LYAPUNOV EXPONENTS, ENTROPY, AND ZIMMER'S CONJECTURE FOR ACTIONS OF COCOMPACT LATTICES

AARON BROWN

These notes present a self-contained proof of the following recent result from [16] which verifies Zimmer's conjecture on the finiteness of actions by cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$ for $n \geqslant 3$.

Theorem 3.4. *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a cocompact lattice. Let M be a compact manifold.*

(1) If $\dim(M) < n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}^2(M)$ has finite image.
(2) In addition, if vol is a volume form on M and if $\dim(M) = n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ has finite image.

See Conjecture 3.2 and Conjecture 3.3 for precise statements of the conjecture.

Outside of this text, we also refer the reader to the excellent exposition by Serge Cantat [25] that also presents (in French) a complete proof of Theorem 3.4.

These notes are organized as follows. In the introduction, Section 1, we discuss group actions, some general rigidity programs, and discuss and motivate the Zimmer program. In Part 1, we present background on lattices in Lie groups, examples of standard algebraic actions,, and state the main conjecture, Conjecture 3.2, as well as the main results, Theorem 3.4 and Theorem 3.5, discussed in this text. We reduce the proof of Theorem 3.4 to Theorem 5.2 in Section 5. In Part 2 we present a number of tools and constructions from smooth ergodic theory with an emphasis on Lyapunov exponents, metric entropy, and geometry of conditional measures along foliations for actions of higher-rank abelian groups. In Part 3 we use the tools developed in Part 2 to prove Theorem 3.5 and Theorem 5.2.

CONTENTS

Preliminary version. Updated March 26, 2019.
Send comments to `awb@uchicago.edu`

Introduction

1. Groups acting on manifolds and the Zimmer program

In the classical theory dynamical systems, one typically studies actions of 1-parameter groups: Given a compact manifold M, a diffeomorphism $f\colon M \to M$ generates an action of the group $\mathbb{Z}$; a smooth vector field X on M generates a flow $\phi^t\colon M \to M$ or an action of the group $\mathbb{R}$. However, one might consider groups more general than $\mathbb{Z}$ or $\mathbb{R}$ acting on a manifold M. Natural families of group actions arise naturally in many geometric and algebraic settings and the study of group actions connects many areas of mathematics including geometric group theory, representation theory, Lie theory, geometry, and dynamical systems.

This text focuses on various *rigidity programs* for group actions. Roughly, such rigidity results aim to classify all actions or all invariant geometric structures (such as closed subsets, probability measures, etc.) under

(1) suitable algebraic hypotheses on the acting group, and/or
(2) suitable dynamical hypotheses on the action.

This primarily takes the first approach: under certain algebraic conditions on the acting group, we establish certain rigidity properties of the action. Specifically, we will consider actions of lattices Γ in higher-rank simple Lie groups such as $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ for $n \geqslant 3$. In this introduction, we also impose certain dynamical hypotheses for instance, by considering affine Anosov actions.

1.1. **Smooth group actions.** Let M be a compact manifold without boundary and denote by $\mathrm{Diff}^r(M)$ the group of C^r diffeomorphisms $f\colon M \to M$. Recall that if $r \geqslant 1$ is not integral then, writing

$$r = k + \beta \quad \text{for } k \in \mathbb{N} \text{ and } \beta \in (0,1),$$

we say that $f\colon M \to M$ is C^r or is $C^{k+\beta}$ if it is C^k and if the kth derivatives of f are β-Hölder continuous.

For $r \geqslant 1$, the set $\mathrm{Diff}^r(M)$ has a group structure given by composition of maps. Given a (typically countably infinite, finitely generated) discrete group Γ, a $\boldsymbol{C^r}$ **action** of Γ on M is a homomorphism

$$\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$$

from the group Γ into the group $\mathrm{Diff}^r(M)$; that is, for each $\gamma \in \Gamma$ the image $\alpha(\gamma)$ is a C^r diffeomorphism $\alpha(\gamma)\colon M \to M$ and for $x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\alpha(\gamma_1\gamma_2)(x) = \alpha(\gamma_1)\big(\alpha(\gamma_2)(x)\big).$$

If the discrete group Γ is instead replaced by a Lie group G, we also require that the map $G \times M \to M$ given by $(g, x) \mapsto \alpha(g)(x)$ be C^r.

If vol is some fixed volume form on M we write $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ for the group of C^r-diffeomorphisms preserving vol. A **volume-preserving** action is a homomorphism $\alpha \colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$ for some volume form vol.

As discussed above, actions of the group of integers $\mathbb{Z}$ are generated by iteration of a single diffeomorphism $f \colon M \to M$ and its inverse. For instance, given an integer $n > 1$, the diffeomorphism $\alpha(n) \colon M \to M$ is defined as the nth iterate of f: for $x \in M$

$$\alpha(n)(x) = f^n(x) := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x).$$

Given a manifold M, any pair of diffeomorphisms $f, g \in \mathrm{Diff}(M)$ naturally induces an action of F_2, the free group on two generators, by associating to every reduced word in $\{f, g, f^{-1}, g^{-1}\}$ the diffeomorphism obtained by composing elements of the word. If a pair of diffeomorphisms $f \colon M \to M$ and $g \colon M \to M$ commute, we naturally obtain a $\mathbb{Z}^2$-action $\alpha \colon \mathbb{Z}^2 \to \mathrm{Diff}(M)$ given by

$$\alpha(n, m)(x) = f^n \circ g^m(x).$$

1.2. **Representations of higher-rank lattices and the Zimmer program.** The primary family of discrete groups considered in this text are lattices Γ in (typically higher-rank, see Section 2.2) simple Lie groups G. That is, we consider discrete subgroups $\Gamma \subset G$ such that G/Γ has finite volume. Examples of such groups include $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ where $G = \mathrm{SL}(n, \mathbb{R})$ (which is higher-rank if $n \geqslant 3$) and the free group F_2 on two generators where $G = \mathrm{SL}(2, \mathbb{R})$ (which has rank 1.) See Section 2.1 for background and additional details.

1.2.1. *Linear representations.* To motivate the results and conjectures concerning smooth actions of such Γ, first consider the setting of linear representations $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$. A linear representation $\pi \colon \mathbb{Z} \to \mathrm{GL}(d, \mathbb{R})$ of the group of integers is determined by a choice of a matrix $A \in \mathrm{GL}(d, \mathbb{R})$; similarly, a linear representation $\pi \colon F_2 \to \mathrm{GL}(d, \mathbb{R})$ of the free group F_2 is determined by a choice of a pair of matrices $A, B \in \mathrm{GL}(d, \mathbb{R})$. These representations may be perturbed to non-conjugate representations $\tilde{\pi}$. When $G = \mathrm{SL}(2, \mathbb{R})$, the inclusion $\iota \colon \Gamma \to G$ of a lattice subgroup is not locally rigid. Indeed, when Γ is the fundamental group of a closed orientable surface of genus $g \geqslant 2$, the space of deformations is the $(6g - g)$-dimensional Teichmuller space.

In contrast, for groups such as $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ for $n \geqslant 3$ (and other lattices Γ in higher-rank simple Lie groups), linear representations $\pi \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ are very rigid as demonstrated by various classical results including [78, 80, 85, 94, 106, 113]. For instance, for cocompact Γ, local rigidity results in [106, 113] established that any representation $\pi \colon \Gamma \to \mathrm{SL}(n, \mathbb{Z})$ sufficiently close to the inclusion $\iota \colon \Gamma \to \mathrm{SL}(n, \mathbb{Z})$ is conjugate to ι. A cohomological criteria for local rigidity of general linear representations $\pi \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ was given in [114], further studied in [83, 99], and is known to hold for lattices in higher-rank simple Lie groups. Margulis's superrigidity theorem (see Theorem 4.3 below and [80]) establishes that every linear representation $\pi \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ extends to a representation $\overline{\pi} \colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(d, \mathbb{R})$ up to a "compact error;" this effectively classifies all representations $\Gamma \to \mathrm{GL}(d, \mathbb{R})$ up to conjugacy.

1.2.2. *Smooth actions of lattices.* As in the case of linear representations, actions of $\mathbb{Z}$ or F_2 on a manifold M are determined by a choice of diffeomorphism $f \in \mathrm{Diff}^r(M)$ or pair of diffeomorphisms $f, g \in \mathrm{Diff}^r(M)$. Such actions may be perturbed to create new actions that are inequivalent under change of coordinates. In particular, there is no possible classification of all actions of $\mathbb{Z}$ or F_2 on arbitrary manifolds M. The free group on two generators F_2 and the group $\mathrm{SL}(2,\mathbb{Z})$ are lattices in the Lie group $\mathrm{SL}(2,\mathbb{R})$. Both F_2 and $\mathrm{SL}(2,\mathbb{Z})$ admit actions that are "non-algebraic" (i.e. not built from modifications of algebraic constructions) and the algebraic actions of such groups often display less rigidity then actions of higher-rank groups. For instance, there exists a 1-parameter family of deformations (see Example 2.12) of the standard $\mathrm{SL}(2,\mathbb{Z})$-action on the 2-torus $\mathbb{T}^2$ (see Example 2.5) such that no continuous change of coordinates conjugates the deformed actions to the original affine action. See Examples 2.11 and 2.12 for further discussion.

However, as in the case of linear representations, the situation is expected to be very different for actions by lattices in $\mathrm{SL}(n,\mathbb{R})$ for $n \geqslant 3$ and other higher-rank simple Lie groups. In particular, the **Zimmer program** refers to a collection of conjectures and questions which roughly aim to establish analogues of rigidity results for linear representations $\pi\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ in the context of smooth (often volume-preserving) actions

$$\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$$

or "nonlinear representations." In particular, it is expected that all nonlinear actions $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ are, in some sense, of "an algebraic origin." We note that genuinely "non-algebraic" actions exist; see for instance the discussions in Example 2.10 and [40, Sections 9, 10]. In particular, a complete a classification of all actions of higher-rank lattices up to smooth conjugacy is impossible. However, it seems plausible that certain families of actions (Anosov, volume-preserving, low-dimensional, actions on specific manifolds, actions preserving a geometric structure, etc.) are classifiable and that all such actions are constructed from modifications of standard algebraic actions. See Section 2.3 for examples of standard algebraic actions. We refer to the surveys [38, 40, 41, 66] for further discussion on various notions of "algebraic actions," the Zimmer program, and precise statements of related conjectures and results.

For volume-preserving actions, the primary evidence supporting conjectures in the Zimmer program is Zimmer's superrigidity theorem for cocycles, Theorem 4.2 below. This extension of Margulis's superrigidity theorem (for homomorphisms) shows that the derivative cocycle of any volume-preserving action $\alpha\colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$ is—up to a compact error and measurable coordinate change—given by a linear representation $\Gamma \to \mathrm{GL}(d,\mathbb{R})$.

1.3. **Low-dimensional actions.** Precise conjectures in the Zimmer program are easiest to formulate for actions in low dimensions. See in particular Questions 3.1. For instance, if the dimension of M is sufficiently small, **Zimmer's conjecture** states that all actions $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ should have finite image $\alpha(\Gamma)$ (see Definition 2.4). For lattices in $\mathrm{SL}(n,\mathbb{R})$ the precise formulation of the conjecture is as follows.

Conjecture 3.2. *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice. Let M be a compact manifold.*

(1) If $\dim(M) < n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}(M)$ has finite image.
(2) In addition, if vol is a volume form on M and if $\dim(M) = n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ has finite image.

The focus of these notes is to this presents recent progress towards this conjecture made in [16]. See Theorem 3.4. See also Conjecture 3.3 for statement of this conjecture for general Lie groups.

Early results establishing this conjecture in the setting of actions the circle appear in [23, 51, 117] and in the setting of volume-preserving (and more general measure-preserving) actions on surfaces in [47, 48, 93]. See also [50] and [36] for results on real-analytic actions and [24, 26, 27] for results on holomorphic and birational actions. There are also many results (usually in the C^0 setting) for actions of specific lattices on manifolds where there are topological obstructions to the group acting; a partial list of such results includes [12, 13, 90, 115, 116, 120, 121, 128].

1.4. **Rigidity of Anosov diffeomorphisms.** As a prototype for general rigidity results discussed below, we recall certain rigidity properties exhibited by Anosov diffeomorphisms $f\colon M \to M$. We first recall the definition of an Anosov diffeomorphism.

Definition 1.1. A C^1 diffeomorphism $f\colon M \to M$ of a compact Riemannian manifold M is **Anosov** if there is a Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^u$ and constants $0 < \kappa < 1$ and $C \geqslant 1$ such that for every $x \in M$ and every $n \in \mathbb{N}$

$$\begin{aligned} \|D_x f^n(v)\| &\leqslant C\kappa^n \|v\| && \text{for all } v \in E^s(x) \\ \|D_x f^{-n}(w)\| &\leqslant C\kappa^n \|w\| && \text{for all } w \in E^u(x). \end{aligned}$$

As a primary example, consider a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ with all eigenvalues of modulus different from 1. Then, with $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ the n-torus, the induced toral automorphism $L_A\colon \mathbb{T}^n \to \mathbb{T}^n$ given by

$$L_A(x + \mathbb{Z}^n) = Ax + \mathbb{Z}^n$$

is Anosov. More generally, given $v \in \mathbb{T}^n$ we have $f\colon \mathbb{T}^n \to \mathbb{T}^n$ given by

$$f(x) = L_A(x) + v$$

is an affine Anosov map. In dimension 2, a standard example of an Anosov diffeomorphism is given by $L_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ where A is the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

A prototype for *local rigidity* results, it is known (see [2, 84], [58, Corollary 18.2.2]) that Anosov maps are **structurally stable**: if f is Anosov and g is C^1 close to f then g is also Anosov and there is a homeomorphism $h\colon \mathbb{T}^n \to \mathbb{T}^n$ such that

$$h \circ g = f \circ h. \tag{1.1}$$

The map h is always Hölder continuous but in general need not be C^1 even when f and g are C^∞. The map h in (1.1) is called a **topological conjugacy** between f and g.

All known examples of Anosov diffeomorphisms occur on finite factors of tori and nilmanifolds. From [46, 76] we have a complete classification—a prototype *global rigidity* result—of Anosov diffeomorphisms on tori (as well as nilmanifolds) up to a continuous change of coordinates: If $f\colon \mathbb{T}^n \to \mathbb{T}^n$ is Anosov, then f is homotopic to L_A for some $A \in \mathrm{GL}(n, \mathbb{Z})$ with all eigenvalues of modulus different from 1; moreover there is a homeomorphism $h\colon \mathbb{T}^n \to \mathbb{T}^n$ such that

$$h \circ f = L_A \circ h.$$

Again, the topological conjugacy h is Hölder continuous but need not be C^1. Conjecturally, all Anosov diffeomorphisms are, up to finite covers, topologically conjugate to affine maps on tori and nilmanifolds.

1.5. **Local and global rigidity programs.** Though the finiteness of actions in low dimensions is the focus of these notes, there are a number of local and global rigidity problems concerning actions of lattices and higher-rank Lie groups and related problems concerning actions of higher-rank abelian groups.

1.5.1. *Local rigidity.* Local rigidity conjectures aim to classify perturbations of actions. We recall one common definition of local rigidity of a C^∞ group action:

Definition 1.2. An action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ of a finitely generated group Γ is said to be **locally rigid** if, for any action $\tilde\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ sufficiently C^1-close to α, there exists a C^∞ diffeomorphism $h\colon M \to M$ such that

$$h \circ \tilde\alpha(\gamma) \circ h^{-1} = \alpha(\gamma) \qquad \text{for all } \gamma \in \Gamma. \tag{1.2}$$

In Definition 1.2, using that Γ is finitely generated, we define the C^1 distance between α and $\tilde\alpha$ to be

$$\max\{d_{C^1}(\alpha(\gamma), \tilde\alpha(\gamma)) \mid \gamma \in F\}$$

where $F \subset \Gamma$ is a finite, symmetric generating subset.

Local rigidity results have been established for actions of higher-rank lattices in many settings. For instance, local rigidity is known to hold for isometric actions by [8, 43]. In the non-isometric setting, local rigidity has been established for affine Anosov actions.

Definition 1.3. We say an action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ is **Anosov** if $\alpha(\gamma)$ is an Anosov diffeomorphism for some $\gamma \in \Gamma$.

See Example 2.5 and Remark 2.6 for examples of affine Anosov actions of lattices on tori.

For Anosov actions, note that while structural stability (1.1) holds for individual Anosov elements of an action, local rigidity requires that map h in (1.2) intertwines the action of the entire group Γ; moreover, unlike in the case of a single Anosov map where h is typically only Hölder continuous, we ask that the map h in (1.2) be smooth.

There are a number of results establishing local rigidity of affine Anosov actions on tori and nilmanifolds including [52, 54, 59, 61, 95, 98]. The full result on local rigidity of Anosov actions by higher-rank lattices was obtained in [62, Theorem 15]. See also related rigidity results including [54] for results on deformation rigidity and [54, 56, 74, 96] for various infinitesimal rigidity results. Additionally, see [44, 81] for local rigidity of closely related actions and [57] and [62, Theorem 17] for results on the local rigidity of projective actions by cocompact lattices.

1.5.2. *Global rigidity.* Beyond the study of perturbations, there are a number of conjectures and results on the **global rigidity** of smooth actions of higher-rank lattices. Much of the global rigidity results in the literature focus on various families of Anosov actions. Such conjectures and results aim to classify all (typically volume-preserving) Anosov actions by showing they are smoothly conjugate to affine actions on (infra-)tori and nilmanifolds. See for instance [37, 45, 52, 54, 60, 61, 81, 97] for a various global rigidity results for Anosov actions.

Recently, [22] gave a new mechanism to study rigidity of Anosov actions on tori; in particular, it is shown in [22] that all Anosov actions (satisfying a certain lifting condition which holds, for instance, when the lattice is cocompact) of higher-rank lattices are smoothly conjugate to affine actions, even when the action is not assumed to preserve a measure. This provides the most general global rigidity result for Anosov actions on tori and nilmanifolds.

Outside of Anosov actions, we have the following global rigidity conjecture for actions of lattices $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ in the critical dimension $(n-1)$.

Conjecture 3.7. *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a lattice, let M be a $(n-1)$-dimensional manifold, and let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ be an action with infinite image. Then, either $M = S^{n-1}$ or $M = \mathbb{R}P^{n-1}$ and the action α is C^∞ conjugate to the projective action on either S^{n-1} or $\mathbb{R}P^{n-1}$ in Example 2.7.*

Part 1. Actions of lattices in Lie groups and Zimmer's conjecture

2. Smooth actions by lattices in Lie groups

We present some background on lattices in semisimple Lie groups and a number of examples of smooth actions of lattices on manifolds. References with additional details for this and the next section include [6, 40, 65, 66, 80, 119].

2.1. Lattices in semisimple Lie groups.

Recall that a Lie algebra $\mathfrak{g}$ is **simple** if it is non-abelian and has no non-trivial ideal. A Lie algebra $\mathfrak{g}$ is **semisimple** if it is the direct sum $\mathfrak{g} = \bigoplus_{i=1}^{\ell} \mathfrak{g}_i$ of simple Lie algebras $\mathfrak{g}_i$; this is equivalent to the fact that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. We say a Lie group G is **simple** (resp. **semisimple**) if its Lie algebra $\mathfrak{g}$ is simple (resp. semisimple). The main example for this text is the simple Lie group $G = \mathrm{SL}(n, \mathbb{R})$.

Let G be a connected semisimple Lie group with finite center. Semisimple Lie groups are unimodular and hence admit a bi-invariant measure, called the **Haar measure**, which is unique up to normalization. A **lattice** in G is a discrete subgroup $\Gamma \subset G$ with finite co-volume. That is, if D is a measurable fundamental domain for the right-action of Γ on G then D has finite volume. If the quotient G/Γ is compact, we say that Γ is a **cocompact** lattice. If G/Γ has finite volume but is not compact we say that Γ is **nonuniform**. The quotient manifold G/Γ by the right action of Γ admits a left-action by G and the Haar measure on G descends to a finite, G-invariant measure on G/Γ which we normalize to be a probability measure.

