and y. This proves the second assertion above. We extend this path to a path of length s (in $\mathbf{diag}(\mathbb{Z}_m^n)$) from x to $x+se_j$ as follows. Observe that for every $1\le\ell\le t/2$, $\varepsilon_j^\ell=\delta_j^\ell=1$. Thus $-\varepsilon^\ell+2e_j,-\delta^\ell+2e_j\in\{-1,1\}^n$. If we define $c_\ell=y+\sum_{k=1}^{\ell}(-\varepsilon^k+2e_j)+\sum_{k=1}^{\ell-1}(-\delta^k+2e_j)$ and $d_\ell=y+\sum_{k=1}^{\ell}(-\varepsilon^k+2e_j)+\sum_{k=1}^{\ell}(-\delta^k+2e_j)$, then

$d_{t/2} = x + 2te_j$. Observe that by the definition of V, $2t \le s$, and $s - 2t$ is even. Thus we can continue the path from $x + 2te_j$ to $x + se_j$ by alternatively using the directions $e_j + \sum_{\ell \neq j} e_\ell$ and $e_j - \sum_{\ell \neq j} e_\ell$. □

Corollary 6.8. *Assume that $x \in V$. Then for A as in Corollary* 6.6, *we have for all $\varepsilon \in \{-1, 1\}^n$,*

$$\left(1 - 10\sqrt{\eta} \cdot 2^{2sn}\right) A \le d_{\mathcal{M}}(f(x+\varepsilon), f(x)) \le \left(1 + 10\sqrt{\eta} \cdot 2^{2sn}\right) A.$$

Proof. By Claim 6.7 (and its proof), there exist $j \in \{1, \ldots, n\}$ and $\pi \in \mathcal{G}_j^+(0)$ such that $\pi_1 = e = (1, \ldots, 1)$ and for some $k \in \{1, \ldots, s\}$, $\pi_k = x$. Now, by (61) we have that

$$|d_{\mathcal{M}}\left(f(e), f(0)\right) - d_{\mathcal{M}}\left(f(\pi_{k-1}), f(x)\right)| \le 2\sqrt{\eta} \cdot 2^{2sn}.$$

Observe that since $x \in V$, $x + e \in \{0, \ldots, s-1\}^n$. Thus by Claim 6.5

$$\begin{aligned}
|d_{\mathcal{M}}&\left(f(x+\varepsilon), f(x)\right) - d_{\mathcal{M}}\left(f(e), f(0)\right)| \\
&\le |d_{\mathcal{M}}\left(f(e), f(0)\right) - d_{\mathcal{M}}\left(f(\pi_{k-1}), f(x)\right)| \\
&\quad + |d_{\mathcal{M}}\left(f(\pi_{k-1}), f(x)\right) - d_{\mathcal{M}}\left(f(x+e), f(x)\right)| \\
&\quad + |d_{\mathcal{M}}\left(f(x+\varepsilon), f(x)\right) - d_{\mathcal{M}}\left(f(x+e), f(x)\right)| \\
&\le 6\sqrt{\eta} \cdot 2^{2sn},
\end{aligned}$$

so that the required inequalities follow from Corollary 6.6. □

Corollary 6.9. *For every distinct $x, y \in V$,*

$$\left(1 - 12\sqrt{\eta} \cdot 2^{2sn}\right) A \le \frac{d_{\mathcal{M}}(f(x), f(y))}{\|x - y\|_\infty} \le \left(1 + 12\sqrt{\eta} \cdot 2^{2sn}\right) A,$$

where A is as in Corollary 6.6.

Proof. Denote $t = \|x - y\|_\infty$; we may assume that there exists $j \in \{1, \ldots, n\}$ such that $y_j - x_j = t$. By Claim 6.7 there is a path $\pi \in \mathcal{G}_j^+(x)$ of length s such that $\pi_t = y$. By (61) and Corollary 6.8 we have for every $\ell \in \{1, \ldots, s\}$

$$\left| d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1})) - \frac{1}{s} d_{\mathcal{M}}\left(f\left(x + se_j\right), f(x)\right) \right| \le \sqrt{\eta} \cdot 2^{2sn},$$

and

$$\left(1 - 10\sqrt{\eta} \cdot 2^{2sn}\right) A \le d_{\mathcal{M}}(f(\pi_0), f((\pi_1)) \le \left(1 + 10\sqrt{\eta} \cdot 2^{2sn}\right) A.$$