Example 2.1. The standard example of a lattice in $G = \mathrm{SL}(n, \mathbb{R})$ is $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. Note that $\mathrm{SL}(n, \mathbb{Z})$ is not cocompact in $\mathrm{SL}(n, \mathbb{R})$. However, $\mathrm{SL}(n, \mathbb{R})$ and more general simple and semisimple Lie groups possess both nonuniform and cocompact lattices. (See for example [119, Sections 6.7, 6.8] for examples and constructions.)

Example 2.2. In the case $G = \mathrm{SL}(2, \mathbb{R})$, the fundamental group of any finite area hyperbolic surface is a lattice in G. In particular, the fundamental group of a compact hyperbolic surface is a cocompact lattice in G. This can be seen by identifying the fundamental group of S with the deck group of the hyperbolic plane $\mathbb{H} = \mathrm{SO}(2, \mathbb{R}) \backslash \mathrm{SL}(2, \mathbb{R})$. For instance, the free group $\Gamma = F_2$ on two generators is a lattice in G as can be seen by giving the punctured torus $S = \mathbb{T}^2 \smallsetminus \{\mathrm{pt}\}$ a hyperbolic metric.

See [119] for further details on constructions and properties of lattices in Lie groups.

2.2. Rank of G.

Every semisimple matrix group admits an Iwasawa decomposition $G = KAN$ where K is compact, A is a simply connected free abelian group of $\mathbb{R}$-diagonalizable elements, and N is unipotent. For general semisimple Lie groups with finite center, we have a similarly defined Iwasawa decomposition $G = KAN$ where the images of A and N under the adjoint representation are, respectively, $\mathbb{R}$-diagonalizable and unipotent. The dimension of A is the **rank** of G. We call such a group A a maximal split Cartan subgroup.

In the case of $G = \mathrm{SL}(n, \mathbb{R})$, the standard choice of K, A, and N are

$$K = \mathrm{SO}(n, \mathbb{R}), \qquad A = \{\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n}) : t_1 + \cdots + t_n = 0\},$$

and N the group of upper-triangular matrices with all diagonal entries equal to 1. Note that, as elements in $\mathrm{SL}(n, \mathbb{R})$ have determinant 1, we have

$$\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n}) \in \mathrm{SL}(n, \mathbb{R})$$

if and only if $t_1 + \cdots + t_n = 0$. Thus $A \simeq \mathbb{R}^{n-1}$ and the rank of $\mathrm{SL}(n, \mathbb{R})$ is $n - 1$.

We say that a simple Lie group G is **higher-rank** if its rank is at least 2. We will say that a lattice Γ in a higher-rank simple Lie group G is a **higher-rank lattice.** In particular, $G = \mathrm{SL}(n, \mathbb{R})$ and its lattices are higher-rank when $n \geqslant 3$.

In Example 2.9 below, we present an example of a cocompact lattice Γ in the group $G = \mathrm{SO}(n, n)$ when $n \geqslant 4$. The group $\mathrm{SO}(n, n)$ has rank n and thus Γ is a higher-rank, cocompact lattice.

For further examples, see Table 1 for calculations of the rank for various matrix groups and see [65, Section VI.4] for examples of Iwasawa decompositions for various matrix groups.

2.3. **Standard actions of lattices in Lie groups.** We present a number of standard examples of "algebraic" actions of lattices in Lie groups. We also discuss in Example 2.10 some modifications of algebraic actions and constructions of more exotic actions.

Example 2.3 (Finite actions)**.** Let Γ' be a finite-index normal subgroup of Γ. Then $F = \Gamma/\Gamma'$ is finite. Suppose the finite group F acts on a manifold M. Since F is a quotient of Γ we naturally obtain a Γ-action on M.

Note that an action of a finite group preserves a volume simply by averaging any volume form by the action.

Definition 2.4. An action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ is **finite** or **almost trivial** if it factors through the action of a finite group. That is, α is finite if there is a finite-index normal subgroup $\Gamma' \subset \Gamma$ such that $\alpha\restriction_{\Gamma'}$ is the identity.

We remark that by a theorem of Margulis [79], if Γ is a lattice in a higher-rank simple Lie group then all normal subgroups of Γ are either finite or of finite-index.

Example 2.5 (Affine actions)**.** Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ (or any finite-index subgroup of $\mathrm{SL}(n, \mathbb{Z})$). Let $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. We have a natural action $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^n)$ given by

$$\alpha(\gamma)(x + \mathbb{Z}^n) = \gamma \cdot x + \mathbb{Z}^n$$

for any matrix $\gamma \in \mathrm{SL}(n, \mathbb{Z})$.

To generalize this example to other lattices, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be any lattice and let $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{Z})$ be any representation. Then we have a natural action $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ given by

$$\alpha(\gamma)(x + \mathbb{Z}^d) = \rho(\gamma) \cdot x + \mathbb{Z}^d.$$

Note that these examples preserve a volume form, namely, the Lebesgue measure on $\mathbb{T}^d$. Also note that these actions are non-isometric.

Remark 2.6. Both constructions in Example 2.5 give actions $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ that have global fixed points. That is, the coset of 0 in $\mathbb{T}^d$ is a fixed point of $\alpha(\gamma)$ for every $\gamma \in \Gamma$.

The construction can be modified further to obtain genuinely affine actions without global fixed points. Given a lattice $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ and a representation $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{Z})$, there may exist non-trivial elements $c \in H^1_\rho(\Gamma, \mathbb{T}^d)$; that is, $c\colon \Gamma \to \mathbb{T}^d$ is a function with

$$c(\gamma_1\gamma_2) = \rho(\gamma_1)c(\gamma_2) + c(\gamma_1) \tag{2.1}$$

and such that there does not exist any $\eta \in \mathbb{T}^d$ with

$$c(\gamma) = \rho(\gamma)\eta - \eta \tag{2.2}$$

for all $\gamma \in \Gamma$. (Equation (2.1) says that c is a cocycle with coefficients in the Γ-module $\mathbb{T}^d$; (2.2) says c is not a coboundary.) We may then define $\tilde{\alpha}\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ by

$$\tilde{\alpha}(\gamma)(x + \mathbb{Z}^d) = \rho(\gamma) \cdot x + c(\gamma) + \mathbb{Z}^d.$$

Equation (2.1) ensures that $\tilde{\alpha}$ is an action and (2.2) ensures that $\tilde{\alpha}$ is not conjugate to the action α.

In the above construction, any cocycle $c\colon \Gamma \to \mathbb{T}^d$ is necessarily cohomologous to a torsion-valued (that is, $\mathbb{Q}^d/\mathbb{Z}^d$-valued) cocycle. This follows from Margulis's result (see [80, Theorem 3 (iii)]) on the vanishing of $H^1_\rho(\Gamma, \mathbb{R}^d)$. In particular, $\tilde{\alpha}$ and α are conjugate when restricted to a finite-index subgroup of Γ. See [55] for more details.

Example 2.7 (Projective actions)**.** Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be any lattice. Then Γ has a natural linear action on $\mathbb{R}^n$. The linear action of Γ on $\mathbb{R}^n$ induces an action of Γ on the sphere S^{n-1} thought of as the set of unit vectors in $\mathbb{R}^n$: we have $\alpha\colon \Gamma \to \mathrm{Diff}(S^n)$ given by

$$\alpha(\gamma)(x) = \frac{\gamma \cdot x}{\|\gamma \cdot x\|}.$$

Alternatively we could act on the space of lines in $\mathbb{R}^n$ and obtain an action of Γ on the $(n-1)$-dimensional real projective space $\mathbb{R}P^{n-1}$. This action does not preserve a volume; in fact there is no invariant probability measure for this action. Additionally, these actions are not isometric for any Riemannian metric.

Remark 2.8 (Actions on boundaries)**.** Example 2.7 generalizes to actions of lattices Γ in G acting on boundaries of G. Given a semisimple Lie group G with Iwasawa decomposition $G = KAN$, let $M = K \cap C_G(A)$ be the centralizer of A in K. A closed subgroup $Q \subset G$ is **parabolic** if it is conjugate to a group containing MAN. When $G = \mathrm{SL}(n, \mathbb{R})$ we have that M is a finite group and any parabolic subgroup Q is conjugate to a group containing all upper triangular matrices. See [65, Section VII.7] for further discussion on the structure of parabolic subgroups.

Given a semisimple Lie group G, a (finite-index subgroup of a) proper parabolic subgroup $Q \subset G$, and a lattice $\Gamma \subset G$, the coset space $M = G/Q$ is compact and Γ acts on M naturally as

$$\alpha(\gamma)(xQ) = \gamma x Q.$$

These actions never preserve a volume form or any Borel probability measure and are not isometric.

In Example 2.7, the action on the projective space $\mathbb{R}P^{n-1}$ can be seen as the action on $\mathrm{SL}(n, \mathbb{R})/Q$ where Q is the parabolic subgroup

$$Q = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \right\}.$$

Example 2.9 (Isometric actions)**.** Another important family of algebraic actions are isometric actions obtained from embeddings of cocompact lattices in Lie groups into compact groups.

Isometric actions of cocompact lattices in split orthogonal groups of type D_n. For $n \geqslant 4$, consider the quadratic form in $2n$ variables

$$Q(x_1, \dots, x_n, y_1, \dots, y_n) = x_1^2 + \cdots + x_n^2 - \sqrt{2}(y_1^2 + \cdots + y_n^2).$$

Let

$$B = \mathrm{diag}\left(1, \dots, 1, -\sqrt{2}, \dots, -\sqrt{2}\right) \in \mathrm{GL}(2n, \mathbb{R})$$

be the matrix such that $Q(x) = x^T B x$ for all $x \in \mathbb{R}^{2n}$ and let

$$G = \mathrm{SO}(Q) = \{g \in \mathrm{SL}(2n, \mathbb{R}) \mid g^T B g = B\}$$

be the special orthogonal group associated with Q. We have that

$$\mathrm{SO}(Q) \simeq \mathrm{SO}(n, n)$$

is a Lie group of rank n with restricted root system of type D_n when $n \geqslant 4$.[1]

Let $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$ and let $\mathbb{Z}[\sqrt{2}]$ be the ring of integers in $\mathbb{K}$. Let

$$\Gamma = \{g \in \mathrm{SL}(2n, \mathbb{Z}[\sqrt{2}]) \mid g^T B g = B\}.$$

Then Γ is a cocompact lattice in G. (See for example [119], Proposition 5.5.8 and Corollary 5.5.10.)

Let $\tau\colon \mathbb{K} \to \mathbb{K}$ be the nontrivial Galois automorphism with $\tau(\sqrt{2}) = -\sqrt{2}$. Let τ act coordinate-wise on matrices with entries in $\mathbb{K}$. Given $\gamma \in \Gamma$ we have $\tau(\gamma) = \mathrm{Id}$ if and only if $\gamma = \mathrm{Id}$. Moreover, as $\tau^2 = \mathrm{Id}$ we have

$$\tau(\gamma) \in \mathrm{SO}(\tau(Q)) := \{g \in \mathrm{SL}(2n, \mathbb{R}) \mid g^T \tau(B) g = \tau(B)\} \simeq \mathrm{SO}(2n).$$

In particular, the map $\gamma \to \tau(\gamma)$ gives a representation $\Gamma \to \mathrm{SO}(2n)$ with infinite image into the compact group $\mathrm{SO}(2n)$.

As $\mathrm{SO}(2n)$ is the isometry group of the sphere $S^{2n-1} = \mathrm{SO}(2n)/\mathrm{SO}(2n-1)$ we obtain an action of Γ by isometries on a manifold of dimension $2n - 1$.

Isometric actions of cocompact lattices in $\mathrm{SL}(n, \mathbb{R})$. A more complicated construction can be used to build cocompact lattices $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ that possess infinite-image representations $\pi\colon \Gamma \to \mathrm{SU}(n)$ (see discussion in [119, Sections 6.7, 6.8] as well as [119, Warning 16.4.3].) In this case, one obtains isometric actions of certain cocompact lattices Γ in $\mathrm{SL}(n, \mathbb{R})$ on the $(2n-2)$-dimensional homogeneous space

$$\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1)).$$

Example 2.10 (Modifications of standard examples and exotic actions)**.** Beyond the "algebraic actions" discussed in Examples 2.5–2.9, it is possible to modify certain algebraic constructions to construct genuinely new actions; these actions might not be conjugate to algebraic actions and may exhibit much weaker rigidity properties. One such construction starts with the standard action of (finite-index subgroups of) $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{T}^n$ and creates a non-volume-preserving action by blowing-up fixed points or finite orbits of the action. In [60, Section 4], Katok and Lewis showed this example can be modified to obtain volume-preserving, real-analytic actions of $\mathrm{SL}(n, \mathbb{Z})$ that are not C^0 conjugate to an affine action; moreover, these actions are not C^1-locally rigid. In [7, 9, 39], constructions of non-locally rigid, ergodic, volume-preserving actions of any lattice in a simple Lie group are constructed by more general blow-up constructions.

Another example due to Stuck [110] demonstrates that it is impossible to fully classifying all lattice actions. Let $P \subset \mathrm{SL}(n, \mathbb{R})$ be the group of upper triangular matrices. There is a non-trivial homomorphism $\rho\colon P \to \mathbb{R}$. Now consider any flow (i.e. $\mathbb{R}$-action) on a manifold M and view the flow as a P-action via the image of ρ. Then G acts on the induced space $N = (G \times M)/P$ and the restriction induces a non-volume-preserving, non-finite action of Γ. This example shows—particularly in the non-volume-preserving-case—that care is needed in order to formulate any precise conjectures that assert that every action should be "of an algebraic origin." Note, however, that we obtain a natural map $N \to G/P$ that intertwines Γ-actions; in particular, this action has an "algebraic action" as a factor.

We refer to [40, Sections 9 and 10] for more detailed discussion and references to modifications of algebraic actions and exotic actions.

[1]For $n = 1$, $\mathrm{SO}(1, 1)$ is a one-parameter group and for $n = 2$, $\mathrm{SO}(2, 2)$ is not simple (it is double covered by $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$). For $n = 3$, $\mathrm{SO}(3, 3)$ is double covered by $\mathrm{SL}(4, \mathbb{R})$.

2.4. **Actions of lattices in rank-1 groups.** Actions by lattices in higher-rank Lie groups are expected to be rather constrained. Although Example 2.10 shows there exist exotic, genuinely "non-algebraic" actions of such groups, these actions are build from modifying algebraic constructions or factor over algebraic actions. For lattices in rank-one Lie groups such as $\mathrm{SL}(2,\mathbb{R})$, the situation is very different. There exist natural actions that have no algebraic origin and the algebraic actions of such groups seem to exhibit far less rigidity (for example Example 2.12 which is not locally rigid) than those above.

Example 2.11 (Actions of free groups)**.** Let $G = \mathrm{SL}(2,\mathbb{R})$. The free group $\Gamma = F_2$ is a lattice in G. (For instance, F_2 is the fundamental group of the punctured torus; more explicitly, $\mathrm{SL}(2,\mathbb{Z})$ contains a copy of F_2 as an index 12 subgroup.) Let M be any manifold and let $f, g \in \mathrm{Diff}(M)$. Then f and g generate an action of Γ on M that is in general is of an algebraic origin. In particular, there is no expectation that any rigidity phenomena should hold in general for actions by all lattices in $\mathrm{SL}(2,\mathbb{R})$.

For the next example, recall Definitions 1.1 and 1.3 of Anosov actions.

Example 2.12 (Non-standard Anosov actions of $\mathrm{SL}(2,\mathbb{Z})$)**.** Consider the standard action α_0 of $\mathrm{SL}(2,\mathbb{Z})$ on the 2 torus $\mathbb{T}^2$ as constructed in Example 2.5. In [54, Example 7.21], Hurder presents an example of a 1-parameter family of deformations $\alpha_t \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Diff}(\mathbb{T}^2)$ of α_0 with the following properties:

(1) Each α_t is a real-analytic, volume-preserving action;
(2) For $t > 0$, α_t is not topologically conjugate to α_0, (even when restricted to a finite-index subgroup of $\mathrm{SL}(2,\mathbb{Z})$.)

Moreover, since α_0 is an Anosov action and since the Anosov property is an open property we have that

(3) each α_t is an Anosov action.

This shows that even affine Anosov actions of $\mathrm{SL}(2,\mathbb{Z})$ fail to exhibit local rigidity properties and that there exist genuinely exotic Anosov actions of $\mathrm{SL}(2,\mathbb{Z})$. This is in stark contrast to the affine Anosov actions of higher-rank lattices which are known to be locally rigid by [62, Theorem 15].

In contrast, it is expected that all Anosov actions of higher-rank lattices are smoothly conjugate to affine actions as in Example 2.5 or Remark 2.6 (or analogous constructions in infra-nilmanifolds). See Question 3.1(6) below. Recent progress towards this conjecture appears in [22].

Remark 2.13. There are a number of rank-1 Lie groups whose lattices are known to exhibit some rigidity properties relative to linear representations. For instance, Corlette established superrigidity and arithmeticity of lattices in certain rank-1 simple Lie groups in [29]. In particular, Corlette establishes superrigidity for lattices in $\mathrm{Sp}(n,1)$ and F_4^{-20}, the isometry groups of quaternionic hyperbolic space and the Cayley plane. It seems plausible that lattices in certain rank-1 Lie groups would exhibit some rigidly properties for actions by diffeomorphisms; currently, there do not seem to be any results in this direction other than results that hold for all groups with property (T) groups such as [43, 87].

3. Actions in low dimension and Zimmer's conjecture

3.1. **Motivating questions.** For actions by lattices in rank-1 groups, we have seen that it is easy to construct exotic actions of free groups and Example 2.12 shows there are exotic Anosov actions of $\mathrm{SL}(2,\mathbb{Z})$ on tori.

However, for actions of lattices in higher-rank, simple Lie groups, the situation is expected to be far more rigid. In particular, the examples from the previous section lead to a number of more precise questions and conjectures. For concreteness, fix $n \geqslant 3$ and let $G = \mathrm{SL}(n, \mathbb{R})$. Let $\Gamma \subset G$ be a lattice. Recall the action of Γ on S^{n-1} and the volume-preserving Anosov action of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{T}^n$.

Questions 3.1. Consider the following questions:

(1) Is there a non-finite action of Γ on a manifold of dimension at most $n-2$?
(2) If the answer to (1) is unknown, does every action of Γ on a manifold of dimension at most $n-2$ preserve a volume form?
(3) Is there a non-finite, volume-preserving action of Γ on a manifold of dimension at most $n-1$?
(4) Is every non-finite action of Γ on an n-torus of the type considered in Example 2.5? What about volume-preserving actions? That is, if $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^n)$ is a non-finite action is α smoothly conjugate to an affine action as in Example 2.5 (or as in Remark 2.6)?
(5) Are the only non-finite actions of Γ on a connected $(n-1)$-manifold those considered in Example 2.7? That is, if $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ is a non-finite action is M either S^{n-1} or $\mathbb{R}P^{n-1}$ and is α smoothly conjugate to the projective action.

Motivated by various conjectures on the classification of Anosov diffeomorphisms and Question 3.1(4), we also pose the following.

(6) Is every non-finite (volume-preserving) Anosov action of Γ of the type considered in Example 2.5? That is, if $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ is an Anosov action is M a (infra-)nilmanifold and is α smoothly conjugate to an affine action as in Example 2.5 (or as in Remark 2.6)?

Questions 3.1(1) and (3) are referred to as **Zimmer's conjecture**, discussed in the next section. Question 3.1(2) is irrelevant given a negative answer to Question 3.1(1) but motivated the result stated in Theorem 3.5 below and was natural to conjecture before an answer to Question 3.1(1) was known. It may be that answering Question 3.1(2) is possible in dimension ranges where Conjecture 3.3(1) below is expected to hold but is not yet known.

3.2. **Zimmer's conjecture for actions by lattices in** $\mathrm{SL}(n, \mathbb{R})$**.** Recall Example 2.5 and Example 2.7. For lattices in $\mathrm{SL}(n, \mathbb{R})$, **Zimmer's conjecture** asserts that these are the minimal dimensions in which non-finite actions can occur. We have the following precise formulation.

Conjecture 3.2. *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a lattice. Let M be a compact manifold.*

(1) If $\dim(M) < n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}(M)$ has finite image.
(2) In addition, if vol is a volume form on M and if $\dim(M) = n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ has finite image.