Thus, for all $\ell \in \{1, \ldots, s\}$,

$$\left(1 - 12\sqrt{\eta} \cdot 2^{2sn}\right) A \le d_{\mathcal{M}}(f(\pi_\ell), f(\pi_{\ell-1})) \le \left(1 + 12\sqrt{\eta} \cdot 2^{2sn}\right) A.$$

Thus

$$\begin{aligned} d_{\mathcal{M}}(f(x),f(y)) &\le \sum_{\ell=1}^{t} d_{\mathcal{M}}(f(\pi_\ell),f(\pi_{\ell-1})) \le t\cdot\left(1+12\sqrt{\eta}\cdot 2^{2sn}\right)A\\ &= \|x-y\|_\infty\cdot\left(1+12\sqrt{\eta}\cdot 2^{2sn}\right)A. \end{aligned}$$

On the other hand

$$\begin{aligned} d_{\mathcal{M}}(f(x),f(y)) &\ge d_{\mathcal{M}}(f(x+se_j),f(x)) - d_{\mathcal{M}}(f(x+se_j),f(y))\\ &\ge s d_{\mathcal{M}}(f(x),f(\pi_1)) - s\sqrt{\eta}\cdot 2^{2sn} - \sum_{\ell=t+1}^{s} d_{\mathcal{M}}(f(\pi_\ell),f(\pi_{\ell-1}))\\ &\ge s\left(1-10\sqrt{\eta}\cdot 2^{2sn}\right)A - s\sqrt{\eta}\cdot 2^{2sn}\\ &\quad -(s-t)\left(1-12\sqrt{\eta}\cdot 2^{2sn}\right)A\\ &\ge \|x-y\|_\infty\cdot\left(1-12\sqrt{\eta}\cdot 2^{2sn}\right)A. \end{aligned}$$

□

This concludes the proof of Theorem 6.4, since the mapping $x \mapsto x/2$ is a distortion 1 bijection between $(V, d_{\mathbb{Z}_m^n})$ and $[s/4]_\infty^n$. □

We are now in position to prove Theorem 1.5.

Proof of Theorem 1.5. We assume that $\Gamma_q^{(2)}(\mathcal{M}) = \infty$ for all $q < \infty$. By Lemma 6.3 it follows that for every two integers $n, s > 1$, $\mathcal{B}(\mathcal{M}; n, s) = 1$. Now the required result follows from Theorem 6.4. □

LEMMA 6.10. *Let $\mathcal{M}$ be a metric space and $K > 0$. Fix $q < \infty$ and assume that $m := m_q^{(2)}(\mathcal{M}; n, K) < \infty$. Then*

$$c_{\mathcal{M}}\left(\mathbb{Z}_m^n\right) \ge \frac{n^{1/q}}{2K}.$$

Proof. Fix a bijection $f : \mathbb{Z}_m^n \to \mathcal{M}$. Then

$$\begin{aligned} \frac{nm^2}{4\|f^{-1}\|_{\mathrm{Lip}}^2} &\le \sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(f\left(x+\frac{m}{2}e_j\right), f(x)\right)^2 d\mu(x)\\ &\le K^2 m^2 n^{1-\frac{2}{q}} \int_{\{-1,01\}^n}\int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(f(x+\varepsilon),f(x))^2 d\mu(x)d\sigma(\varepsilon)\\ &\le K^2 m^2 n^{1-\frac{2}{q}}\|f\|_{\mathrm{Lip}}^2. \end{aligned}$$

It follows that $\mathrm{dist}(f) \ge \frac{n^{1/q}}{2K}$. □

COROLLARY 6.11. *Let $\mathcal{F}$ be a family of metric spaces and $0 < q, K, c < \infty$. Assume that for all $n \in \mathbb{N}$, $\Gamma_q^{(2)}(\mathcal{M}; n, n^c) \le K$ for every $\mathcal{M} \in \mathcal{F}$. Then for every integer N,*

$$\mathcal{D}_N(\mathcal{F}) \ge \frac{1}{2cK}\left(\frac{\log N}{\log\log N}\right)^{1/q}.$$

We require the following simple lemma, which shows that the problems of embedding $[m]_\infty^n$ and $\mathbb{Z}_m^n$ are essentially equivalent.