We are intentionally vague about the regularity in Conjecture 3.2 (and Conjecture 3.3 below). Zimmer's original conjecture considered the case of C^∞ volume-preserving actions. See [122, 125, 126]. Most evidence for the conjecture requires the action to be at least C^1. It is possible the conjecture holds for actions by homeomorphisms; see for instance [12, 116, 117] for a partial list of results in this direction. The results we discuss below require the action to be at least $C^{1+\beta}$ as we use tools nonuniformly hyperbolic dynamics though some of our results still hold for actions by C^1 diffeomorphisms (see Theorem 3.6 below.)

$\ast$3.3. **Zimmer's conjecture for actions by lattices in other Lie groups.** To formulate Zimmer's conjecture for lattices general Lie groups, to each simple, non-compact Lie group G we associate 3 positive integers $d_0(G), d_{\rm rep}(G), d_{\rm cmt}(G)$ defined roughly as follows:

(1) $d_0(G)$ is the minimal dimension of G/H as H varies over proper closed subgroups $H \subset G$. (We remark that H is necessarily a parabolic subgroup in this case.)
(2) $d_{\rm rep}(G)$ is the minimal dimension of a non-trivial linear representation of (the Lie algebra) of G.
(3) $d_{\rm cmt}(G)$ is the minimal dimension of a non-trivial homogeneous space of a compact real form of G.

See Table 1 where we compute the above numbers for a number of matrix groups, (split) real forms of exceptional Lie algebras, and complex matrix groups. We also include another number $r(G)$ which is defined in [16, 20] and arises from certain dynamical arguments[2]; this number gives the bounds appearing in the most general result, Theorem 3.9 below, towards solving Conjecture 3.3. For complete tables of values of $d_{\rm rep}(G)$, $d_{\rm cmt}(G)$, and $d_0(G)$, we refer to [25].

G	restricted root system	rank	$d_{\rm rep}(G)$	$d_{\rm cmt}(G)$	$d_0(G)$	$r(G)$
$\mathrm{SL}(n,\mathbb{R})$	A_{n-1}	$n-1$	n	$2n-2$	$n-1$	$n-1$
$\mathrm{SO}(n,n+1)$	B_n	n	$2n+1$	$2n$	$2n-1$	$2n-1$
$\mathrm{Sp}(2n,\mathbb{R})$	C_n	n	$2n$	$4n-4$	$2n-1$	$2n-1$
$\mathrm{SO}(n,n)$	D_n	n	$2n$	$2n-1$	$2n-2$	$2n-2$
E_I	E_6	6	27	26	16	16
E_V	E_7	7	56	54	27	27
E_{VIII}	E_8	8	248	112	57	57
F_1	F_4	4	26	16	15	15
G	G_2	2	7	6	5	5
$\mathrm{SL}(n,\mathbb{C})$	A_{n-1}	$n-1$	$2n$	$2n-2$	$2n-2$	$n-1$
$\mathrm{SO}(2n,\mathbb{C})$	D_n	n	$4n$	$2n-1$	$4n-4$	$2n-2$
$\mathrm{SO}(2n+1,\mathbb{C})$	B_n	n	$4n+2$	$2n$	$4n-2$	$2n-1$
$\mathrm{Sp}(2n,\mathbb{C})$	C_n	n	$4n$	$4n-4$	$4n-2$	$2n-1$
$\mathrm{SO}(p,q)$ $p<q$	B_p	p	$p+q$	$p+q-1$	$p+q-2$	$2p-1$

TABLE 1. Numerology in appearing in Zimmer's conjecture for various groups. See also [25] for more complete tables. See Theorem 3.9 where the number $r(G)$ appears and [16, 20] or Footnote 2 for definition.

Given the examples in Section 2.3 and the integers $d_{\rm rep}(G), d_{\rm cmt}(G), d_0(G)$ defined above, is it natural to conjecture the following full conjecture.

Conjecture 3.3 (Zimmer's conjecture)**.** *Let G be a connected, simple Lie group with finite center. Let $\Gamma \subset G$ be a lattice. Let M be a compact manifold and* vol *a volume form on M. Then*

[2]A precise definition that is equivalent to that in [16, 20] is that $r(G)$ is $d_0(G')$ where G' is the largest $\mathbb{R}$-split simple subgroup in G.

(1) *if* $\dim(M) < \min\{d_{\mathrm{rep}}(G), d_{\mathrm{cmt}}(G), d_0(G)\}$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ *has finite image;*
(2) *if* $\dim(M) < \min\{d_{\mathrm{rep}}(G), d_{\mathrm{cmt}}(G)\}$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ *has finite image;*
(3) *if* $\dim(M) < \min\{d_0(G), d_{\mathrm{rep}}(G)\}$ *then for any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}(M)$*, the image* $\alpha(\Gamma)$ *preserves a Riemannian metric;*
(4) *if* $\dim(M) < d_{\mathrm{rep}}(G)$ *then for any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$*, the image* $\alpha(\Gamma)$ *preserves a Riemannian metric.*

3.4. **Recent progress on Zimmer's conjecture.** Recently, the author with David Fisher and Sebastian Hurtado answered Questions 3.1(1) and (3) for actions by cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$ (and other higher-rank simple Lie groups) in [16]. They also announced the analogous result for actions by $\mathrm{SL}(n,\mathbb{Z})$ in [17] and for actions by general lattices in [18]. This solves Conjecture 3.2 for actions by C^2 (or even $C^{1+\beta}$) diffeomorphisms. See Remarks below for discussion of actions by C^1 diffeomorphisms.

We also refer the reader to the excellent article by Serge Cantat [25] that presents (in French) a complete proof of Theorem 3.4.

Theorem 3.4 ([16, Theorem 1.1])**.** *For* $n \geqslant 3$*, let* $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ *be a cocompact lattice. Let* M *be a compact manifold.*

(1) *If* $\dim(M) < n-1$ *then any homomorphism* $\Gamma \to \mathrm{Diff}^2(M)$ *has finite image.*
(2) *In addition, if* vol *is a volume form on* M *and if* $\dim(M) = n-1$ *then any homomorphism* $\Gamma \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ *has finite image.*

Before Theorem 3.4 gave answers to Questions 3.1(1) and (3) above, the author together with Federico Rodriguez Hertz and Zhiren Wang studied Question 3.1(2) and were able to show that all such actions preserve some probability measure.

Theorem 3.5 ([20, Theorem 1.6])**.** *For* $n \geqslant 3$*, let* $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ *be a lattice. Let* M *be a manifold with* $\dim(M) < n-1$*. Then, for any* $C^{1+\beta}$ *action* $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$*, there exists an* α*-invariant Borel probability measure.*

For actions on the circle, an analogue of Theorem 3.5 is shown in [51, Theorem 3.1] for actions by homeomorphisms.

The proof of Theorem 3.4 uses ideas and results from [20], particularly the proof of Theorem 3.5, as ingredients. Thus, while Theorem 3.5 follows trivially from Theorem 3.4, we include the proof of Theorem 3.5 below as key ideas (namely, Theorem 11.1, Theorem 11.1′, and Proposition 11.5) will be needed in the proof of Theorem 3.4.

Remarks on Theorem 3.4. We give a number of remarks on extensions of Theorem 3.4. See also the discussion in Section 3.5.

(1) Recently, the authors announced in [17] that the conclusion of Theorem 3.4 holds for actions of $\mathrm{SL}(n,\mathbb{Z})$ for $n \geqslant 3$. The result for general lattices in $\mathrm{SL}(n,\mathbb{Z})$ as well as analogous results for lattices in other higher-rank simple Lie groups, has been announced [18]. See Theorem 3.9. The results for actions of $\mathrm{SL}(n,\mathbb{Z})$ and of general nonuniform lattices use many of the ideas presented in this text but also require a number of new techniques (including the structure of arithmetic groups, reduction theory, and ideas from [75]) and will not be discussed.
(2) We state Theorem 3.4 for actions by C^2 diffeomorphisms though the proof can be adapted for actions by $C^{1+\beta}$ actions. Our proof below will assume the action is by C^∞ diffeomorphisms to simplify certain Sobolev space arguments.

(3) The result for actions by lattices in general Lie groups is stated in Theorem 3.9 below. In particular, by Theorem 3.9, Conjecture 3.3(1) and (2) hold for all $C^{1+\beta}$ actions by lattices in all simple Lie groups that are non-exceptional, split real forms. For $C^{1+\beta}$ actions by lattices in all simple Lie groups that are exceptional split real forms, Conjecture 3.3(1) is known to hold by Theorem 3.9.

(4) D. Damjanovich and Z. Zhang observed that the proof of Theorem 3.4 can be adapted to the setting of actions by C^1-diffeomorphisms. Together with the author, the have announced the following theorem.

Theorem 3.6 (Brown-Damjanovich-Zhang, announced)**.** *Let $\Gamma \subset G$ be a lattice in a higher-rank simple Lie group G with finite center. Let M be a compact manifold.*

(a) If $\dim(M) < \operatorname{rank}(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^1(M)$ has finite image.

(b) In addition, if vol is a volume form on M and if $\dim(M) = \operatorname{rank}(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^1_{\operatorname{vol}}(M)$ has finite image.

For actions by lattices on other higher-rank groups there is a gap between what is known for C^1 versus $C^{1+\beta}$-actions. Indeed, our number $r(G)$ in Theorem 3.9 always satisfies $r(G) \geqslant \operatorname{rank}(G)$ and is a strict inequality unless G has restricted root system of type A_n.

Extension of Theorem 3.5 and rigidity conjecture in dimension $n-1$. In the critical dimension, $\dim(M) = n - 1$, the projective action on $\mathbb{R}P^{n-1}$ discussed in Example 2.7 gives an example of an action that does not preserve any Borel probability measure. If α is an action of Γ on a space X, we say that a Borel probability measure μ is **nonsingular** for α if the measure class of μ is preserved by the action. In particular, any smooth volume on $\mathbb{R}P^{n-1}$ is nonsingular for the projective action. In [20, Theorem 1.7], it is shown that all non-measure-preserving actions on manifolds of the critical dimension $(n-1)$ have the projective action on $\mathbb{R}P^{n-1}$ equipped with a smooth volume as a measurable factor. Precisely, for any action $\alpha\colon \Gamma \to \operatorname{Diff}^{1+\beta}(M)$ where $\dim(M) = n - 1$ it is shown that either

(1) there exists an α-invariant Borel probability measure μ on M; or
(2) there exists a Borel probability measure μ on M that is nonsingular for the action α; moreover the action α on (M, μ) is measurably isomorphic to a finite extension of the projective action in Example 2.7 and the image of μ factors to a smooth volume form on $\mathbb{R}P^{n-1}$.

This gives strong evidence for a positive answer to Question 3.1(5) which we pose as a formal conjecture.

Conjecture 3.7. *For $n \geqslant 3$, let $\Gamma \subset \operatorname{SL}(n, \mathbb{R})$ be a lattice, let M be a $(n-1)$-dimensional manifold, and let $\alpha\colon \Gamma \to \operatorname{Diff}^\infty(M)$ be an action with infinite image. Then, either $M = S^{n-1}$ or $M = \mathbb{R}P^{n-1}$ and the action α is C^∞ conjugate to the projective action on either S^{n-1} or $\mathbb{R}P^{n-1}$ in Example 2.7.*

$\ast$3.5. **Results on Zimmer's conjecture for lattices in other Lie groups.** Consider a connected, simple Lie group G with finite center. Let $\Gamma \subset G$ be a cocompact lattice. The proof of Theorem 3.4 discussed above, particularly the use of Theorem 11.1 in Section 12.3 can be adapted almost verbatim to show the following. See also [25] where Theorem 3.8 is stated and given a mostly self-contained proof.

Theorem 3.8. *Let G be a connected, simple Lie group G with finite center and rank at least 2. Let $\Gamma \subset G$ be a cocompact lattice and let M be a compact manifold.*

1. *If $\dim(M) < \operatorname{rank}(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^2(M)$ has finite image.*
2. *In addition, if vol is a volume form on M and if $\dim(M) \leqslant \operatorname{rank}(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^2_{\operatorname{vol}}(M)$ has finite image.*

As mentioned in Section 3.4, Theorem 3.8 holds for C^1 actions; see Theorem 3.6.

Theorem 3.8 fails to give the optimal dimension bounds for the analogue of Conjecture 3.2 given in Conjecture 3.3 for actions by lattices in Lie groups other than $\mathrm{SL}(n,\mathbb{R})$. See Table 1 for various conjectured critical dimensions arising in Zimmer's conjecture for other Lie groups.

To state the most general (as of 2018) result towards solving Conjecture 3.3, to any simple Lie group G, we associate a non-negative integer $r(G)$. See [16, Section 2.2] or Footnote 2 for equivalent definitions of $r(G)$ and Table 1 for values of $r(G)$ in various examples of G. For actions of lattices in a general Lie group G, the main result of [16], as well as the announced extension, gives finiteness of the action up to the critical dimension $r(G)$.

Theorem 3.9 ([16] cocompact case; [18] nonuniform case)**.** *Let $\Gamma \subset G$ be a lattice in a higher-rank simple Lie group G with finite center. Let M be a compact manifold.*

1. *If $\dim(M) < r(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^{1+\beta}(M)$ has finite image.*
2. *In addition, if vol is a volume form on M and if $\dim(M) = r(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^{1+\beta}_{\operatorname{vol}}(M)$ has finite image.*

When G is exceptional or not a split real form, our number $r(G)$ is lower than the conjectured critical dimension in Conjecture 3.3(1) and (2). However, for lattices in all Lie groups that are non-exceptional, split real forms Theorem 3.9 confirms Conjecture 3.3(1) and (2). For instance, for actions by lattices in symplectic groups we have the following.

Theorem 3.10 ([16, Theorem 1.3] cocompact case; [18] nonuniform case)**.** *For $n \geqslant 2$, if M is a compact manifold with $\dim(M) < 2n-1$ and if $\Gamma \subset \mathrm{Sp}(2n,\mathbb{R})$ is a lattice then any homomorphism $\alpha\colon \Gamma \to \operatorname{Diff}^2(M)$ has finite image. In addition, if $\dim(M) = 2n-1$ then any homomorphism $\alpha\colon \Gamma \to \operatorname{Diff}^2_{\operatorname{vol}}(M)$ has finite image.*

Similarly, for actions by lattices in split orthogonal groups we have the following.

Theorem 3.11 ([16, Theorem 1.4] cocompact case; [18] nonuniform case)**.** *Let M be a compact manifold.*

1. *For $n \geqslant 4$, if $\Gamma \subset \mathrm{SO}(n,n)$ is a lattice and if $\dim(M) < 2n-2$ then any homomorphism $\alpha\colon \Gamma \to \operatorname{Diff}^2(M)$ has finite image. If $\dim(M) = 2n-2$ then any homomorphism $\alpha\colon \Gamma \to \operatorname{Diff}^2_{\operatorname{vol}}(M)$ has finite image.*
2. *For $n \geqslant 3$, if $\Gamma \subset \mathrm{SO}(n,n+1)$ is a lattice and if $\dim(M) < 2n-1$ then any homomorphism $\alpha\colon \Gamma \to \operatorname{Diff}^2(M)$ has finite image. If $\dim(M) = 2n-1$ then any homomorphism $\alpha\colon \Gamma \to \operatorname{Diff}^2_{\operatorname{vol}}(M)$ has finite image.*

For actions by lattices Γ in simple Lie groups that are not split real forms such as $G = \mathrm{SL}(n,\mathbb{C})$, $\mathrm{SO}(n,m)$ for $m \geqslant n+2$, or $\mathrm{SU}(n,m)$, Theorem 3.9 above (the main result of [16] for cocompact case, [18] in general) gives finiteness of all actions on manifolds whose dimension is below a certain critical dimension. However, this critical dimension may be below the dimension conjectured by the analogue of Conjecture 3.3 for these groups. See Table 1.

4. Superrigidity and heuristics for Conjecture 3.2

The original conjecture (as formulated for actions by lattices in $\mathrm{SL}(n, \mathbb{R})$) posed by Zimmer was Conjecture 3.2(2) (see for example [125, Conjecture II]). Conjecture 3.2(1) was formulated later and first appears in print in [36, Conjecture I]. The reason Zimmer posed his conjecture as Conjecture 3.2(2) is that the strongest evidence for the conjecture—Zimmer's *cocycle superrigidity theorem*—requires that the action preserve some Borel probability measure. Zimmer's cocycle superrigidity theorem also provides strong evidence for local and global rigidity conjectures related to Questions 3.1(4) and (6) and is typically used in proofs of results towards solving such conjectures.

In this section we state a version of Zimmer's cocycle superrigidity theorem and some consequences. We also state and give a number of consequences of a version of Margulis's superrigidity theorem (for linear representations). We finally end with some heuristics for Conjecture 3.2 that follow from the superrigidity theorems. General references for this section include [80, 119, 123].

4.1. **Cocycles over group actions.** Consider a standard probability space (X, μ). Let G be a locally compact topological group and let $\alpha\colon G \times X \to X$ be a measurable action of G by μ-preserving transformations. In particular, $\alpha(g)$ is a μ-preserving, measurable transformation of X for each $g \in G$. We will always assume that μ is **ergodic**; that is, we assume the only G-invariant sets are null or conoll. A d-dimensional **measurable linear cocycle** over α is a measurable map

$$\mathcal{A}\colon G \times X \to \mathrm{GL}(d, \mathbb{R})$$

satisfying for a.e. $x \in X$ the cocycle condition: for all $g_1, g_2 \in G$

$$\mathcal{A}(g_1 g_2, x) = \mathcal{A}\left(g_1, \alpha(g_2)(x)\right) \mathcal{A}(g_2, x). \tag{4.1}$$

If e is the identity element of G, then (4.1) implies that

$$\mathcal{A}(e, x) = \mathcal{A}(e, x)\mathcal{A}(e, x)$$

whence $\mathcal{A}(e, x) = \mathrm{Id}$ for a.e. x

We say two cocycles $\mathcal{A}, \mathcal{B}\colon G \times X \to \mathrm{GL}(d, \mathbb{R})$ are (measurably) **cohomologous** if there is a measurable map $\Phi\colon X \to \mathrm{GL}(d, \mathbb{R})$ such that for a.e. x and every $g \in G$

$$\mathcal{B}(g, x) = \Phi(\alpha(g)(x))^{-1} \mathcal{A}(g, x) \Phi(x). \tag{4.2}$$

We say a cocycle $\mathcal{A}\colon G \times X \to \mathrm{GL}(d, \mathbb{R})$ is **constant** if $\mathcal{A}(g, x)$ is independent of x, that is, if $\mathcal{A}\colon G \times X \to \mathrm{GL}(d, \mathbb{R})$ coincides with a representation $\pi\colon G \to \mathrm{GL}(d, \mathbb{R})$ on a set of full measures.

As a primary example, let $\alpha\colon G \to \mathrm{Diff}^1_\mu(M)$ be an action of G by C^1 diffeomorphisms of a compact manifold M preserving some Borel probability measure μ. Although the tangent bundle TM may not be a trivial bundle, we may choose a Borel measurable trivialization $\Psi\colon TM \to M \times \mathbb{R}^d$ of the vector-bundle TM where $d = \dim(M)$. We have that Ψ factors over the identity map on M and, writing $\Psi_x\colon T_x \to \mathbb{R}^d$ for the identification of the fiber over x with $\mathbb{R}^d$, we moreover assume that $\|\Psi_x\|$ and $\|\Psi_x^{-1}\|$ are uniformly bounded in x.

Fix such an trivialization Ψ and define $\mathcal{A}$ to be the derivative cocycle relative to this trivialization:

$$\mathcal{A}(g, x) = D_x \alpha(g)$$

where, we view $D_x\alpha(g)$ as an element of $\mathrm{GL}(d, \mathbb{R})$ transferring the fiber $\{x\} \times \mathbb{R}^d$ to $\{\alpha(g)(x)\} \times \mathbb{R}^d$ via the measurable trivialization Ψ. To be precise, if $\Psi\colon TM \to M \times \mathbb{R}^d$

is the measurable vector-bundle trivialization then

$$\mathcal{A}(g,x) := \Psi(\alpha(g)(x))D_x\alpha(g)\Psi(x)^{-1}.$$

In this case, the cocycle relation (4.1) is simply the chain rule. Note that if we choose another Borel measurable trivialization $\Psi'\colon TM \to M\times\mathbb{R}^d$ then we obtain a cohomologous cocycle $\mathcal{A}'$. Indeed, we have

$$\mathcal{A}'(g,x) = \Psi'(\alpha(g)(x))\Psi(\alpha(g)(x))^{-1}\mathcal{A}(g,x)\Psi(x)\Psi'(x)^{-1}$$

so we may take $\Phi(x) = \Psi(x)\Psi'(x)^{-1}$ in (4.2).

We have the following elementary fact which we frequently use in the case of volume-preserving actions.

Claim 4.1. *Let $\alpha\colon G \to \mathrm{Diff}^1_{\mathrm{vol}}(M)$ be an action by volume-preserving diffeomorphisms. Then, for any α-invariant measure μ, the derivative cocycle $\mathcal{A}$ is cohomologous to a $\mathrm{SL}^\pm(d,\mathbb{R})$-valued cocycle.*

Above, $\mathrm{SL}^\pm(d,\mathbb{R})$ is the subgroup of $\mathrm{GL}(d,\mathbb{R})$ defined by $\det(A) = \pm 1$.

4.2. **Cocycle superrigidity.** We formulate the statement of Zimmer's cocycle superrigidity theorem when G is either $\mathrm{SL}(n,\mathbb{R})$ or a lattice $\mathrm{SL}(n,\mathbb{R})$ for $n \geqslant 3$. Note that the version formulated by Zimmer (see [123]) had a slightly weaker conclusion. We state the stronger version formulated and proved in [42].

Theorem 4.2 (Cocycle superrigidity [42, 123])**.** *For $n \geqslant 3$, let G be either $G = \mathrm{SL}(n,\mathbb{R})$ or let G be a lattice in $\mathrm{SL}(n,\mathbb{R})$. Let $\alpha\colon G \to \mathrm{Aut}(X,\mu)$ be an ergodic, measurable action of G by μ-preserving transformations of a standard probability space (X,μ). Let $\mathcal{A}\colon G\times X \to \mathrm{GL}(d,\mathbb{R})$ be a bounded,[3] measurable linear cocycle over α.*

Then there exist

(1) a linear representation $\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$;
(2) a compact subgroup $K \subset \mathrm{GL}(d,\mathbb{R})$ that commutes with the image of ρ;
(3) a K-valued cocycle $\mathcal{C}\colon G\times X \to K$;
(4) and a measurable function $\Phi\colon X \to \mathrm{GL}(d,\mathbb{R})$

such that for a.e. $x \in X$ and every $g \in G$

$$\mathcal{A}(g,x) = \Phi(\alpha(g)(x))^{-1}\rho(g)\mathcal{C}(g,x)\Phi(x). \tag{4.3}$$

In particular, Theorem 4.2 states that any bounded measurable linear cocycle $\mathcal{A}\colon G\times X \to \mathrm{GL}(d,\mathbb{R})$ over the action α is cohomologous to the product of a constant cocycle $\rho\colon G \to \mathrm{SL}(d,\mathbb{R})$ and a compact-valued cocycle $\mathcal{C}\colon G\times X \to K \subset \mathrm{GL}(d,\mathbb{R})$.

When $\mathcal{A}$ is the derivative cocycle associated to a smooth volume-preserving action $\alpha\colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$, Theorem 4.2 says that the derivative $(\gamma,x) \mapsto D_x\alpha(\gamma)$ coincides—up to a compact group and measurable trivialization of TM—with a representation $\rho\colon G \to \mathrm{SL}(\dim(M),\mathbb{R})$. This, in particular, suggests that non-isometric, volume-preserving actions $\alpha\colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$ on low dimensional manifolds should be "derived from" affine actions of Γ. An example of a "derived from" affine action is the example of Katok in Lewis mentioned in Example 2.10. In [61], such a philosophy is carried out for volume-preserving Anosov actions of $\mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{T}^n$.

[3]Here, **bounded** means that for every compact $K \subset G$, the map $K\times X \to \mathrm{GL}(d,\mathbb{R})$ given by $(g,x) \mapsto \mathcal{A}(g,x)$ is bounded. More generally, we may replace the boundedness hypothesis with the hypothesis that the function $x \mapsto \sup_{g\in K}\log\|\mathcal{A}(g,x)\|$ is $L^1(\mu)$. See [42].

4.3. **Superrigidity for linear representations.** Zimmer's cocycle superrigidity theorem is an extension of Margulis's superrigidity theorem for linear representations. We formulate a version of this theorem for linear representations of lattices in $\mathrm{SL}(n,\mathbb{R})$.

Theorem 4.3 (Margulis superrigidity [80])**.** *For $n \geqslant 3$, let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$. Given a representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ there are*

(1) a linear representation $\hat\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$;
(2) a compact subgroup $K \subset \mathrm{GL}(d,\mathbb{R})$ that commutes with the image of $\hat\rho$

such that

$$\hat\rho(\gamma)\rho(\gamma)^{-1} \in K$$

for all $\gamma \in \Gamma$.

That is, $\rho = \hat\rho \cdot c$ is the product of the restriction of a representation

$$\hat\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$$

to Γ and a compact-valued representation $c\colon \Gamma \to K$. Moreover the image of $\hat\rho$ and c commute.

In the case that Γ is nonuniform, one can show that all compact-valued representations $c\colon \Gamma \to K$ have finite image. See for instance the discussion in [119, Section 16.4], especially [119, Exercise 16.4.1].

For certain cocompact $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$, there exists compact-valued representations $c\colon \Gamma \to \mathrm{SU}(n)$ with infinite image. (See discussion Example 2.9.) The next theorem, characterizing all homomorphisms from lattices in $\mathrm{SL}(n,\mathbb{R})$ into compact Lie groups, shows that representations into $\mathrm{SU}(n)$ are more-or-less the only such examples. The proof uses the p-adic version of Margulis's superrigidity theorem and some algebra. See [80, Theorem VII.6.5] and [119, Corollary 16.4.2].

Theorem 4.4. *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice. Let K be a compact Lie group and $\pi\colon \Gamma \to K$ a homomorphism.*

(1) If Γ is nonuniform then $\pi(\Gamma)$ is finite.
(2) If Γ is cocompact and $\pi(\Gamma)$ is infinite then there is a closed subgroup $K' \subset K$ with

$$\pi(\Gamma) \subset K' \subset K$$

and the Lie algebra of K' is of the form $\mathrm{Lie}(K') = \mathfrak{su}(n) \times \cdots \times \mathfrak{su}(n)$.

The appearance of $\mathfrak{su}(n)$ in (2) of Theorem 4.4 is due to the fact that $\mathfrak{su}(n)$ is the compact real form of $\mathfrak{sl}(n,\mathbb{R})$, the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$. For a cocompact lattice Γ in $\mathrm{SO}(n,n)$ as in Example 2.9, the analogue of Theorem 4.4 states that

$$\mathrm{Lie}(K') = \mathfrak{so}(2n) \times \cdots \times \mathfrak{so}(2n).$$

4.4. **Heuristic evidence for Conjecture 3.2.** We present a number of heuristics that motivate the conclusions of Conjectures 3.2 and 3.3.

4.4.1. *Analogy with linear representations.* Note that if $d < n$, there is no non-trivial representation $\hat\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$; moreover, by a dimension count, there is no embedding of $\mathfrak{su}(n)$ in $\mathfrak{sl}(d,\mathbb{R})$. We thus immediately obtain as corollaries of Theorems 4.3 and 4.4 the following.

Corollary 4.5. *For $n \geqslant 3$, let Γ be a lattice in $G = \mathrm{SL}(n,\mathbb{R})$. Then, for $d < n$, the image of any representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ is finite.*

Conjecture 3.2 can be seen as a "nonlinear" analogue of this corollary. That is, we aim to prove the same result when the linear group $\mathrm{GL}(d, \mathbb{R})$ is replaced by certain diffeomorphism groups $\mathrm{Diff}(M)$.

4.4.2. *Invariant measurable metrics.* For $n \geqslant 3$, let Γ be a lattice in $G = \mathrm{SL}(n, \mathbb{R})$ and consider a measure-preserving action $\alpha \colon \Gamma \to \mathrm{Diff}^1_\mu(M)$ where M is a compact manifold of dimension at most $d \leqslant n-1$ and μ is an arbitrary Borel probability measure on M preserved by α. The derivative cocycle of the action α is then $\mathrm{GL}(d, \mathbb{R})$-valued. Since there are no representations $\rho \colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(d, \mathbb{R})$ for $d < n$, Theorem 4.2 implies that the derivative cocycle is cohomologous to a compact-valued cocycle. In particular, we have the following:

Corollary 4.6. *For Γ, M, μ and $\alpha \colon \Gamma \to \mathrm{Diff}^1_\mu(M)$ as above*

(1) α preserves a 'μ-measurable Riemannian metric,' i.e. there is a μ-measurable, α-invariant, positive-definite symmetric two-form on TM;

(2) for any $\epsilon > 0$ and $\gamma \in \Gamma$, the set of $x \in M$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \log \|D_x \alpha(\gamma^n)\| \geqslant \epsilon$$

has zero μ-measure.

For (1), suppose the derivative cocycle is cohomologous to a K-valued cocycle for some compact group $K \subset \mathrm{GL}(d, \mathbb{R})$. One may then pull-back any K-invariant inner product on $\mathbb{R}^d$ to $T_x M$ via the map $\Phi(x)$ in Theorem 4.2 to an $\alpha(\Gamma)$-invariant inner product. Conclusion (2) follows from Poincaré recurrence to sets on which the function $\Phi \colon M \to \mathrm{GL}(d, \mathbb{R})$ in Theorem 4.2 has bounded norm and conorm. Note from (2) that all Lyapunov exponents (see Section 6.1 below) for individual elements of the action must vanish.

From Corollary 4.6, given $n \geqslant 3$ and a lattice Γ in $G = \mathrm{SL}(n, \mathbb{R})$, we have that every action $\alpha \colon \Gamma \to \mathrm{Diff}^1_{\mathrm{vol}}(M)$ preserves a Lebesgue-measurable Riemannian metric g whenever M is a compact manifold of dimension at most $n-1$. Suppose one could show that g was continuous or C^ℓ. As we discuss in Step 3 of Section 5 below, this combined with Theorem 4.4 implies the image $\alpha(\Gamma)$ is finite. Thus, Conjecture 3.2(2) follows if one can promote the measurable invariant metric g guaranteed by Corollary 4.6 of Theorem 4.2 to a continuous Riemannian metric.

The discussion in the previous paragraphs suggests the following variant of Conjecture 3.2(2) might hold:

> *For $n \geqslant 3$, if $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ is a lattice and if μ is any fully supported Borel probability measure on a compact manifold M of dimension at most $(n-1)$ then any homomorphism*
>
> $$\Gamma \to \mathrm{Diff}_\mu(M)$$
>
> *has finite image.*

Our method of proof of Conjecture 3.2(2) does not establish this conjecture. However, the conjecture would follow (even allowing for μ to have partial support) if the global rigidity result in Conjecture 3.7 holds.

4.4.3. *Actions with discrete spectrum.* Upgrading the measurable invariant Riemannian metric in Corollary 4.6 to a continuous Riemannian metric in the above heuristic seems quite difficult and is not the approach we take in the proof of Theorem 3.4. In [124], Zimmer was able to upgrade the measurable metric to a continuous metric for volume-preserving actions that are very close to isometries. This result now follows from the local rigidity of isometric actions in [8, 43].

Zimmer later established a much stronger result in [127] which provides very strong evidence for the volume-preserving cases in Conjecture 3.3. Using the invariant, measurable metric discussed above and that higher-rank lattices have Property (T), Zimmer showed that any volume-preserving action appearing in Conjecture 3.3 has discrete spectrum. In particular, this result implies that (the ergodic components of) all volume-preserving actions appearing in Conjecture 3.3 are measurably isomorphic to isometric actions.

5. PROOF OUTLINE OF THEOREM 3.4

We outline the proof of Theorem 3.4 for the case of C^∞ actions of cocompact lattice in $\mathrm{SL}(n,\mathbb{R})$. That is, for $n \geqslant 3$, we consider a cocompact lattice Γ in $\mathrm{SL}(n,\mathbb{R})$ and show that every homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ has finite image when

(1) M is a compact manifold of dimension at most $(n-2)$, or
(2) M is a compact manifold of dimension at most $(n-1)$ and α preserves a volume form vol.

The broad outline of the proof consists of 3 steps.

5.1. Step 1: Subexponential growth.

In the case that $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ is cocompact, using its action on $\mathrm{SL}(n,\mathbb{R})$ and that $\mathrm{SL}(n,\mathbb{R})$ is a proper length space one may show that Γ is finitely generated (see for example [35, Theorem 8.2]). More generally, it is a classical fact that all lattices Γ in semisimple Lie groups are finitely generated.

Fix a finite symmetric generating set S for Γ. Given $\gamma \in \Gamma$, let $|\gamma| = |\gamma|_S$ denote the word-length of γ relative to this generating set; that is,

$$|\gamma| = \min\{k : \gamma = s_k \cdots s_1, s_i \in S\}.$$

Note that if we replace the finite generating set S with a different finite generating set S', there is a uniform constant C such that the word-lengths are uniformly distorted:

$$|\gamma|_{S'} \leqslant C|\gamma|_S.$$

Thus all definitions below will be independent of the choice of S.

Equip TM with a Riemannian metric and corresponding norm.

Definition 5.1. We say that an action $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ has **uniform subexponential growth of derivatives** if for every $\epsilon > 0$ there is a $C = C_\epsilon$ such that for every $\gamma \in \Gamma$,

$$\sup_{x\in M} \|D_x\alpha(\gamma)\| \leqslant Ce^{\epsilon|\gamma|}.$$

Note that if $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ has uniform subexponential growth of derivatives it follows for every $\epsilon > 0$ that there is a $C = C_\epsilon$ such that

$$\sup_{x\in M} \|D_x\alpha(\gamma)\| \geqslant Ce^{-\epsilon|\gamma|}$$

for every $\gamma \in \Gamma$.

The following is the main result of [16], formulated here only for the case of (cocompact) lattices in $\mathrm{SL}(n,\mathbb{R})$.

Theorem 5.2 ([16, Theorem 2.8])**.** *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a cocompact lattice. Let $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ be an action. Suppose that either*

(1) $\dim(M) \leqslant n-2$*, or*

(2) $\dim(M) = n-1$ *and* α *preserves a smooth volume.*

Then α *has uniform subexponential growth of derivatives.*

Remark 5.3. The proof of Theorem 5.2 is the only place in the proof of Theorem 3.4 where cocompactness of Γ is used. It is not required for Steps 2 or 3 below. For $\Gamma = \mathrm{SL}(m,\mathbb{Z})$, the analogue of Theorem 5.2 is established in [17] and has been announced for general lattices [18].

5.2. **Step 2: Strong property (T) and averaging Riemannian metrics.** Assume $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ is an action by C^∞ diffeomorphisms.[4] The action α of Γ on M induces an action $\alpha_\#$ of Γ on tensor powers of the cotangent bundle of M by pull-back: Given $\omega \in (T^*M)^{\otimes k}$ write

$$\alpha_\#(\gamma)\omega = \alpha(\gamma^{-1})^*\omega;$$

that is, if $v_1, \dots v_k \in T_xM$ then

$$\alpha_\#(\gamma)\omega(x)(v_1,\dots,v_k) = \omega(x)(D_x\alpha(\gamma^{-1})v_1,\dots,D_x\alpha(\gamma^{-1})v_k).$$

In particular, we obtain an action of Γ on the set of Riemannian metrics which naturally sits as a half-cone inside $S^2(T^*M)$, the vector space of all symmetric 2-forms on M. Note that $\alpha_\#$ preserves $C^\ell(S^2(T^*(M)))$, the subspace of all C^ℓ sections of $S^2(T^*M)$ for any $\ell \in \mathbb{N}$.

Fix a volume form vol on M. The norm on TM induced by the background Riemannian metric induces a norm on each fiber of $S^2(T^*M)$. We then obtain a natural notion of measurable and integrable sections of $S^2(T^*M)$ with respect to vol. Let $\mathcal{H}^k = W^{2,k}(S^2(T^*M))$ be the Sobolev space of symmetric 2-forms whose weak derivatives of order ℓ are bounded with respect to the $L^2(\mathrm{vol})$-norm for $0 \leqslant \ell \leqslant k$. Then $\mathcal{H}^k$ is a Hilbert space. Let $\|\cdot\|_{\mathcal{H}^k}$ denote the corresponding Sobolev norm on $\mathcal{H}^k$ as well as the induced operator norm on the space $B(\mathcal{H}^k)$ of bounded operators on $\mathcal{H}^k$. Working in local coordinates, the Sobolev embedding theorem implies that

$$\mathcal{H}^k \subset C^\ell(S^2(T^*(M)))$$

as long as

$$\ell < k - \dim(M)/2.$$

In particular, for k sufficiently large, an element ω of $\mathcal{H}^k$ is a C^ℓ section of $S^2(T^*M)$ which will be a C^ℓ Riemannian metric on M if it is positive definite.

The action $\alpha_\#$ is a representation of Γ by bounded operators on $\mathcal{H}^k$. From Theorem 5.2, we obtain strong control on the norm growth of the induced representation $\alpha_\#$. In particular, we obtain that the representation $\alpha_\#\colon \Gamma \to B(\mathcal{H}^k)$ has **subexponential norm growth:**

Lemma 5.4. *Let* $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ *have uniform subexponential growth of derivatives. Then, for all* $\epsilon' > 0$ *there is* $C > 0$ *such that*

$$\|\alpha_\#(\gamma)\|_{\mathcal{H}^k} \leqslant Ce^{\epsilon'|\gamma|}$$

for all $\gamma \in \Gamma$.

The proof of Lemma 5.4 follows from the chain rule, Leibniz rule, and computations that bound the growth of higher-order derivatives by polynomial functions in the growth of the first derivative. See [43, Lemma 6.4] and discussion in [16, Section 6.3].

[4]For C^2 actions, one replaces the Hilbert Sobolev spaces $W^{2,k}(S^2(T^*M)))$ below with appropriate Banach Sobolev spaces $W^{p,1}(S^2(T^*M)))$ and verifies such spaces are of the type $\mathcal{E}_{10}$ considered in [31].

We use the main result from [31, 67]: cocompact lattices Γ in higher-rank simple Lie groups (such as $\mathrm{SL}(n,\mathbb{R})$ for $n \geqslant 3$) satisfy Lafforgue's **strong Banach property (T)** first introduced in [67]. The result for $\mathrm{SL}(n,\mathbb{R})$ and its cocompact lattices (as well as most other higher-rank simple Lie groups) is established by Lafforgue in Corollary 4.1 and Proposition 4.3 of [67]; for cocompact lattices in certain other higher-rank Lie groups, the results of [31] are needed. See also [30] for the case of nonuniform lattices. Strong Banach property (T) considers representations π of Γ by bounded operators on certain Banach spaces E (of type $\mathcal{E}_{10}$). If such representations have sufficiently slow exponential norm growth, then there exists a sequence of operators p_n converging to a projection p_∞ such that for any vector $v \in E$, the limit $p_\infty(v)$ is π-invariant. In the case that E is a Hilbert space (which we may assume when α is an action by C^∞ diffeomorphisms) we have the following formulation. Note that Lemma 5.4 (which follows from Theorem 5.2) ensures our representation $\alpha_\#$ satisfies the hypotheses of the theorem.

Theorem 5.5 ([30, 31, 67])**.** *Let $\mathcal{H}$ be a Hilbert space and for $n \geqslant 3$, let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$.*

There exists $\epsilon > 0$ such that for any representation $\pi\colon \Gamma \to B(\mathcal{H})$, if there exists $C_\epsilon > 0$ such that

$$\|\pi(\gamma)\| \leqslant C_\epsilon e^{\epsilon|\gamma|}$$

for all $\gamma \in \Gamma$ then there exists a sequence of operators $p_n = \sum w_i \pi(\gamma_i)$ in $B(\mathcal{H})$—where $w_i \geqslant 0$, $\sum w_i = 1$, and $w_i = 0$ for every $\gamma_i \in \Gamma$ of word-length larger than n—such that for any vector $v \in \mathcal{H}$, the sequence $v_n = p_n(v) \in \mathcal{H}$ converges to an invariant vector $v^* = p_\infty(v)$.

Moreover, the convergence is exponentially fast: there exist $0 < \lambda < 1$ and $C = C_\lambda$ such that $\|v_n - v^*\| \leqslant C\lambda^n \|v\|$.