LEMMA 6.12. *The grid $[m]_\infty^n$ embeds isometrically into $\mathbb{Z}_{2m}^n$. Conversely, $\mathbb{Z}_{2m}^n$ embeds isometrically into $[m+1]_\infty^{2mn}$. Moreover, for each $\varepsilon > 0$, $\mathbb{Z}_{2m}^n$ embeds with distortion $1+6\varepsilon$ into $[m+1]^{(\lceil 1/\varepsilon\rceil+1)n}$.*

Proof. The first assertion follows by consideration of only elements of $\mathbb{Z}_{2m}^n$ whose coordinates are at most $m-1$. Next, the Fréchet embedding $x \mapsto (d_{\mathbb{Z}_{2m}}(x,0), d_{\mathbb{Z}_{2m}}(x,1), \dots, d_{\mathbb{Z}_{2m}}(x,2m-1)) \in [m+1]_\infty^{2m}$, is an isometric embedding of $\mathbb{Z}_{2m}$. Thus $\mathbb{Z}_{2m}^n$ embeds isometrically into $[m+1]_\infty^{2mn}$. The final assertion is proved analogously by showing that $\mathbb{Z}_{2m}$ embeds with distortion $1+\varepsilon$ into $[m+1]_\infty^{\lceil 1/\varepsilon\rceil+1}$. This is done by consideration of the embedding

$$x \mapsto (d_{\mathbb{Z}_m}(x,0), d_{\mathbb{Z}_m}(x,\lfloor 2\varepsilon m\rfloor), d_{\mathbb{Z}_m}(x,\lfloor 4\varepsilon m\rfloor), d_{\mathbb{Z}_m}(x,\lfloor 6\varepsilon m\rfloor), \dots \\ \dots, d_{\mathbb{Z}_m}(x,\lfloor 2\lceil 1/\varepsilon\rceil \varepsilon m\rfloor)),$$

which is easily seen to have distortion at most $1+6\varepsilon$. □

We are now in position to prove Theorem 1.6.

Proof of Theorem 1.6. We first prove the implication 1) $\implies$ 2). Let Z be the disjoint union of all finite subsets of members of $\mathcal{F}$, i.e.

$$Z = \bigsqcup \{\mathcal{N} :\ |\mathcal{N}| < \infty \text{ and } \exists\ \mathcal{M} \in \mathcal{F},\ \mathcal{N} \subseteq \mathcal{M}\}.$$

For every $k > 1$ we define a metric d_k on Z by

$$d_k(x,y) = \begin{cases} \frac{d_{\mathcal{N}}(x,y)}{\operatorname{diam}(\mathcal{N})} & \exists\ \mathcal{M} \in \mathcal{F},\ \exists\ \mathcal{N} \subseteq \mathcal{M} \text{ s.t. } |\mathcal{N}| < \infty \text{ and } x,y \in \mathcal{N} \\ k & \text{otherwise.} \end{cases}$$

Clearly d_k is a metric. Moreover, by construction, for every $K, k > 1$,

$$q^{(2)}_{(Z,d_k)}(K) \geq q^{(2)}_{\mathcal{F}}(K).$$

Assume for the sake of contradiction that for every $K, k > 1$, $q^{(2)}_{(Z,d_k)}(K) = \infty$. In other words, for every $q < \infty$, and $k \geq 1$, $\Gamma_q^{(2)}(Z,d_k) = \infty$. By Lemma 6.3 it follows that for every $k \geq 1$, and every two integers $n, s > 1$,

$$\mathcal{B}\left((Z,d_k); n, s\right) = 1.$$

Theorem 6.4 implies that $c_{(Z,d_k)}\left([m]_\infty^n\right) = 1$.

By our assumption there exists a metric space X such that $c_{\mathcal{F}}(X) := D > 1$. Define a metric space $X' = X \times \{1,2\}$ via $d_{X'}((x,1),(y,1)) = d_{X'}((x,2),(y,2)) = d_X(x,y)$ and $d_{X'}((x,1),(y,2)) = 2\operatorname{diam}(X)$. For large enough s we have that $c_{[2^{s-3}]_\infty^{2^s}}(X') < D$. Thus $c_{(Z,d_k)}(X') < D$ for all k. Define

$$k = \frac{4\,\mathrm{diam}(X)}{\min_{\substack{x,y\in X\\ x\neq y}} d_X(x,y)}.$$

Then there exists a bijection $f : X' \to (Z, d_k)$ with $\mathrm{dist}(f) < \min\{2, D\}$. Denote $L = \|f\|_{\mathrm{Lip}}$.