Though we only use convergence in the strong operator topology, the convergence in Theorem 5.5 actually holds in the norm topology.

Theorem 5.5 as stated in [67] (and its extension in [31]) requires that Γ be cocompact. The extension to nonuniform lattices is announced in [30]. The exponential convergence in Theorem 5.5 is often not explicitly stated in the definition of strong property (T) or in statements of theorems establishing that the property holds for lattices in higher-rank simple Lie groups; however, the exponential convergence follows from the proofs.

We complete Step 2 with the following computation.

Proposition 5.6. *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice and let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ be an action with uniform subexponential growth of derivatives. Then for any ℓ, there is a C^ℓ Riemannian metric g on M such that*

$$\alpha(\Gamma) \subset \mathrm{Isom}_g(M).$$

Proof. Consider an arbitrary C^∞ Riemannian metric g. For any k, we have $g \in \mathcal{H}^k$. We apply Theorem 5.5 and its notation to the representation $\alpha_\#\colon \Gamma \to B(\mathcal{H}^k)$ with g the initial vector v. As averages of finitely many Riemannian metrics are still Riemannian metrics we have that $g_n := p_n(g)$ is positive definite for every n. In particular, the limit $g_\infty = p_\infty(g)$ is in the closed cone of positive (possibly indefinite) symmetric 2-tensors in $\mathcal{H}^k$. Having taken k sufficiently large we have that g_∞ is C^ℓ; in particular, g_∞ is continuous, everywhere defined, and positive everywhere. We need only confirm that g_∞ is non-degenerate, i.e. is positive definite on T_xM for every $x \in M$.

Given any $x \in M$ and unit vector $\xi \in T_xM$, for any $\epsilon > 0$ we have from Definition 5.1 that there is a $C_\epsilon > 0$ such that

$$\begin{aligned} p_n(g)(\xi,\xi) &= \left(\sum w_i \alpha_\#(\gamma_i)g\right)(\xi,\xi) \\ &= \sum w_i g(D_x\alpha(\gamma_i^{-1})\xi, D_x\alpha(\gamma_i^{-1})\xi) \\ &\geqslant \frac{1}{C_\epsilon^2}e^{-2\epsilon n} \end{aligned}$$

where we use that $w_i > 0$ only when γ_i has word-length at most n.

On the other hand, from the exponential convergence in Theorem 5.5 we have

$$|p_n(g)(\xi,\xi) - p_\infty(g)(\xi,\xi)| \leqslant C_\lambda \lambda^n.$$

Thus

$$p_\infty(g)(\xi,\xi) \geqslant \frac{1}{C_\epsilon^2}e^{-2\epsilon n} - C_\lambda\lambda^n$$

for all $n \geqslant 0$. Taking $\epsilon > 0$ sufficiently small we can ensure that

$$C_\epsilon^2 e^{2\epsilon n} < \frac{1}{C_\lambda}\lambda^{-n}$$

for all sufficiently large n. Then, for all sufficiently large n we have

$$\frac{1}{C_\epsilon^2}e^{-2\epsilon n} > C_\lambda\lambda^n$$

and thus $p_\infty(g)(\xi,\xi) > 0$. □

5.3. **Step 3: Margulis superrigidity with compact codomain.** From Steps 1 and 2 we have that any action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ as in Theorem 3.4 preserves a C^ℓ Riemannian metric g. In the general case of C^2-actions (or even $C^{1+\beta}$-actions), we have that any action $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ preserves a continuous Riemannian metric g. See [16, Theorem 2.7]. We thus have

$$\alpha\colon \Gamma \to \mathrm{Isom}^2_g(M) \subset \mathrm{Diff}^2(M).$$

Let $\dim(M) = m$. The group $\mathrm{Isom}_g(M)$ of isometries of a continuous Riemannian metric is a compact Lie group with

$$\dim(\mathrm{Isom}_g(M)) \leqslant \frac{m(m+1)}{2}. \tag{5.1}$$

Indeed, the orbit of any point $p \in M$ under $\mathrm{Isom}_g(M)$ has dimension at most m and the dimension of the stabilizer of a point is at most $\frac{m(m-1)}{2}$, the dimension of $\mathrm{SO}(m)$; thus

$$\dim(\mathrm{Isom}_g(M)) \leqslant m + \frac{m(m-1)}{2}.$$

With $K = \mathrm{Isom}^2_g(M) \subset \mathrm{Diff}^2(M)$ we thus obtain a compact-valued representation $\alpha\colon \Gamma \to K$. By equation (5.1), if $m < \frac{1}{2}\sqrt{8n^2-7} - \frac{1}{2}$ then $\dim(\mathfrak{su}(n)) = n^2 - 1 > \dim(K)$; by conclusion (2) of Theorem 4.4, $\alpha(\Gamma)$ is thus contained in a 0-dimensional subgroup of K. This holds in particular if $m \leqslant n-1$. We thus conclude that the image

$$\alpha(\Gamma) \subset K = \mathrm{Isom}^2_g(M) \subset \mathrm{Diff}^2(M)$$

is finite.

Summarizing the arguments from Steps (2) and (3), we obtain the following.

Theorem 5.7. *For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice. Let $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ be an action with uniform subexponential growth of derivatives.*

Then, if

$$\dim(M) < \frac{1}{2}\sqrt{8n^2 - 7} - \frac{1}{2},$$

the image $\alpha(\Gamma)$ *is finite.*

Part 2. Primer: smooth ergodic theory for actions of $\mathbb{Z}^d$

We pause the proof of Theorem 3.4 to introduce a number of constructions and results from smooth ergodic theory. Of particular interest will be Lyapunov exponents, metric entropy, and the relationships between entropy, exponents, and conditional measures along unstable manifolds for single diffeomorphisms and for actions of higher-rank, torsion-free abelian groups. We state all our results for actions of $\mathbb{Z}^d$ though all constructions and results naturally extend to groups of the form $\mathbb{R}^d$ or $\mathbb{Z}^k \times \mathbb{R}^\ell$.

6. Lyapunov exponents and Pesin manifolds

6.1. **Lyapunov exponents for diffeomorphisms.** Let $f\colon M \to M$ be a C^1 diffeomorphism of a compact manifold M. Let μ be an ergodic, f-invariant Borel probability measure.

We recall Oseledec's Theorem [89]; see also [100, 112].

Theorem 6.1 (Oseledec [89])**.** *There are*

(1) a measurable set Λ *with* $\mu(\Lambda) = 1$*;*
(2) numbers $\lambda^1 > \lambda^2 > \cdots > \lambda^p$*;*
(3) a μ*-measurable,* Df*-invariant splitting* $T_xM = \bigoplus_{i=1}^p E^i(x)$ *defined for* $x \in \Lambda$

such that for every $x \in \Lambda$

(a) for every $v \in E^i(x) \setminus \{0\}$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda^i;$$

(b) if Jf *denotes the Jacobian determinant of* f *then*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |Jf^n| = \sum_{i=1}^p m_i \lambda^i$$

where $m_i = \dim E^i(x)$*;*

(c) for every $i \neq j$ *we have*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\Big(\sin\angle\Big(E^i(f^n(x)), E^j(f^n(x))\Big)\Big) = 0.$$

The numbers λ^i are called **Lyapunov exponents** of f with respect to μ and the subspaces $E^i(x)$ are called the **Oseledec's subspaces**. Above, m^i denotes the almost-surely constant value of $\dim E^i(x)$, called the **multiplicity** of λ^i.

Given any f-invariant measure μ on M (which may be nonergodic) the **average top fiberwise Lyapunov exponent** of f with respect to μ is

$$\lambda_{\text{top}}(f,\mu) = \inf_{n\to\infty} \frac{1}{n} \int \log \|D_x f^n\|\, d\mu(x). \tag{6.1}$$

Since μ is f-invariant, the subadditive ergodic theorem implies the infimum in (6.1) can be replace by a limit (see [49, 64] and [111, Chapter 3]).

When $f\colon M \to M$ preserves a possibly non-ergodic measure μ we recall the following construction.

Definition 6.2. Let $f\colon M \to M$ be a Borel map of a metric space M preserving a Borel probability measure μ. Then, there exists a measurable partition $\mathcal{E}$ of (M, μ) such that—writing $\{\mu_x^{\mathcal{E}}\}$ for a family of conditional measures of μ relative to $\mathcal{E}$ (see Definition 7.1)—for μ-a.e. x the measure $\mu_x^{\mathcal{E}}$ is an ergodic, f-invariant Borel probability measure. The partition $\mathcal{E}$ is called the **ergodic decomposition** or the **partition into ergodic components** of μ with respect to f. The measures $\{\mu_x^{\mathcal{E}}\}$ are called the **ergodic components** of μ.

Given any f-invariant measure μ on M if $\{\mu_x^{\mathcal{E}}\}$ is the ergodic decomposition of μ and if $\lambda_x^1 > \lambda_x^2 > \cdots > \lambda_x^{p(x)}$ denote the Lyapunov exponents of f with respect to the ergodic invariant measure $\mu_x^{\mathcal{E}}$ then we have

$$\lambda_{\text{top}}(f, \mu) = \int \lambda_x^1 \, d\mu(x).$$

6.2. Lyapunov exponents and (sub)exponential growth of derivatives.

To motivate our interest in Lyapunov exponents, recall that in order to prove Theorem 3.4 it remains to show Theorem 5.2. For actions of the group $\mathbb{Z}$, the following proposition characterizes uniform subexponential growth of derivatives in terms of the vanishing of Lyapunov exponents.

Let M be a compact manifold and equip TM with a background Riemannian metric and associated norm. Let $f\colon M \to M$ be a C^1 diffeomorphism. We say $f\colon M \to M$ has **uniform subexponential growth of derivatives** if for all $\epsilon > 0$ there is a $C_\epsilon > 0$ such that

$$\|Df^n\| := \sup_{x \in M} \|D_x f^n\| < C_\epsilon e^{\epsilon |n|} \qquad \text{for all } n \in \mathbb{Z}.$$

Note that we allow that $C_\epsilon \to \infty$ as $\epsilon \to 0$.

Proposition 6.3. *A diffeomorphism $f\colon M \to M$ has uniform subexponential growth of derivatives if and only if for any f-invariant Borel probability measure μ, all Lyapunov exponents of f with respect to μ are zero.*

That is, $f\colon M \to M$ has uniform subexponential growth of derivatives if and only if $\lambda_{\text{top}}(f, \mu) = \lambda_{\text{top}}(f^{-1}, \mu) = 0$ for every f-invariant Borel probability measure μ.

Proof. We show that vanishing of all Lyapunov exponents for all f-invariant probability measures implies that f has uniform subexponential growth of derivatives; the converse is clear.

Suppose that $f\colon M \to M$ fails to have uniform subexponential growth of derivatives. Then there is an $\epsilon > 0$ and sequences of iterates $n_j \in \mathbb{Z}$ with $|n_j| \to \infty$, base points $x_j \in M$, and unit vectors $v_j \in T_{x_j} M$ such that

$$\|D_{x_j} f^{n_j} v_j\| \geqslant e^{\epsilon |n_j|}. \tag{6.2}$$

Replacing f with f^{-1}, we may assume without loss of generality that $n_j \to \infty$.

Let $UM \subset TM$ denote the unit-sphere bundle. We represent an element of UM by a pair (x, v) where $v \in T_x M$ with $\|v\| = 1$. Note that UM is compact. Note also that $Df\colon TM \to TM$ induces a map $Uf\colon U \to U$ given by the renormalized derivative:

$$Uf(x, v) := \left(f(x), \frac{D_x f(v)}{\|D_x f(v)\|} \right).$$

Define $\Phi\colon UM \to \mathbb{R}$ as follows: given $(x, v) \in UM$, let

$$\Phi(x, v) := \log \|D_x f(v)\|.$$

By the chain rule, we have

$$\log \|D_x f^n(v)\| = \sum_{j=0}^{n-1} \Phi(Uf^j(x,v)).$$

For each j, let ν^j denote the empirical measure along the orbit segment

$$(x_j, v_j), Uf(x_j, v_j), \dots, Uf^{n_j-1}(x_j, v_j)$$

in UM given by

$$\nu^j = \frac{1}{n_j} \sum_{k=0}^{n_j-1} \delta_{Uf^k(x_j,v_j)}.$$

From (6.2) we have for every j that

$$\int \Phi \, d\nu^j \geqslant \epsilon.$$

Claim 6.4. *Let ν be any weak-$*$ subsequential limit of $\{\nu^j\}$. Then*

(a) ν is Uf-invariant;
(b) $\int \Phi \, d\nu \geqslant \epsilon$.

Proof. Conclusion (a) follows as in the proof of the Krylov-Bogolyubov theorem: if $\phi\colon M \to \mathbb{R}$ is any (bounded) continuous function then

$$\lim_{j\to\infty} \left| \int \phi \, d\nu^j - \int \phi \circ f \, d\nu^j \right| \leqslant \lim_{j\to\infty} \frac{2\|\phi\|_{C^0}}{n_j} = 0 \tag{6.3}$$

showing that ν is f-invariant. Conclusion (b) follows from continuity of Φ and weak-$*$ convergence. $\square$

From Claim 6.4(b), we may replace ν with an ergodic component (see Definition 6.2) ν' of ν such that $\int \Phi \, d\nu' \geqslant \epsilon$.

Take μ to be the push-forward of ν' under the natural projection $UM \to M$. Then μ is an f-invariant, ergodic measure on M. Let $\{\nu'_x\}$ denote a family of conditional measures of ν' for the partition of UM into fibers over M. By the pointwise ergodic theorem, for μ-a.e. $x \in M$ and ν'_x-a.e. $v \in UM(x)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(Uf^j(x,v)) = \int \Phi \, d\nu' \geqslant \epsilon.$$

On the other hand,

$$\begin{aligned}
\lambda_{\mathrm{top}}(f,\mu) &= \lim_{n\to\infty} \frac{1}{n} \int \log \|D_x f^n\| \, d\mu(x) \\
&= \lim_{n\to\infty} \frac{1}{n} \int \sup_{v\in UM(x)} \sum_{j=0}^{n-1} \Phi(Uf^j(x,v)) \, d\mu(x) \\
&\geqslant \lim_{n\to\infty} \int \int \frac{1}{n} \sum_{j=0}^{n-1} \Phi(Uf^j(x,v)) \, d\nu_x(v) \, d\mu(x) \\
&= \int \Phi \, d\nu' \geqslant \epsilon.
\end{aligned}$$

Above, the inequality follows from comparing the maximal growth with the average growth (averaged by ν'_x.) It follows that the largest Lyapunov exponent of f with respect to μ is at least $\epsilon > 0$. □

6.3. **Lyapunov exponents for nonuniformly hyperbolic $\mathbb{Z}^d$-actions.** How does the theory of Lyapunov exponents change for actions of more general abelian groups? We state a version of Oseledec's theorem for actions of $\mathbb{Z}^d$ which can easily be extended to actions of $\mathbb{R}^\ell \times \mathbb{Z}^k$.

Let M be a compact manifold, let $\alpha\colon \mathbb{Z}^d \to \mathrm{Diff}^1(M)$ be a $\mathbb{Z}^d$-action, and let μ be an ergodic, α-invariant measure.

Theorem 6.5 (Higher-rank Oseledec's theorem (see [19]))**.** *There are*

(1) a measurable set Λ with $\mu(\Lambda) = 1$;
(2) linear functionals $\lambda^1, \lambda^2, \dots, \lambda^p\colon \mathbb{R}^d \to \mathbb{R}$;
(3) a μ-measurable, $D\alpha$-invariant splitting $T_xM = \bigoplus_{i=1}^p E^i(x)$ defined for $x \in \Lambda$

such that for every $x \in \Lambda$

(a) for every $v \in E^i(x) \smallsetminus \{0\}$
$$\lim_{|n|\to\infty} \frac{\log \|D_x\alpha(n)(v)\| - \lambda^i(n)}{|n|} = 0;$$
(b) if Jf denotes the Jacobian determinant of f then
$$\lim_{|n|\to\infty} \frac{\log |J\alpha(n)| - \sum_{i=1}^p m^i\lambda^i(n)}{|n|};$$
(c) for every $i \neq j$
$$\lim_{n\to\infty} \frac{1}{|n|} \log\Big(\sin\angle\Big(E^i(\alpha(n)(x)), E^j(\alpha(n)(x))\Big)\Big) = 0.$$

In (b), m^i is the almost-surely constant value of $\dim E^i(x)$, called the **multiplicity** of λ^i. Note that (a) implies convergence along rays: for any $n \in \mathbb{Z}^d$ and $v \in E^i(x) \smallsetminus \{0\}$
$$\lim_{k\to\infty} \frac{1}{k} \log \|D_x\alpha(kn)(v)\| = \lambda^i(n). \tag{6.4}$$
The convergence in (a) is taken along **any** sequence $n \to \infty$; this is stronger than (6.4) and is typically needed in applications.

6.4. **Unstable manifolds and coarse Lyapunov manifolds.**

6.4.1. *Unstable subspaces and unstable manifolds for a single diffeomorphism.* Let $f\colon M \to M$ be a C^1 diffeomorphism of M and let μ be an ergodic, f-invariant measure. Let λ^i be the Lyapunov exponents for f with respect to μ. For $x \in \Lambda \subset M$ where Λ is as in Theorem 6.1, define
$$E^u(x) := \bigoplus_{\lambda^i > 0} E^i(x)$$
to be the **unstable subspace** through x. We have that
$$E^u(x) := \{v \in T_xM : \limsup_{n\to\infty} \frac{1}{n} \log \|D_xf^{-n}(v)\| < 0\}.$$
We may similarly define **stable** and **neutral** (or center) subspaces through x, respectively, by
$$E^s(x) := \bigoplus_{\lambda^i < 0} E^i(x)$$

and
$$E^c(x) := \bigoplus_{\lambda^i=0} E^i(x).$$
We now assume that $f\colon M \to M$ is $C^{1+\beta}$ for $\beta > 0$. Through μ-almost every point x the set
$$W^u(x) := \left\{ y : \limsup_{n\to\infty} \frac{1}{n} \log(d(f^{-n}(x), f^{-n}(y))) < 0 \right\}$$
is a connected $C^{1+\beta}$ injectively immersed manifold with $T_xW^u(x) = E^u(x)$ (see [91]) called the **(global) unstable Pesin manifold of f through x**. The collection of all $W^u(x)$ forms a partition of (a full measure subset of) M; in general, this partition does not have the structure of a nice foliation. However, restricted to sets of large measure the partition into local unstable manifolds has the structure of a continuous lamination. That is, for almost every $x \in M$ and any $\epsilon > 0$ there is a neighborhood U of x such that, on a set Ω of relative measure $(1 - \epsilon)$ in U, the local leaves of W^u-manifolds form a partition of Ω by embedded $\dim(E^u)$-dimensional balls that vary continuously in the $C^{1+\beta}$-topology.

Given the Lyapunov exponents $\lambda^1 > \lambda^2 > \cdots > \lambda^p$ of μ fix $j \in \{1, \cdots, p\}$ such that $\lambda^j > 0$. Then for almost every x the set
$$W^j(x) := \left\{ y : \limsup_{n\to\infty} \frac{1}{n} \log(d(f^{-n}(x), f^{-n}(y))) \leqslant -\lambda^j \right\}$$
is again a connected $C^{1+\beta}$ injectively immersed manifold with
$$T_xW^j(x) = \bigoplus_{\lambda^i \geqslant \lambda^j} E^i(x)$$
called the **(global) jth unstable manifold** through x.

6.4.2. *Coarse Lyapunov exponents and subspaces.* Let $\alpha\colon \mathbb{Z}^d \to \mathrm{Diff}^1(M)$ be an action and let μ be an ergodic, α-invariant probability measure. We introduce objects that play the role of unstable subspaces and unstable manifolds for the $\mathbb{Z}^d$-action α.

Given Lyapunov exponents $\lambda^1, \lambda^2, \ldots, \lambda^p\colon \mathbb{R}^d \to \mathbb{R}$ we say λ^i and λ^j are **positively proportional** if there is a $c > 0$ with
$$\lambda^i = c\lambda^j.$$
Note that this defines an equivalence relation on the linear functionals
$$\lambda^1, \lambda^2, \ldots, \lambda^p\colon \mathbb{R}^d \to \mathbb{R}.$$
The positive proportionality classes are called **coarse Lyapunov exponents**. For a $\mathbb{Z}$-action generated by a single diffeomorphism f, the coarse Lyapunov exponents are simply the collections of positive, zero, and negative Lyapunov exponents.

Let $\chi = \{\lambda^i\}$ be a coarse Lyapunov exponent. While the size of $\chi(n)$ is not well defined, the sign of χ is well defined. Write
$$E^\chi(x) = \oplus_{\lambda^i \in \chi} E^i(x)$$
called the corresponding **coarse Lyapunov subspace.**

6.4.3. *Coarse Lyapunov manifolds for $\mathbb{Z}^d$-actions.* Analogous to the existence and properties of unstable Pesin manifolds for nonuniformly hyperbolic diffeomorphisms we have the following for actions of higher-rank abelian groups.

Let $\alpha\colon \mathbb{Z}^d \to \mathrm{Diff}^{1+\beta}(M)$ be an action and let μ be an ergodic, α-invariant probability measure. Let Λ be as in Theorem 6.5.

Proposition 6.6. *For almost every $x \in \Lambda$ and for every coarse Lyapunov exponent χ there is a connected, $C^{1+\beta}$, injectively immersed manifold $W^\chi(x)$ satisfying the following:*
(1) $T_x W^\chi(x) = E^\chi(x)$;
(2) $\alpha(n)W^\chi(x) = W^\chi(\alpha(n)(x))$ for all $n \in \mathbb{Z}^d$;
(3) $W^\chi(x)$ is the set of all $y \in M$ satisfying

$$\limsup_{k\to\infty} \frac{1}{k} \log d(\alpha(-kn)(y), \alpha(-kn)(x)) < 0 \text{ for all } n \in \mathbb{Z}^d \text{ with } \chi(n) > 0.$$

To construct W^χ-manifolds, given $n \in \mathbb{Z}^d$ with $\chi(n) > 0$ let $W^u_{\alpha(n)}(x)$ denote the unstable manifold for the diffeomorphism $\alpha(n)\colon M \to M$ through x. Then, for almost every $x \in M$ the manifold $W^\chi(x)$ is the path component of the intersection

$$\bigcap_{n\in\mathbb{Z}^d, \chi(n)>0} W^u_{\alpha(n)}(x)$$

containing x.

7. Metric entropy

7.1. **Metric entropy.** Let (X, μ) be a **standard probability space**. That is, (X, μ) equipped with the σ-algebra of μ-measurable sets is measurably isomorphic to an interval equipped with the Lebesgue measure and a countable number of point masses.

7.1.1. *Measurable partitions and conditional measures.* Recall that a partition ξ of (X, μ) is **measurable** if the quotient $(Y, \hat\mu) := (X, \mu)/\xi$ is a standard probability space. This is a technical but crucial condition. For more discussion and other characterizations of measurability see [28] and [104].

A key property of measurable partitions is the existence and uniqueness of a family of conditional measures (or a **disintegration**) of μ relative to this partition. Given a partition ξ of X, for $x \in X$ we write $\xi(x)$ for the element of ξ containing x.

Definition 7.1. Let ξ be a measurable partition of (X, μ). Then there is a family of Borel probability measure $\{\mu^\xi_x\}_{x\in X}$, called a **family of conditional measures** of μ relative to ξ, with the following properties: For almost every x
(1) μ^ξ_x is a Borel probability measure on X with $\mu^\xi_x(\xi(x)) = 1$;
(2) if $y \in \xi(x)$ then $\mu^\xi_y = \mu^\xi_x$.
Moreover, if $D \subset X$ is a Borel subset then
(3) $x \mapsto \mu^\xi_x(D)$ is measurable and
(4) $\mu(D) = \int \mu^\xi_x(D)\, d\mu(x)$.
Such a family is unique modulo μ-null sets.

For construction and properties of $\{\mu^\xi_x\}$ see for instance [104].

7.1.2. *Conditional information and conditional entropy.* Given a measurable partition η of a standard probability space (X, μ), write $\{\mu^\xi_x\}$ for a family of conditional measures of μ with respect to the partition ξ. Given two measurable partitions η, ξ of (X, μ) the **mean conditional information** of η relative to ξ is

$$I_\mu(\eta \mid \xi)(x) = -\log(\mu^\xi_x(\eta(x)))$$

and the **mean conditional entropy** of η relative to ξ is

$$H_\mu(\eta \mid \xi) = \int I_\mu(\eta \mid \xi)(x)\, d\mu(x).$$

The **join** $\eta \vee \xi$ of two partitions η and ξ is

$$\eta \vee \xi = \{A \cap B \mid A \in \eta, B \in \xi\}.$$

The **entropy** of η is $H_\mu(\eta) = H_\mu(\eta \mid \{\emptyset, X\})$. Note that if $H_\mu(\eta) < \infty$ then η is necessarily countable.

7.1.3. *Metric entropy of a transformation.* Let $f\colon (X,\mu) \to (X,\mu)$ be an invertible, measurable, measure-preserving transformation. Let η be an arbitrary measurable partition of (X,μ). We define

$$\eta^+ := \bigvee_{i=0}^{\infty} f^i\eta, \qquad \eta^f := \bigvee_{i\in\mathbb{Z}}^{\infty} f^i\eta.$$

We define the **entropy of f given the partition** η to be

$$h_\mu(f,\eta) := H_\mu(\eta \mid f\eta^+) = H_\mu(\eta^+ \mid f\eta^+) = H_\mu(f^{-1}\eta^+ \mid \eta^+).$$

We define the **μ-metric entropy of f** to be $h_\mu(f) = \sup\{h_\mu(f,\eta)\}$ where the supremum is taken over all measurable partitions of (X,μ). If

$$\mu = \alpha\mu_1 + \beta\mu_2$$

where $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta = 1$ and μ_1 and μ_2 are f-invariant Borel probability measures then

$$h_\mu(f) = \alpha h_{\mu_1}(f) + \beta h_{\mu_2}(f). \tag{7.1}$$

7.2. **Entropy under factor maps.** Let (X,μ) and (Y,ν) be standard probability spaces. Let $f\colon X \to X$ and $g\colon Y \to Y$ be measure-preserving transformations. Suppose there is a measurable map $\psi\colon X \to Y$ with

$$\psi_*\mu = \nu$$

and

$$\psi \circ f = g \circ \psi.$$

In this case, we say that $g\colon (Y,\nu) \to (Y,\nu)$ is a **measurable factor** of $f\colon (X,\mu) \to (X,\mu)$.

We note that entropy only decreases under measurable factors: if $g\colon (Y,\nu) \to (Y,\nu)$ is a measurable factor of $f\colon (X,\mu) \to (X,\mu)$ then

$$h_\nu(g) \leqslant h_\mu(f).$$

The difference between the entropies $h_\nu(g)$ and $h_\mu(f)$ is captured by the **Abramov–Rohlin theorem**. Let ζ be the measurable partition of (X,μ) into level sets of $\psi\colon X \to Y$. Note that ζ is an f-invariant partition: $\zeta = \zeta^f$. Define the conditional entropy $h_\mu(f \mid \zeta)$ of f relative to ζ to be

$$h_\mu(f \mid \zeta) = \sup_\xi h_\mu(f, \xi \vee \zeta)$$

where, as usual, the supremum is over all measurable partitions ξ of (X,μ). We call $h_\mu(f \mid \zeta)$ the **fiberwise entropy** of f. The Abramov–Rohlin theorem (see [1, 10, 71]) states the following:

$$h_\mu(f) = h_\nu(g) + h_\mu(f \mid \zeta). \tag{7.2}$$

7.3. **Unstable entropy of a diffeomorphism.** Let $f\colon M \to M$ be a $C^{1+\beta}$ diffeomorphism and let μ be an ergodic, f-invariant measure.

7.3.1. *Partitions subordinate to a foliation.* For the following discussion and in most applications considered in this text, we may take $\mathcal{F}$ to be an f-invariant foliation of M with

$C^{1+\beta}$ leaves. More generally, we may take $\mathcal{F}$ to be, in the terminology introduced in [19], an f-invariant, **tame measurable foliation**; that is, $\mathcal{F}$ a partition of a full measure set by $C^{1+\beta}$ manifolds with the property that locally, restricting to sets of large measure, $\mathcal{F}$ has the structure of a continuous family of $C^{1+\beta}$ discs. The primary examples of such measurable foliations include the partition into global jth unstable Pesin manifolds and the partition into global coarse Lyapunov manifolds in the setting of $\mathbb{Z}^d$-actions. Note that the partition into leaves of measurable foliation is not necessarily a measurable partition; rather the transverse structure of the foliation is measurable.

Write $\mathcal{F}(x)$ for the leave of $\mathcal{F}$ through x. We say $\mathcal{F}$ is **expanding** (for f) if $\mathcal{F}(x) \subset W^u(x)$, i.e. if $\mathcal{F}(x)$ is a subset of the global unstable manifold through x for f discussed in Section 6.4. As a key example, one should consider $\mathcal{F}^u$, the partition of M into full global unstable manifolds.

Definition 7.2. We say a measurable partition ξ is **subordinate to** $\mathcal{F}$ if

(1) $\xi(x) \subset \mathcal{F}(x)$ for μ-a.e. x;
(2) $\xi(x)$ contains an open (in the immersed topology) neighborhood of x in $\mathcal{F}(x)$ for μ-a.e. x;
(3) $\xi(x)$ is precompact in (the immersed topology of) $\mathcal{F}(x)$ for μ-a.e. x;

7.3.2. *Partial ordering on the set of partitions.* We recall the partial order on partitions of (M, μ). Let ξ and η be partitions of the probability space (M, μ). We write

$$\eta \prec \xi$$

and say that ξ is **finer** than η (or that η is **coarser** than ξ) if there is a subset $X \subset M$ with $\mu(X) = 1$ such that for almost every x,

$$\xi(x) \cap X \subset \eta(x) \cap X.$$

We say $\eta = \xi$ if $\eta \prec \xi$ and $\xi \prec \eta$.

7.3.3. *Entropy conditioned on a foliation.* We say that a partition ξ is **increasing** if $f\xi \prec \xi$ where $f\xi$ denotes the partition $f\xi = \{f(C) \mid C \in \xi\}$.

Definition 7.3. Given an expanding, f-invariant foliation $\mathcal{F}$ we define the **entropy of** f **conditioned on** $\mathcal{F}$ to be

$$h_\mu(f \mid \mathcal{F}) = h_\mu(f, \xi)$$

where ξ is any increasing, measurable partition subordinate to $\mathcal{F}$.

There are two small claims in Definition 7.3: First we have that $h_\mu(f, \xi_1) = h_\mu(f, \xi_2)$ for any two increasing partitions ξ_1 and ξ_2 subordinate to $\mathcal{F}$; see for example [72, Lemma 3.1.2]. Second, such a partition ξ always exists. This was shown when $\mathcal{F} = \mathcal{F}^u$ is the partition into global unstable Pesin manifolds for a $C^{1+\beta}$ diffeomorphism in [70] (see also discussion in [72, (3.1)]) extending a construction due to Sinai for uniformly hyperbolic dynamics [108, 109]; the proof in [70] can be adapted for general invariant expanding $\mathcal{F}$.

When $\mathcal{F} = \mathcal{F}^u$ is the partition into full unstable manifolds, define the **unstable metric entropy of** f to be

$$h^u_\mu(f) := h_\mu(f \mid \mathcal{F}^u).$$

The principal result (Corollary 5.3) of [72] shows that for C^2 diffeomorphisms we have equality of the metric entropy of f and the unstable metric entropy of f:

$$h_\mu(f) = h^u_\mu(f). \tag{7.3}$$

For $C^{1+\beta}$-diffeomorphism without zero Lyapunov exponents equality (7.3) was shown by Ledrappier in [68]; for the general case of $C^{1+\beta}$-diffeomorphisms, (7.3) holds from [14].

7.4. **Entropy, exponents, and geometry of conditional measures.** In this section, we consider the relationships between metric entropy $h_\mu(f)$, Lyapunov exponents, and the geometry of conditional measures along unstable manifolds.

Let $f\colon M \to M$ be a $C^{1+\beta}$ diffeomorphism and let μ be an ergodic, f-invariant measure. At one extreme, we have the following Lemma characterizing invariant measures with zero entropy.

Lemma 7.4. *Let μ be an ergodic, f-invariant measure on M and let ξ be a measurable partition of (M, μ) subordinate to the partition into unstable manifolds. The following are equivalent:*

(1) $h_\mu(f) = 0$;
(2) for μ-a.e. x, the conditional measure μ^ξ_x has at least one atom;
(3) for μ-a.e. x, the conditional measure μ^ξ_x is a single atom supported at x;
(4) the partition of (M, μ) into full W^u-manifolds is a measurable partition.

Proof sketch. The implications (1) $\implies$ (4) and (1) $\implies$ (3) are a consequence of [72, Theorem B] (see also [14] for $C^{1+\beta}$ setting.) Indeed, if $h_\mu(f) = 0$, then the Pinsker partition is the point partition. From [72, Theorem B] we have that the Pinsker partition is the measurable hull of (and in particular is coarser than) the partition into full unstable manifolds. As the point partition is the finest partition, it follows that the partition into full unstable manifolds is measurably equivalent to the point partition and (3) and (4) follow.

The implications (4) $\implies$ (3) and (2) $\implies$ (3) follow from the dynamics on unstable manifolds and ergodicity of the measure. For instance, to see (4) $\implies$ (3), assume the partition of (M, μ) into full W^u-manifolds is measurable and let $\{\mu^u_x\}$ denote a family of conditional probability measures for this partition. As μ is f-invariant and as the partition into full unstable leaves is f-invariant, we have $f_*\mu^u_x = \mu^u_{f(x)}$ for almost every x.

Given $x \in M$, let $W^u(x, R)$ denote the metric ball of radius R centered at x in the internal metric of $W^u(x)$. Given $\delta > 0$ and $R > 0$, define the set $G_{\delta,R}$ of (δ, R)-good points to be
$$G_{\delta,R} := \{x \in M \mid \mu^u_x(W^u(x, R)) \geqslant 1 - \delta\}.$$
Fix $R > 0$ such that $\mu(G_{\delta,R}) > 0$. Take a subset $G' \subset G_{\delta,R}$ with $\mu(G') > 0$ such that the function
$$x' \mapsto \mathrm{diam}^u_{f^{-n}(x')}(f^{-n}(W^u(x', R))$$
converges to 0 uniformly on G' as $n \to \infty$ where $\mathrm{diam}^u_x(B)$ denotes the diameter of $B \subset W^u(x)$ with respect to the internal metric on $W^u(x)$. For almost every x, we have $f^n(x) \in G'$ for infinitely many $n \in \mathbb{N}$. For such x and any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ with $f^n(x) \in G'$ we have
$$f^{-n}(W^u(f^n(x), R)) \subset W^u(x, \epsilon)$$
whence
$$\mu^u_x(W^u(x, \epsilon)) \geqslant \mu^u_{f(x)}(W^u(f(x), R)) \geqslant 1 - \delta.$$
Taking $\epsilon \to 0$ we have $\mu^u_x(\{x\}) \geqslant 1 - \delta$ and, as δ was arbitrary, (3) follows.

Finally, the implication (3) $\implies$ (1) follows from Corollary 5.3 of [72] (see (7.3) below) and the computation of unstable entropy in Definition 7.3. □

At the other extreme, we have the following definition.

Definition 7.5. We say μ is an **SRB** measure (or satisfies the **SRB property**) if, for any measurable partition ξ of (M, μ) subordinate to the partition into unstable manifolds, for almost every x the conditional measure μ_x^ξ is absolutely continuous with respect to Riemannian volume on $W^u(x)$.

We have the following summary of a number of important results.

Theorem 7.6. *Let $f\colon M \to M$ be a $C^{1+\beta}$ diffeomorphism and let μ be an ergodic, f-invariant measure. Then*

(1) $h_\mu(f) \leqslant \sum_{\lambda^i>0} m^i\lambda^i$;
(2) if μ is absolutely continuous with respect to volume then

$$h_\mu(f) = \sum_{\lambda^i>0} m^i\lambda^i;$$

(3) if μ is SRB then $h_\mu(f) = \sum_{\lambda^i>0} m^i\lambda^i$.

Theorem 7.6(1), known as the **Margulis–Ruelle inequality**, is proven in [105]. Theorem 7.6(2), known as the **Pesin entropy formula**, is shown in [92]. Theorem 7.6(3) was established by Ledrappier and Strelcyn in [70]. In the next section, we will complete Theorem 7.6 with Ledrappier's Theorem, Theorem 8.3, which provides a converse to Theorem 7.6(3).

For general measures invariant under a C^2-diffeomorphism (for the case of $C^{1+\beta}$-diffeomorphisms, see [14]), Ledrappier and Young explain explicitly the defect from equality in Theorem 7.6(1). This captures the intermediate geometry of measures with positive entropy (and hence non-atomic unstable conditional measures) but entropy strictly smaller than the sum of positive exponents.

Let δ^i denote the (almost-surely constant value of the) pointwise dimension of μ along the ith unstable manifolds. With $\delta^0 = 0$, let

$$\gamma^i = \delta^i - \delta^{i-1}.$$

The coefficients γ^i reflect the transverse geometry (in particular the transverse dimension) of the measure μ inside of the ith unstable manifold transverse to the collection of $(i-1)$th unstable manifolds. In particular, we have $\gamma^i \leqslant m^i$ (see [73, Proposition 7.3.2].)

Theorem 7.7 ([73]). *Let $f\colon M \to M$ be a $C^{1+\beta}$ diffeomorphism and let μ be an ergodic, f-invariant measure. Then*

$$h_\mu(f) = \sum_{\lambda^i>0} \gamma^i\lambda^i.$$

(Note that the proof in [73] required f to be C^2; following [14] and [4, Appendix], the theorem holds when $f \in C^{1+\beta}$.)

$*$7.5. **Coarse-Lyapunov entropy and entropy product structure.** Consider now $\alpha\colon \mathbb{Z}^d \to \mathrm{Diff}^{1+\beta}(M)$ a smooth $\mathbb{Z}^d$-action on a compact manifold M. Let μ be an ergodic, α-invariant measure. Recall that a coarse Lyapunov exponent χ is a positive-proportionality class of Lyapunov exponents of α. For almost every $x \in M$ there is a coarse Lyapunov subspace $E^\chi(x) \subset T_xM$ and a coarse Lyapunov manifold $W^\chi(x)$ tangent to $E^\chi(x)$ at x.

Let $\mathcal{F}^\chi$ denote the partition of M into full W^χ-manifolds. Given $n \in \mathbb{Z}^d$ with $\chi(n) > 0$, following the construction from [70] we can find a measurable partition ξ of (M, μ) that is subordinate to $\mathcal{F}^\chi$ and increasing for $\alpha(n)$. We then define the χ-entropy of $\alpha(n)$ to be

$$h_\mu^\chi(\alpha(n)) = h_\mu(\alpha(n) \mid \chi) := h_\mu(\alpha(n) \mid \mathcal{F}^\chi) = h_\mu(\alpha(n), \xi).$$

The main result of [21] is the following "product structure of entropy" for $\mathbb{Z}^d$-actions.

Theorem 7.8 ([21, Corollary 13.2]). *Let $\alpha\colon \mathbb{Z}^d \to \mathrm{Diff}^{1+\beta}(M)$ be a smooth $\mathbb{Z}^d$-action on a compact manifold M and let μ be an ergodic, α-invariant measure. Then for any $n \in \mathbb{Z}^d$*

$$h_\mu(\alpha(n)) = \sum_{\chi(n)>0} h_\mu(\alpha(n) \mid \chi).$$

Fix $n \in \mathbb{Z}^d$ and let $f = \alpha(n)$. The formulas in Theorem 7.7 and Theorem 7.8 then look quite similar. However, the contribution of each Lyapunov exponent λ^i to the total entropy in Theorem 7.7 is a "transverse entropy" (the coefficient γ^i is a measure of "transverse dimension"). In Theorem 7.8, the entropy of each coarse Lyapunov exponent χ is a "tangential entropy" $h_\mu(\alpha(n) \mid \chi)$ obtained by conditioning along W^χ-manifolds. Thus, Theorem 7.7 does not immediately imply Theorem 7.8. To show Theorem 7.8, one first shows that the total "transverse entropy" Theorem 7.7 contributed by all $\lambda^i \in \chi$ is equal to the total conditional entropy $h_\mu(\alpha(n) \mid \chi)$. This is done in [21]. The idea is to first establish and analogue of Theorem 7.7 for the conditional entropy $h_\mu(f \mid \chi)$; this is done in [15] where a formula of the form

$$h_\mu(f \mid \chi) = h_\mu(\alpha(n) \mid \chi) = \sum_{\lambda^i \in \chi} \gamma_n^{\chi,i} \lambda^i(n)$$

is shown. Then, (following [53]) one uses that $n \mapsto h_\mu(\alpha(n) \mid \chi)$ is linear on any half-cone where no coarse Lyapunov exponent χ' changes sign to show that the transverse dimensions $\gamma_n^{\chi,i}$ of each $\lambda^i \in \chi$ are independent of n and coincide with the transverse dimensions γ^i appearing in Theorem 7.7 for $f = \alpha(n)$.