We first claim that there exists $\mathcal{M} \in \mathcal{F}$, and a finite subset $\mathcal{N} \subseteq \mathcal{M}$, such that $|f(X') \cap \mathcal{N}| \geq 2$. Indeed, otherwise, by the definition of d_k, for all $x', y' \in X'$, $d_k(f(x'), f(y')) = k$. Choosing distinct $x, y \in X$, we deduce that

$$k = d_k(f(x,1), f(y,1)) \leq L d_X(x,y) \leq L\,\mathrm{diam}(X),$$

and

$$\begin{aligned} k = d_k(f(x,1), f(y,2)) &\geq \frac{L}{\mathrm{dist}(f)} \cdot d_{X'}((x,1),(y,2)) \\ &> \frac{L}{2} \cdot 2\,\mathrm{diam}(X) = L\,\mathrm{diam}(X), \end{aligned}$$

which is a contradiction.

Thus, there exists $\mathcal{M} \in \mathcal{F}$ and a finite subset $\mathcal{N} \subseteq \mathcal{M}$ such that $|f(X') \cap \mathcal{N}| \geq 2$. We claim that this implies that $f(X') \subseteq \mathcal{N}$. This will conclude the proof of 1) $\implies$ 2), since the metric induced by d_k on $\mathcal{N}$ is a re-scaling of $d_{\mathcal{N}}$, so that X embeds with distortion smaller than D into $\mathcal{N} \subseteq \mathcal{M} \in \mathcal{F}$, which is a contradiction of the definition of D.

Assume for the sake of a contradiction that there exists $x' \in X'$ such that $f(x') \notin \mathcal{N}$. By our assumption there are distinct $a', b' \in X'$ such that $f(a'), f(b') \in \mathcal{N}$. Now,

$$1 \geq d_k(f(a'), f((b')) \geq \frac{L}{\mathrm{dist}(f)} \cdot d_{X'}(a', b') > \frac{L}{2} \cdot \min_{\substack{u,v\in X\\ u\neq v}} d_X(u,v),$$

while

$$\begin{aligned} \frac{4\,\mathrm{diam}(X)}{\min_{\substack{u,v\in X\\ u\neq v}} d_X(u,v)} &= k = d_k(f(x'), f((a')) \\ &\leq L d(x', a') \leq L\,\mathrm{diam}(X') = 2L\,\mathrm{diam}(X), \end{aligned}$$

which is a contradiction.

To prove the implication 2) $\implies$ 3) observe that in the above argument we have shown that there exists $k, q < \infty$ such that $\Gamma_q^{(2)}(Z, d_k) < \infty$. It follows that for some integer n_0, $\mathcal{B}((Z, d_k); n_0, n_0) < 1$, since otherwise by Theorem 6.4 we would get that (Z, d_k) contains, uniformly in n, bi-Lipschitz copies of $[n]_\infty^n$. Combining Lemma 6.12 and Lemma 6.10 we arrive at a contradiction. By Lemma 6.3, the fact that $\mathcal{B}((Z, d_k); n_0, n_0) < 1$, combined with Corollary 6.11, implies that $\mathcal{D}_n(Z, d_k) = \Omega((\log n)^\alpha)$ for some $\alpha > 0$. By the definition of (Z, d_k), this implies the required result. $\square$

We end this section by proving Theorem 1.8:

Proof of Theorem 1.8. Denote $|X| = n$ and

$$\Phi = \frac{\operatorname{diam}(X)}{\min_{x\neq y} d(x,y)}.$$

Write $t = 4\Phi/\varepsilon$ and let s be an integer divisible by 4 such that $s \geq \max\{n, t\}$. Then $c_{[s]_\infty^s}(X) \leq 1 + \frac{\varepsilon}{4}$. Fix a metric space Z and assume that $c_Z(X) > 1 + \varepsilon$. It follows that $c_Z([s]_\infty^s) \geq 1 + \frac{\varepsilon}{2}$. By Theorem 6.4 we deduce that

$$\mathcal{B}(Z, s, 4s) \leq 1 - \frac{\varepsilon^2}{2^{s^2}}.$$

By Lemma 6.3 we have that $m_q^{(2)}(\mathcal{M}; n, 3s) \leq 8sn^{\log_s(4s)}$, where $q \leq \frac{10^s}{\varepsilon^2}$. Thus by Lemma 6.10 and Lemma 6.12 we see that for any integer $n \geq 8s$,

$$c_Z\left(\left[n^5\right]_\infty^n\right) \geq \frac{n^{1/q}}{4s} = \frac{n^{\varepsilon^2/10^s}}{4s}.$$

Choosing $N \approx (C\gamma)^{\frac{2^{4s}}{\varepsilon^2}}$, for an appropriate universal constant C, yields the required result. $\square$

7. Applications to bi-Lipschitz, uniform, and coarse embeddings

Let $(\mathcal{N}, d_{\mathcal{N}})$ and $(\mathcal{M}, d_{\mathcal{M}})$ be metric spaces. For $f : \mathcal{N} \to \mathcal{M}$ and $t > 0$ we define

$$\Omega_f(t) = \sup\{d_{\mathcal{M}}(f(x), f(y));\ d_{\mathcal{N}}(x,y) \leq t\},$$

and

$$\omega_f(t) = \inf\{d_{\mathcal{M}}(f(x), f(y));\ d_{\mathcal{N}}(x,y) \geq t\}.$$

Clearly Ω_f and ω_f are nondecreasing, and for every $x, y \in \mathcal{N}$,

$$\omega_f\left(d_{\mathcal{N}}(x,y)\right) \leq d_{\mathcal{M}}(f(x), f(y)) \leq \Omega_f\left(d_{\mathcal{N}}(x,y)\right)$$

With these definitions, f is uniformly continuous if $\lim_{t\to 0} \Omega_f(t) = 0$, and f is a uniform embedding if f is injective and both f and f^{-1} are uniformly continuous. Also, f is a coarse embedding if $\Omega_f(t) < \infty$ for all $t > 0$ and $\lim_{t\to\infty} \omega_f(t) = \infty$.

Lemma 7.1. *Let* $(\mathcal{M}, d_{\mathcal{M}})$ *be a metric space,* n *an integer,* $\Gamma > 0$, *and* $0 < p \leq q \leq r$. *Then for every function* $f : \ell_r^n \to \mathcal{M}$, *and every* $s > 0$,

$$n^{1/q}\omega_f(2s) \leq \Gamma m_q^{(p)}(\mathcal{M}; n, \Gamma) \cdot \Omega_f\left(\frac{2\pi s n^{1/r}}{m_q^{(p)}(\mathcal{M}; n, \Gamma)}\right).$$

Proof. Denote $m = m_q^{(p)}(\mathcal{M}; n, \Gamma)$, and define $g : \mathbb{Z}_m^n \to \mathcal{M}$ by

$$g(x_1, \dots, x_n) = f\left(\sum_{j=1}^n s e^{\frac{2\pi i x_j}{m}} e_j\right).$$

Then

$$\int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}(g(x+\varepsilon), g(x))^p d\mu(x) d\sigma(\varepsilon)$$
$$\le \max_{\varepsilon \in \{-1,0,1\}^n} \Omega_f\left(s\left(\sum_{j=1}^n \left|e^{\frac{2\pi i \varepsilon_j}{m}} - 1\right|^r\right)^{1/r}\right)^p \le \Omega_f\left(\frac{2\pi s n^{1/r}}{m}\right)^p.$$

On the other hand,

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_{\mathcal{M}}\left(g\left(x + \frac{m}{2} e_j\right), g(x)\right)^p d\mu(x) \ge n\omega_f(2s)^p.$$

By the definition of $m_q^{(p)}(\mathcal{M}; n, \Gamma)$ it follows that

$$n\omega_f(2s)^p \le \Gamma^p m^p n^{1-\frac{p}{q}} \Omega_f\left(\frac{2\pi s n^{1/r}}{m}\right)^p,$$

as required. □

Corollary 7.2. *Let $\mathcal{M}$ be a metric space and assume that there exist constants $c, \Gamma > 0$ such that for infinitely many integers n, $m_q^{(p)}(\mathcal{M}; n, \Gamma) \le c n^{1/q}$. Then for every $r > q$, ℓ_r does not uniformly or coarsely embed into $\mathcal{M}$.*