8. Entropy, invariance, and the SRB property

In dissipative dynamical systems, ergodic SRB measures μ without zero Lyapunov exponents provide examples of **physical measures**: there is a set B of positive Lebesgue measure such that for any continuous function ϕ, the forwards time average of ϕ along the orbit of points in B converges to $\int \phi \, d\mu$. In applications and specific examples, a recurring problem is to establish the existence of physical and SRB measures. We pose a related question that arises naturally in the settings considered in this text:

Question 8.1. Given a diffeomorphism $f\colon M \to M$ and an f-invariant measure μ, how do you verify that μ is an SRB measure?

Seemingly unrelated, consider a group G acting smoothly on a manifold M. We pose the following:

Question 8.2. Given a Borel probability measure μ on M and a subgroup $H \subset G$, is it is possible to verify that μ is H-invariant?

One method to answer both of these questions is given in Theorem 8.3 and Theorem 8.5 below.

8.1. Ledrappier's theorem.

We outline one approach that solves both Question 8.1 and 8.2 in a number of settings. We discuss other approaches towards verifying the existence of SRB measures below.

We recall Section 7.3 where the notion of unstable entropy was introduced. The main result (Corollary 5.3) of [72] shows for a C^2 (see [14] for the $C^{1+\beta}$ case) diffeomorphism $f\colon M \to M$ preserving an ergodic probability measure μ that the metric entropy of f and the unstable metric entropy of f coincide:

$$h_\mu(f) = h_\mu^u(f).$$

Using this fact, Ledrappier gave a geometric characterization of all measures satisfying equality $h_\mu(f) = \sum_{\lambda^i>0} m^i\lambda^i$ in the Margulis-Ruelle inequality, giving a converse of Theorem 7.6(3).

Theorem 8.3 (Ledrappier's Theorem [68])**.** *Let f be a $C^{1+\beta}$ diffeomorphism and let μ be an ergodic, f-invariant, Borel probability measure. Then μ is SRB if and only if*

$$h^u_\mu(f) = \sum_{\lambda^i>0} m^i\lambda^i. \tag{8.1}$$

In the proof of Theorem 8.3, Ledrappier actually proves something much stronger than the SRB property: if $h^u_\mu(f) = \sum_{\lambda^i>0} m^i\lambda^i$ then the leaf-wise measures μ^u_x of μ along unstable manifolds are *equivalent* to the Riemannian volume with a Hölder continuous density. That is, if m^u_x the Riemannian volume along $W^u(x)$ then for a.e. x there is a Hölder continuous, nowhere vanishing function $\rho\colon W^u(x) \to (0,\infty)$ with

$$\mu^u_x = \rho\, m^u_x. \tag{8.2}$$

In particular, the leaf-wise measure μ^u_x has full support in $W^u(x)$. Moreover, Ledrappier explicitly computes the density function ρ; see [72, Corollary 6.1.4].

We make use of the explicit formula for the density ρ in the following setup. Consider a Lie group G and a smooth, locally free, action of G on a manifold M. We denote the action by $g\cdot x$ for $g \in G$ and $x \in M$. Consider a Lie subgroup $H \subset G$ and $s \in G$ that normalizes H. Let $f\colon M \to M$ be the diffeomorphism given by s; that is $f(x) = s\cdot x$. Let μ be an f-invariant Borel probability measure and suppose that the orbit $H\cdot x$ is contained in the unstable manifold $W^u(x)$ for μ-almost every x.

Since s normalizes H, the partition of M into H-orbits is preserved by f; in particular, the partition into H-orbits is a subfoliation of the partition into unstable manifolds. Given a Borel probability measure μ on M and a measurable partition ξ subordinate to the partition into H-orbits we can define conditional measures μ^ξ_x of μ. Given $x \in M$ (using that the action is locally free) we can push forward the left-Haar measure on H onto the orbit $H\cdot x$ via the parametrization $H\cdot x = \{h\cdot x : h \in H\}$.

Lemma 8.4. *μ is H-invariant if and only if for any measurable partition ξ subordinate to the partition into H-orbits and μ-a.e. x the conditional measure μ^ξ_x coincides—up to normalization—with the restriction of the left-Haar measure on $H\cdot x$ to $\xi(x)$.*

Similar to the definition of metric entropy of f conditioned on unstable manifolds, we can define the metric **entropy** of f **conditioned on H-orbits**, written $h_\mu(f \mid H)$, by

$$h_\mu(f \mid H) := h_\mu(f, \xi)$$

where ξ is any increasing, measurable partition ξ subordinate to H-orbits. Let λ^i, $E^i(x)$, and m^i be as in 6.1 for the dynamics of f and the measure μ. We define the **multiplicity of λ^i relative to H** to be (the almost surely constant value of)

$$m^{i,H} = \dim(E^i(x) \cap T_x(H\cdot x)).$$

Generalizing Theorem 7.6(1) we have (see for instance [15])

$$h_\mu(f \mid H) \leqslant \sum_{\lambda^i>0} \lambda^i m^{i,H}. \tag{8.3}$$

From the proof of Theorem 8.3, (in particular, the explicit formula for the density function ρ in (8.2)) we obtain the following.

Theorem 8.5. *With the above setup, the following are equivalent:*

(1) $h_\mu(f \mid H) = \sum_{\lambda^i > 0} \lambda^i m^{i,H}$*;*
(2) for any measurable partition ξ *subordinate to the partition into* H*-orbits and almost every* x*,* μ_x^ξ *is absolutely continuous with respect to the Riemannian volume on the* H*-orbit* $H \cdot x$*;*
(3) for any measurable partition ξ *subordinate to the partition into* H*-orbits and almost every* x*,* μ_x^ξ *is equivalent to with the Riemannian volume on the* H*-orbit* $H \cdot x$*;*
(4) for any measurable partition ξ *subordinate to the partition into* H*-orbits and almost every* x*,* μ_x^ξ *is equal, up to normalization, to the left-Haar measure on the* H*-orbit* $H \cdot x$*;*
(5) μ *is* H*-invariant.*

We give a proof of the implication (1) $\implies$ (4) in the next subsection.

Note that as Theorem 8.5 only concerns the entropy and dynamics inside H-orbits, the result holds for C^1 or even C^0 actions since the dynamics permuting H-orbits is affine and hence C^∞. See for instance [33] where related entropy results are shown for C^0 actions of Lie groups.

A possible critique of Theorem 8.3 is that in examples it seems nearly impossible to verify equality in (8.1) without first knowing that the measure is SRB. However, in a number of settings of group actions on manifolds, it turns out one can, in fact, verify equality in (8.1) (or typically, equality in Theorem 8.5(1)) and thus derive the SRB property or gain additional invariance of the measure only from entropy considerations. This is one key idea in this text, the papers [16, 20], and also appears as a main tool in [34, 82].

Remark 8.6. The statement and proof of Theorem 8.3, especially the reformulation in Theorem 8.5, is very similar to the *invariance principle* for fiberwise disintegrations of measures invariant under skew products. The earliest version of this invariance principle is due to Ledrappier [69] for projectivized linear cocycles. Avila–Viana extended this to cocycles taking values in the group of C^1 diffeomorphisms in [3]. See Proposition 11.5 for a related invariance principle in the setting of actions of lattices on manifolds.

8.2. **Proof of Theorem 8.5.** We only prove the implication (1) $\implies$ (4) in the proposition. Given (1) $\implies$ (4), the only other non-trivial implication is (2) $\implies$ (1). This implication follows, for instance, from [70] (see for instance Theorem 7.6(3)) and can be shown using calculations similar to those in the following proof. That (4) $\implies$ (5) is a standard fact that was stated in Lemma 8.4.

Our proof essentially follows [68, 72] though we make certain simplifications using that the dynamics along H-orbits is affine.

Proof that (1) $\implies$ *(4).* We introduce some notation.

We may assume μ is ergodic for f. Indeed, from (8.3) we have that

$$h_\mu(f \mid H) \leqslant \sum_{\lambda^i > 0} \lambda^i m^{i,H}$$

for any f-ergodic component μ' of μ (see Definition 6.2). As entropy is convex (see (7.1), page 32), it follows that $h_\mu(f \mid H) = \sum_{\lambda^i > 0} \lambda^i m^{i,H}$ for almost every ergodic component μ' of μ.

Given a measurable partition ξ of M, write $f^{-1}\xi$ for the partition

$$f^{-1}\xi := \{f^{-1}(C) \mid C \in \xi\}.$$

Then the atom of the partition $f^{-1}\xi$ containing x is

$$f^{-1}\xi(x) = f^{-1}(\xi(f(x))).$$

Recall that the H-orbits are assumed to be contained in the unstable manifolds for f. It follows μ-almost every x that the H-orbit of x is free. Given $x \in M$, let m^H_x denote the left-Haar measure on H-orbit through x: if m_H is the left-Haar measure on H and if $C \subset H \cdot x$ is a measurable subset then

$$m^H_x(C) = m_H\{h \in H \mid h \cdot x \in C\}.$$

Since m_H is left-invariant, if $x' \in H \cdot x$ then $m^H_x = m^H_{x'}$.

For $x \in M$ le

$$J^H(x) = |\det(D_x f\restriction_{H\cdot x})|$$

be the Jacobian determinant of the restriction of $f\colon H\cdot x \to H\cdot f(x)$ to the H-orbit through x; that is, J^u is the function such that for any precompact measurable subset $C \subset H \cdot x$ we have

$$m^H_{f(x)}(f(C)) = \int_C J^u(x)\, dm^H_x.$$

As the dynamics of f is affine along W^i-leaves, we have that $J^u(x)$ is constant in x. Explicitly, we have

$$J^u(x) = e^{\sum_{\lambda^i>0} \lambda^i m^{i,H}}$$

and

$$m^H_{f(x)}(f(C)) = e^{\sum_{\lambda^i>0} \lambda^i m^{i,H}} m^H_x(C).$$

For the remainder of the proof, fix ξ to be a measurable partition of (M, μ) such that

(1) ξ is subordinate to the partition into H-orbits (see Definition 7.2 above), and
(2) ξ is **increasing** under f: for a.e. x we have $f^{-1}\xi(x) \subset \xi(x)$.

A partition with the above properties can be constructed by adapting the construction in [70] where such partitions are built along unstable manifolds for $C^{1+\beta}$ diffeomorphisms preserving a probability measure.

Let $\{\mu^\xi_x\}$ denote a family of conditional measures for this partition. Also let

$$m^\xi_x = \frac{1}{m^H_x(\xi(x))} m^H_x\restriction_{\xi(x)}$$

denote the normalized restriction of the Haar measure m^H_x to the atom $\xi(x) \subset H \cdot x$ of the partition ξ containing x. Note that we have $m^H_x(\xi(x)) > 0$ for μ-a.e. x since each atom $\xi(x)$ contains a neighborhood of x in the H-orbit of x; in particular, the measure m^ξ_x is well-defined for μ-a.e. x.

We have

$$\log\left(\int \frac{m^\xi_x(f^{-1}\xi(x))}{\mu^\xi_x(f^{-1}\xi(x))}\, d\mu(x)\right) \leqslant 0. \tag{8.4}$$

Indeed,

$$\begin{aligned}
&\log\left(\int \frac{m^\xi_x(f^{-1}\xi(x))}{\mu^\xi_x(f^{-1}\xi(x))}\, d\mu(x)\right)\\
&= \log\left(\int \int_{\xi(x)} \frac{m^\xi_x(f^{-1}\xi(y))}{\mu^\xi_x(f^{-1}\xi(y))}\, d\mu^\xi_x(y) d\mu(x)\right)\\
&\leqslant \log 1 = 0
\end{aligned}$$

where the inequality follows as

$$\int_{\xi(x)} \frac{m_x^\xi(f^{-1}\xi(y))}{\mu_x^\xi(f^{-1}\xi(y))}\, d\mu_x^\xi(y) = \sum_{\substack{C\in f^{-1}\xi\\ \mu_x^\xi(C)>0}} m_x^\xi(C) \leqslant m_x^\xi(\xi(x)) = 1.$$

We claim that

$$\int \log\left(\frac{m_x^\xi(f^{-1}\xi(x))}{\mu_x^\xi(f^{-1}\xi(x))}\right)\, d\mu(x) = 0. \tag{8.5}$$

Indeed, write

$$\begin{aligned}\int \log&\left(\frac{m_x^\xi(f^{-1}\xi(x))}{\mu_x^\xi(f^{-1}\xi(x))}\right)\, d\mu(x)\\ &= \int \log\left(m_x^\xi(f^{-1}\xi(x))\right)\, d\mu(x) - \int \log\left(\mu_x^\xi(f^{-1}\xi(x))\right)\, d\mu(x).\end{aligned}$$

From the properties of ξ (see Section 7.3), we have

$$-\int \log(\mu_x^\xi(f^{-1}\xi(x)))\, d\mu(x) = h_\mu(f^{-1}\xi \mid \xi) = h_\mu(f \mid H).$$

On the other hand, we claim that

$$\int \log\left(m_x^\xi(f^{-1}\xi(x))\right)\, d\mu(x) = -\int \log J^u(x)\, d\mu(x) = -\sum_{\lambda^i>0} \lambda^i m^{i,H} \tag{8.6}$$

To establish (8.6), let

$$q(x) := m_x^i(\xi(x)).$$

As $f^{-1}\xi(x) \subset \xi(x) \subset f\xi(x)$ we have

$$\frac{q(f(x))}{q(x)} = \frac{m^i_{f(x)}(\xi(f(x)))}{m^i_x(\xi(x))} \leqslant \frac{m^i_{f(x)}(f(\xi(x)))}{m^i_x(\xi(x))} = \frac{\int_{\xi(x)} J^u(x)\, dm^i_x}{m^i_x(\xi(x))} = \sum_{\lambda^i>0} \lambda^i m^{i,H}$$

and

$$\frac{q(f(x))}{q(x)} = \frac{m^i_{f(x)}(\xi(f(x)))}{m^i_x(\xi(x))} \geqslant \frac{m^i_{f(x)}(\xi(f(x)))}{m^i_x(f^{-1}(\xi(f(x))))} = \frac{1}{\sum_{\lambda^i>0} \lambda^i m^{i,H}}.$$

It follows that the function

$$\log \frac{q\circ f}{q}$$

is $L^\infty(\mu)$ (in particular $L^1(\mu)$); from [70, Proposition 2.2] we have that

$$\int \log \frac{q\circ f}{q}\, d\mu = 0.$$

We then have that

$$\begin{aligned}\int \log\left(m_x^\xi(f^{-1}\xi(x))\right)\,d\mu(x) &= \int \log\left(\frac{m_x^i(f^{-1}\xi(x))}{m_x^i(\xi(x))}\right)\,d\mu(x)\\ &= \int \log\left(\frac{e^{-\sum_{\lambda^i>0}\lambda^i m^{i,H}}\, m^i_{f(x)}(\xi(f(x)))}{m_x^i(\xi(x))}\right)\,d\mu(x)\\ &= \int \left(-\sum_{\lambda^i>0}\lambda^i m^{i,H}\right) + \log\frac{q\circ f}{q}\,d\mu\\ &= -\sum_{\lambda^i>0}\lambda^i m^{i,H}\end{aligned}$$

and (8.6) follows.

As we assumed $h_\mu(f \mid H) = \sum_{\lambda^i>0}\lambda^i m^{i,H}$, equation (8.5) follows. From the strict concavity of $\log$ we have

$$\int \log\left(\frac{m_x^\xi(f^{-1}\xi(x))}{\mu_x^\xi(f^{-1}\xi(x))}\right)\,d\mu(x) \leqslant \log\left(\int \frac{m_x^\xi(f^{-1}\xi(x))}{\mu_x^\xi(f^{-1}\xi(x))}\,d\mu(x)\right)$$

with equality if and only if the function

$$x \mapsto \frac{m_x^\xi(f^{-1}\xi(x))}{\mu_x^\xi(f^{-1}\xi(x))}$$

is constant off a μ-null set. From (8.4) and (8.5), it thus follows that

$$\mu_x^\xi(f^{-1}\xi(x)) = m_x^\xi(f^{-1}\xi(x))$$

for μ-almost every x. In particular, if $C \subset \xi(x)$ is a union of elements of $f^{-1}\xi$, then $\mu_x^\xi(C) = m_x^\xi(C)$.

We may repeat the above calculations with f replaced by f^n for $n \geqslant 1$ and obtain that

$$\mu_x^\xi(f^{-n}\xi(x)) = m_x^\xi(f^{-n}\xi(x))$$

for μ-a.e. x. As the partitions $\{f^{-n}\xi(x') \mid x' \in \xi(x)\}$ generate the point partition on each $\xi(x)$, it follows for μ-a.e. x that

$$\mu_x^\xi = m_x^\xi$$

and (4) follows. □

Remark 8.7. When $f\colon M \to M$ is an Anosov diffeomorphism or, more generally, a non-uniformly hyperbolic $C^{1+\beta}$ diffeomorphism we still have equivalence of (1), (2), and (3) in Theorem 8.5 when the right-hand side of (1) is replaced by the sum of all positive Lyapunov exponents counted with multiplicity and the measures are conditional measures along unstable manifolds. See Theorem 8.3. The proof is nearly identical to the above except for the analogue of computation (8.6). Multiplying the measures m_x^ξ by an appropriate dynamically defined density, a computation analogous to (8.6) still holds. See [72, Lemma 6.1.2].

The extra conclusion (4) in Theorem 8.5 follows in our setting from the fact that the dynamics of f acts homogeneously along H-orbits are orbits. The density function guaranteed by (2) is then constant and equality up to normalization in (4) follows.

Part 3. Proofs of Theorem 5.2 and Theorem 3.5

We return to the proof of Theorem 3.4. Specifically, it remains to prove Theorem 5.2. We introduce some notation and tools in this and the next section. In Section 11 we derive an invariance principle and establish Theorem 3.5. We then prove Theorem 5.2 in Section 12 and Section 13.

9. Structure theory of $\mathrm{SL}(n,\mathbb{R})$ and Cartan flows on $\mathrm{SL}(n,\mathbb{R})/\Gamma$

Let $G = \mathrm{SL}(n,\mathbb{R})$ and let $\Gamma \subset G$ be a lattice. Recall we write $G = KAN$ for the Iwasawa decomposition where

$$K = \mathrm{SO}(n,\mathbb{R}), \qquad A = \{\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n}) : t_1 + \cdots + t_n = 0\},$$

and N is the group of upper triangular matrices with 1s on the diagonal.

We will be interested in certain subgroups of G and how they capture dynamical information of the action of the Cartan subgroup A on the homogeneous space G/Γ.

9.1. Roots and root subgroups.

We consider the following linear functionals

$$\beta^{i,j} \colon A \to \mathbb{R}$$

given as follows: for $i \neq j$,

$$\beta^{i,j}\left(\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n})\right) = t_i - t_j.$$

The linear functionals $\beta^{i,j}$ are the **roots** of G.