Proof. To rule out the existence of a coarse embedding choose $s = n^{\frac{1}{q} - \frac{1}{r}}$ in Lemma 7.1. Using Lemma 2.3 we get that

$$\omega_f\left(2n^{\frac{1}{q} - \frac{1}{r}}\right) \le c\Gamma\Omega_f(2\pi\Gamma).$$

Since $q < r$, it follows that $\liminf_{t\to\infty} \omega_f(t) < \infty$, so that f is not a coarse embedding.

To rule out the existence of a uniform embedding, assume that $f : \ell_r \to X$ is invertible and f^{-1} is uniformly continuous. Then there exists $\delta > 0$ such that for $x, y \in \ell_r$, if $d_{\mathcal{M}}(f(x), f(y)) < \delta$ then $\|x - y\|_r < 2$. It follows that $\omega_f(2) \ge \delta$. Choosing $s = 1$ in Lemma 7.1, and using Lemma 2.3, we get that

$$0 < \delta \le \omega_f(2) \le c\Gamma\Omega_f\left(2\pi\Gamma \cdot n^{\frac{1}{r} - \frac{1}{q}}\right).$$

Since $r > q$ it follows that $\limsup_{t\to 0} \Omega_f(t) > 0$, so that f is not uniformly continuous. □

The following corollary contains Theorem 1.9, Theorem 1.10 and Theorem 1.11.

Corollary 7.3. *Let X be a K-convex Banach space. Assume that Y is a Banach space which coarsely or uniformly embeds into X. Then $q_Y \le q_X$. In particular, for $p, q > 0$, L_p embeds uniformly or coarsely into L_q if and only if $p \le q$ or $q \le p \le 2$.*

Proof. By the Maurey-Pisier theorem [46], for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, Y contains a $(1+\varepsilon)$ distorted copy of $\ell_{q_Y}^n$. By Theorem 4.1, since X is K-convex, for every $q > q_X$ there exists $\Gamma < \infty$ such that $m_q(\mathcal{M}; n, \Gamma) = O\left(n^{1/q}\right)$. Thus, by the proof of Corollary 7.2, if Y embeds coarsely or uniformly into X then $q_Y \le q$, as required.

The fact that if $p \le q$ then L_p embeds coarsely and uniformly into L_q follows from the fact that in this case L_p, equipped with the metric $\|x-y\|_p^{p/q}$, embeds *isometrically* into L_q (for $p \le q \le 2$ this is proved in [12], [70]. For the remaining cases see Remark 5.10 in [47]). If $2 \ge p \ge q$ then L_p is linearly isometric to a subspace of L_q (see e.g. [71]). It remains to prove that if $p > q$ and $p > 2$ then L_p does not coarsely or uniformly embed into L_q. We may assume that $q \ge 2$, since for $q \le 2$, L_q embeds coarsely and uniformly into L_2. But, now the required result follows from the fact that L_q is K convex and $q_{L_q} = q$, $q_{L_p} = p$ (see [50]). □

We now pass to the proof of Theorem 1.12. Before doing so we remark that Theorem 1.12 is almost optimal in the following sense. The identity mapping embeds $[m]_\infty^n$ into ℓ_q^n with distortion $n^{1/q}$. By the Maurey-Pisier theorem [46], Y contains a copy of $\ell_{q_Y}^n$ with distortion $1+\varepsilon$ for every $\varepsilon > 0$. Thus $c_Y([m]_\infty^n) \le n^{1/q_Y}$. Additionally, $[m]_\infty^n$ is m-equivalent to an equilateral metric. Thus, if Y is infinite dimensional then $c_Y([m]_\infty^n) \le m$. It follows that

$$c_Y([m]_\infty^n) \le \min\left\{n^{1/q_Y}, m\right\}.$$

Proof of Theorem 1.12. Assume that m is divisible by 4 and

$$m \ge \frac{2n^{1/q}}{C_q(Y)K(Y)}.$$

By Theorem 4.1, for every $f : \mathbb{Z}_m^n \to Y$,

$$\begin{aligned}\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|f\left(x+\frac{m}{2}\right) - f(x)\right\|_Y^q d\mu(x) \\ \le [15C_q(Y)K(Y)]^q\, m^q \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x)\|_Y^q d\mu(x) d\sigma(\varepsilon).\end{aligned}$$

Thus, assuming that f is bi-Lipschitz we get that

$$\frac{nm^q}{2^q\|f^{-1}\|_{\mathrm{Lip}}^q} \le [15C_q(Y)K(Y)]^q\, m^q \cdot \|f\|_{\mathrm{Lip}}^q,$$

i.e.

$$\mathrm{dist}(f) \ge \frac{n^{1/q}}{30C_q(Y)K(Y)}.$$

By Lemma 6.12 this shows that for $m \ge \frac{2n^{1/q}}{C_q(Y)K(Y)}$, such that m is divisible by 4, $c_Y([m]_\infty^n) = \Omega\left(n^{1/q}\right)$. If $m < \frac{2n^{1/q}}{C_q(Y)K(Y)}$ then the required lower bound follows from the fact that $[m]_\infty^n$ contains an isometric copy of $[m_1]_\infty^{n_1}$, where m_1 is an integer divisible by 4, $m_1 \ge \frac{2n_1^{1/q}}{C_q(Y)K(Y)}$, and $m_1 = \Theta(m)$, $n_1 = \Theta(m^q)$. Passing to integers m which are not necessarily divisible by 4 is just as simple. □

Remark 7.4. Similar arguments yield bounds on $c_Y([m]_p^n)$, which strengthen the bounds in [48].

Remark 7.5. Although L_1 is not K-convex, we can still show that

$$c_1([m]_\infty^n) = \Theta\left(\min\left\{\sqrt{n}, m\right\}\right).$$

This is proved as follows. Assume that $f:\mathbb{Z}_m^n \to L_1$ is bi-Lipschitz. If m is divisible by 4, and $m \ge \pi\sqrt{n}$, then the fact that L_1, equipped with the metric $\sqrt{\|x-y\|_1}$, is isometric to a subset of Hilbert space [70], [14], together with Proposition 3.1, shows that

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|f\left(x+\frac{m}{2}\right) - f(x)\right\|_1 d\mu(x) \le m^2 \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon)-f(x)\|_1 d\mu(x) d\sigma(\varepsilon).$$

Arguing as in the proof of Theorem 1.12, we see that for $m \approx \sqrt{n}$, $c_1([m]_\infty^n) = \Omega\left(\sqrt{n}\right)$. This implies the required result, as in the proof of Theorem 1.12.

8. Discussion and open problems

1. Perhaps the most important open problem related to the nonlinear cotype inequality on Banach spaces is whether for every Banach space X with cotype $q < \infty$, for every $1 \le p \le q$ there is a constant $\Gamma < \infty$ such that $m_q^{(p)}(X; n, \Gamma) = O\left(n^{1/q}\right)$. By Lemma 2.3 this is best possible. In Theorem 4.1 we proved that this is indeed the case when X is K-convex, while our proof of Theorem 1.2 only gives $m_q^{(p)}(X; n, \Gamma) = O\left(n^{2+1/q}\right)$.

2. L_1 is not K-convex, yet we do know that $m_2^{(1)}(L_1;n,4) = O(\sqrt{n})$. This follows directly from Remark 7.5, Lemma 2.4 and Lemma 2.5. It would be interesting to prove the same thing for $m_2(L_1;n,\Gamma)$.

3. We conjecture that the K-convexity assumption in Theorem 1.9 and Theorem 1.11 is not necessary. Since L_1 embeds coarsely and uniformly into L_2, these theorems do hold for L_1. It seems to be unknown whether any Banach space with finite cotype embeds uniformly or coarsely into a K-convex Banach space. The simplest space for which we do not know the conclusion of these theorems is the Schatten trace class C_1 (see [71]. In [67] it is shown that this space has cotype 2). The fact that C_1 does not embed uniformly into Hilbert space follows from the results of [2], together with [56], [32]. For more details we refer to the discussion in [6] (a similar argument works for coarse embeddings of C_1 into Hilbert space, by use of [61]). We remark that the arguments presented here show that a positive solution of the first problem stated above would yield a proof of Theorem 1.9 and Theorem 1.11 without the K-convexity assumption.

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