Associated to each root $\beta^{i,j}$ is a 1-parameter unipotent subgroup $U^{i,j} \subset G$. For instance, in $G = \mathrm{SL}(3,\mathbb{R})$ we have the following 1-parameter flows

$$u^{1,2}(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u^{1,3}(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u^{2,3}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

$$u^{2,1}(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u^{3,1}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{pmatrix}, \quad u^{3,2}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

We let $U^{i,j}$ denote the associated 1-parameter unipotent subgroups of G:

$$U^{i,j} := \{u^{i,j}(t) : t \in \mathbb{R}\}. \tag{9.1}$$

The groups $U^{i,j}$ have the property that conjugation by $s \in A$ dilates their parametrization by $e^{\beta^{i,j}(s)}$:

$$su^{i,j}(t)s^{-1} = u^{i,j}(e^{\beta^{i,j}(s)}t). \tag{9.2}$$

In particular, if $g' = u^{i,j}(t) \cdot g$ is in the $U^{i,j}$-orbit of g and $s \in A$ then

$$s \cdot g' = u^{i,j}(e^{\beta^{i,j}(s)}t) \cdot s \cdot g.$$

9.2. Cartan flows.

For concreteness, consider $G = \mathrm{SL}(3,\mathbb{R})$ and let Γ be a lattice in $\mathrm{SL}(3,\mathbb{R})$ such as $\mathrm{SL}(3,\mathbb{Z})$. Let X denote the coset space $X = G/\Gamma$. This is an 8 dimensional manifold (which is noncompact when Γ is a nonuniform lattice such as $\mathrm{SL}(3,\mathbb{Z})$.) G acts on X on the left: given $g \in G$ and $x = g'\Gamma \in X$ we have

$$g \cdot x = gg'\Gamma \in X.$$

The Cartan subgroup $A \subset G$ is the subgroup of diagonal matrices with positive entries

$$A := \left\{ \begin{pmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{pmatrix} : t_1 + t_2 + t_3 = 0 \right\}.$$

The group A is isomorphic to $\mathbb{R}^2$, for instance, via the embedding

$$(s,t) \mapsto \operatorname{diag}(e^s, e^t, e^{-s-t}).$$

We consider the action $\alpha\colon A \times X \to X$ of A on X given by

$$\alpha(s)(x) = sx.$$

For $x \in X$ let $W^{i,j}(x)$ be the orbit of x under the 1-parameter group $U^{i,j}$:

$$W^{i,j}(x) = \{u^{i,j}(t)x : t \in \mathbb{R}\}\}.$$

For $s \in A$, we claim that the s-action on X dilates the natural parametrization of each $W^{i,j}(x)$ by exactly $\beta^{i,j}(s)$. Indeed, if $x \in X$ and if $x' = u^{i,j}(v) \cdot x \in W^{i,j}(x)$ then for $s \in A$ we have

$$\begin{aligned}
\alpha(s)(x') &= su^{i,j}(v)x \\
&= su^{i,j}(v)s^{-1}sx \\
&= u^{i,j}(v')\alpha(s)(x)
\end{aligned}$$

where, using (9.2), we have that have

$$v' = e^{\beta^{i,j}(s)}v.$$

In particular, we interpret the functionals $\beta^{i,j}$ as the (non-zero) Lyapunov exponents for the A-action on X (with respect to any A-invariant measure). Note that the zero functional is a Lyapunov exponent of multiplicity two corresponding to the A-orbits. The tangent spaces to each $W^{i,j}(x)$ as well as the tangent space to the orbit $A \cdot x$ gives the A-invariant splitting guaranteed by Theorem 6.5. Note that no two roots $\beta^{i,j}$ are positively proportional and hence are their own coarse Lyapunov exponents for the action (see Section 6.4.2).

10. Suspension space and fiberwise exponents

We now begin the proofs of Theorem 3.5 and Theorem 3.4 with a technical but crucial construction. Here, we induce from an action α of a lattice $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ on a manifold M to an action of $G = \mathrm{SL}(n,\mathbb{R})$ on an auxiliary manifold denoted by M^α. The properties of the G-action on M^α mimic the properties of the Γ-action on M. However, for a number of reasons it is much more convenient to study the G-action on M^α.

10.1. **Suspension space and induced G-action.** Fix $G = \mathrm{SL}(n,\mathbb{R})$ and let $\Gamma \subset G$ be a lattice. Let M be a compact manifold and let $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ be an action.

On the product $G \times M$ consider the right Γ-action

$$(g,x) \cdot \gamma = (g\gamma, \alpha(\gamma^{-1})(x))$$

and the left G-action

$$a \cdot (g,x) = (ag, x).$$

Define the quotient manifold $M^\alpha := (G \times M)/\Gamma$. As the G-action on $G \times M$ commutes with the Γ-action, we have an induced left G-action on M^α. For $g \in G$ and $x \in M^\alpha$ we denote this action by $g \cdot x$ and denote the derivative of the diffeomorphism $x \mapsto g \cdot x$ by a $x \in M^\alpha$ by $D_x g\colon T_x M^\alpha \to T_{g \cdot x} M^\alpha$.

We write

$$\pi\colon M^\alpha \to \mathrm{SL}(n,\mathbb{R})/\Gamma$$

for the natural projection map. Note that M^α has the structure of a fiber-bundle over $\mathrm{SL}(n,\mathbb{R})/\Gamma$ induced by the map π with fibers diffeomorphic to M. The G-action permutes the M-fibers of M^α. We let $F = \ker(D\pi)$ be the **fiberwise tangent bundle:** for $x \in M^\alpha$, $F(x) \subset T_x M^\alpha$ is the $\dim(M)$-dimensional subspace tangent to the fiber through x.

Equip M^α with a continuous Riemannian metric. For convenience, we moreover assume the restriction of the metric to G-orbits coincides under push-forward by the projection $\pi\colon M^\alpha \to \mathrm{SL}(n,\mathbb{R})/\Gamma$ with the metric on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ induced by a right-invariant (and left K-invariant) metric on G. (We note that if Γ is cocompact, M^α is compact and all metrics are equivalent. In the case that Γ is not cocompact, some additional care is needed to ensure the metric is well behaved in the fibers. We will not discuss the technicalities of this case here.)

To construct such a metric, first fix a C^∞ Riemannian metric $\langle\cdot,\cdot\rangle$ on TM. Let $\{\hat\psi_i, i = 1,\dots,m\}$ be a finite C^∞ partition of unity on the symmetric space $K\backslash G/\Gamma$ subordinate to finitely many coordinate charts. Lift each $\hat\psi_i$ to a K-invariant function defined on G/Γ and then select a lift $\psi_i\colon G \to [0,1]$ of each $\hat\psi_i$ whose support intersects some fixed compact fundamental domain containing the identity. Write $\psi_{i,\gamma}\colon G \to [0,1]$ for the function

$$\psi_{i,\gamma}(g) = \psi_i(g\gamma^{-1}).$$

The supports satisfy $\mathrm{supp}(\psi_{i,\gamma}) \cap \mathrm{supp}(\psi_{i,\gamma'}) = \varnothing$ whenever $\gamma \neq \gamma'$ and the collection $\{\psi_{i,\gamma} \mid i \in \{1,\dots,m\}, \gamma \in \Gamma\}$ is a partition of 1 on G. Given $v, w \in \{g\} \times T_x M$ set

$$\langle v, w\rangle_{g,x} := \sum_{i=1}^{m} \sum_{\gamma\in\Gamma} \phi_{i,\gamma}(g) \langle D_x\alpha(\gamma)(v), D_x\alpha(\gamma)(w)\rangle_x.$$

Equip $T_{(g,x)}(G \times M) = T_g G \times T_x M$ with the product of the left K-invariant, right Γ-invariant metric on G and $\langle v, w\rangle_{g,x}$. Note that this metric is β-Hölder continuous if α is an action by $C^{1+\beta}$ diffeomorphisms. We then verify that Γ acts by isometries and thus the metric descends to a metric on M^α. Indeed, writing $\|\cdot\|_{g,x}$ for the norm associated to $\langle\cdot,\cdot\rangle_{g,x}$, for $v \in \{g\hat\gamma\} \times T_x M$ we have

$$\begin{aligned}
\|v\|^2_{g\hat\gamma,x} &= \sum_{i=1}^{m} \sum_{\gamma\in\Gamma} \phi_{i,\gamma}(g\hat\gamma) \|D_x\alpha(\gamma)(v)\|_0^2 \\
&= \sum_{i=1}^{m} \sum_{\gamma\in\Gamma} \phi_{i,\gamma\hat\gamma^{-1}}(g) \|D_x\alpha(\gamma)(v)\|_0^2 \\
&= \sum_{i=1}^{m} \sum_{\gamma\in\Gamma} \phi_{i,\gamma\hat\gamma^{-1}}(g) \|D_x\alpha(\gamma\hat\gamma^{-1}\hat\gamma)(v)\|_0^2 \\
&= \sum_{i=1}^{m} \sum_{\gamma\in\Gamma} \phi_{i,\gamma\hat\gamma^{-1}}(g) \|D_{\alpha(\hat\gamma)(x)}\alpha(\gamma\hat\gamma^{-1}, \alpha(\hat\gamma)(x)) D_x\alpha(\hat\gamma, x)(v)\|_0^2 \\
&= \|D_x\alpha(\hat\gamma)v\|^2_{g,\alpha(\hat\gamma)(x)}.
\end{aligned}$$

10.2. **Fiberwise Lyapunov exponents.** Recall that $A \subset G$ is the subgroup

$$A = \{\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n})\} \simeq \mathbb{R}^{n-1}.$$

The G-action on M^α restricts to an A-action on M^α. Let μ be any ergodic, A-invariant Borel probability measure on M^α. The G-action (and hence the A-action) permutes the fibers of M^α and hence the derivatives of the G- and A-actions preserve the fiberwise tangent subbundle $F \subset TM^\alpha$.

We equip $A \simeq \mathbb{R}^{n-1}$ with a norm $|\cdot|$. We may restrict Theorem 6.5 to the A-invariant subbundle $F \subset TM^\alpha$ and obtain Lyapunov exponent functionals for the fiberwise derivative cocycle. We thus obtain

(1) an A-invariant set $\Lambda \subset M^\alpha$ with $\mu(\Lambda) = 1$;
(2) linear functionals $\lambda^F_{1,\mu}, \lambda^F_{2,\mu}, \dots, \lambda^F_{p,\mu} \colon A \to \mathbb{R}$; and
(3) a μ-measurable, A-invariant splitting $F(x) = \bigoplus_{i=1}^p E^F_i(x)$ defined for $x \in \Lambda$

such that for every $x \in \Lambda$ and $v \in E^F_i(x) \smallsetminus \{0\}$

$$\lim_{|a|\to\infty} \frac{\log \|D_x a(v)\| - \lambda^F_{i,\mu}(a)}{|a|} = 0.$$

In particular, for any $a \in A$ and $v \in F(x) \smallsetminus \{0\}$ we have

$$\lim_{k\to\infty} \frac{1}{k} \log \|D_x a^k(v)\| = \lambda^F_{i,\mu}(a).$$

A **coarse fiberwise Lyapunov exponent** χ^F_μ is a positive proportionality class of fiberwise Lyapunov exponents.

11. Invariance principle and Proof of Theorem 3.5

11.1. **Proof of Theorem 3.5.** Given the constructions in Section 10 and Ledrappier's theorem as formulated in Theorem 8.5, we are now in a position to prove Theorem 3.5. In fact, we prove the following invariance principle:

Theorem 11.1. *Let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice. Let $\alpha \colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be an action and let M^α denote the suspension space with induced G-action. Let μ be an ergodic, A-invariant Borel probability measure on M^α whose projection to $\mathrm{SL}(n,\mathbb{R})/\Gamma$ is the Haar measure.*

Then, if $\dim(M) \leqslant n-2$ the measure μ is G-invariant. Moreover, if α preserves a volume form vol *and if $\dim(M) \leqslant n-1$ then the measure μ is G-invariant.*

Note that Theorem 11.1 does not require that Γ be cocompact.[5] Theorem 3.5 follows immediately from Theorem 11.1: since A is abelian (in particular amenable) and the space of probability measures on M^α projecting to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ is nonempty, A-invariant, and weak-$*$ compact, the Krylov-Bogolyubov theorem implies there is an A-invariant Borel probability measure μ on M^α projecting to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$. Theorem 11.1 implies μ is G-invariant and Theorem 3.5 then follows from the following elementary claim.

Claim 11.2. *The Γ-action α on M preserves a Borel probability measure if and only if the induced G-action on M^α preserves a Borel probability measure (which necessarily projects to the Haar measure on G/Γ).*

Indeed, if μ is a G-invariant measure on M^α then conditioning on the fiber of M^α over $e\Gamma \in G/\Gamma$ gives an α-invariant measure on M viewed as the fiber of M^α over $e\Gamma$. On the other hand, if $\hat\mu$ is an α-invariant measure on M then, writing m_G for the Haar measure on G, we have $m_G \times \hat\mu$ is a (right) Γ-invariant and (left) G-invariant measure on $G \times M$ and hence descends to a (finite) G-invariant measure on M^α.

[5] However, in the case that Γ is nonuniform, the space M^α is not compact and some care is needed to define Lyapunov exponents; in particular, we must specify a Riemannian metric on M^α. A Riemannian metric on M^α adapted to this setting is constructed in [20].

Remark 11.3. For more general semisimple Lie groups G we have the following theorem which follows from the proof of Theorem 11.1. In this setting, we take A to be a maximal split Cartan subgroup; that is, A is a maximal, connected, abelian subgroup of $\mathbb{R}$-diagonalizable elements.

Theorem 11.1′. *Let G be a simple Lie group and let $\Gamma \subset G$ be any lattice. Let $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be an action and let M^α denote the suspension space with induced G-action. Let μ be an ergodic, A-invariant Borel probability measure on M^α whose projection to G/Γ is the Haar measure.*

Then, if $\dim(M) < \mathrm{rank}(G)$ then the measure μ is G-invariant. Moreover, if α preserves a volume form vol and if $\dim(M) \leqslant \mathrm{rank}(G)$ then the measure μ is G-invariant.

Remark 11.4. In fact, Theorem 11.1 and 11.1′ hold for actions by C^1-diffeomorphisms. This can be shown by the *invariance principle* of Avila and Viana [3] (generalizing a result of Ledrappier [69].) We present below a proof that uses (mildly) the $C^{1+\beta}$ hypotheses as this motivates the proof of Proposition 11.5 (which allows us to establish Theorem 11.1′ for manifolds of higher critical dimension) in the next section which requires the higher-regularity of the action.

We proceed with the proof of Theorem 11.1, adapted from [25], which is somewhat simpler than the arguments in [16, 20].

Proof of Theorem 11.1. Let μ be an ergodic, A-invariant Borel probability measure on M^α whose projection to $\mathrm{SL}(n,\mathbb{R})/\Gamma$ is the Haar measure.

Recall that $A \simeq \mathbb{R}^{n-1}$. In the non-volume-preserving case, since $\dim(M) \leqslant n-2$ there are at most $n-2$ fiberwise Lyapunov exponents. In particular, the intersection of the kernels of the fiberwise Lyapunov exponents is a subspace of A whose dimension is at least 1. In the volume-preserving case, there are at most $(n-1)$ fiberwise Lyapunov exponents; however, these satisfy the linear relation they necessarily sum to zero since the cocycle is cohomologous to an $\mathrm{SL}^{\pm}(n-1,\mathbb{R})$-valued cocycle (recall Claim 4.1) whence for every $g \in G$,

$$0 = \int \log|\det(Dg\restriction_F)|\,d\mu = \sum \lambda^F_{i,\mu}.$$

Thus, if $\dim(M) \leqslant n-1$ and if α is a volume-preserving action, then the intersection of the kernels of all fiberwise Lyapunov exponents again has dimension at least 1. In particular, in either case we may find a nonzero $s_0 \in A$ such that

$$\lambda^F_{i,\mu}(s_0) = 0 \text{ for every fiberwise Lyapunov exponent } \lambda^F_{i,\mu}. \tag{11.1}$$

Recall that entropy can only decrease under a factor. Thus

$$h_\mu(s_0) \geqslant h_{\mathrm{Haar}}(s_0)$$

where $h_{\mathrm{Haar}}(s_0)$ denotes the entropy of translating by s_0 on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ with respect to the Haar measure.

Recall we interpret the roots β of $\mathrm{SL}(n,\mathbb{R})$ as the (non-zero) Lyapunov exponents for the A-action on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ with respect to any A-invariant measures and hence also as Lyapunov exponents for the A-action on the fiber bundle M^α transverse to the fibers and tangential to the local G-orbits. See discussion in Section 9.2. Let $N_+ \subset G$ be the subgroup generated by all root subgroups U^β with $\beta(s_0) > 0$. Similarly, let $N_- \subset G$ be the subgroup generate by all root subgroups U^β with $\beta(s_0) < 0$. The orbits of N_+ and N_- in $\mathrm{SL}(n,\mathbb{R})/\Gamma$ correspond, respectively, to the unstable and stable manifolds for the action of translation by s_0 on G/Γ. Since s_0 is in the kernel of all Fiberwise Lyapunov exponents, each tangent space $F(x)$ to the fibers of M^α is contained in the neutral Lyapunov subspace $E^c_{s_0}(x)$ for the action of s_0 on (M^α, μ) for almost every x. Thus, the orbits or N_+ and N_- in M^α also correspond, respectively, to the unstable and stable manifolds for the action of s_0 on M^α.

We have that

$$h_{\mathrm{Haar}}(s_0) = \sum_{\beta(s_0)>0} \beta(s_0) = h_{\mathrm{Haar}}(s_0^{-1}) = \sum_{\beta(s_0)<0} (-\beta(s_0)).$$

In particular, from the choice of s_0, the Margulis-Ruelle inequality (Theorem 7.6(1)), and the Ledrappier–Young Theorem (7.3) (page 33)

$$\sum_{\beta(s_0)>0} \beta(s_0) = h_{\mathrm{Haar}}(s_0) \leqslant h_\mu(s_0) = h_\mu(s_0 \mid N_+) \leqslant \sum_{\beta(s_0)>0} \beta(s_0).$$

It follows that

$$h_\mu(s_0 \mid N_+) = \sum_{\beta(s_0)>0} \beta(s_0).$$

By Theorem 8.5, it follows that μ is N_+-invariant. Similarly, we have that μ is N_--invariant.

In particular, μ is invariant by the subgroups N_-, N_+, and A of G. To end the proof, we claim the following standard fact: the subgroups N_- and N_+ generate all of $\mathrm{SL}(n, \mathbb{R})$. It follows from the claim that the measure μ is G-invariant.

To prove the claim, it is best to work with Lie algebras. Let $\mathfrak{n}_+$, $\mathfrak{n}_-$, and $\mathfrak{a}$ be the Lie algebras of N_- and N_+, and A, respectively. Let $\mathfrak{h}$ be the Lie algebra generated by $\mathfrak{n}_+$ and $\mathfrak{n}_-$. For any $X \in \mathfrak{a}$ we have

$$[X, \mathfrak{h}] = \mathfrak{h}$$

since $\mathfrak{a}$ normalizes each root space $\mathfrak{g}^\beta$. For roots $\beta, \hat\beta$ with $\beta(s_0) \neq 0$ and $\hat\beta(s_0) \neq 0$ we have

$$[\mathfrak{g}^{\hat\beta}, \mathfrak{g}^\beta] \subset \mathfrak{h}$$

by definition. For roots $\beta, \hat\beta$ with $\beta(s_0) > 0$ and $\hat\beta(s_0) = 0$ we have

$$[\mathfrak{g}^{\hat\beta}, \mathfrak{g}^\beta] = \mathfrak{g}^{\beta+\hat\beta} \subset \mathfrak{h}$$

since either $\mathfrak{g}^{\beta+\hat\beta} = 0$ (if $\beta + \hat\beta$ is not a root) or $(\hat\beta + \beta)(s_0) = \beta(s_0) > 0$ (if $\beta + \hat\beta$ is a root). Similarly, for roots $\beta, \hat\beta$ with $\beta(s_0) < 0$ and $\hat\beta(s_0) = 0$ we have

$$[\mathfrak{g}^{\hat\beta}, \mathfrak{g}^\beta] \subset \mathfrak{h}.$$

It follows that $\mathfrak{h}$ is an ideal of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ of $\mathrm{SL}(n, \mathbb{R})$. But $\mathfrak{sl}(n, \mathbb{R})$ is simple (i.e. has no nontrivial ideals). Since $\mathfrak{h} \neq \{0\}$, it follows that $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{R})$ and the claim follows. □

$*$11.2. **Advanced invariance principle: nonresonance implies invariance.** Theorem 11.1 gives the optimal dimension count in Theorem 3.5 for actions by lattices Γ in $\mathrm{SL}(n, \mathbb{R})$. However, for lattices in other simple Lie groups, the critical dimension in Theorem 11.1′ falls below the critical dimension expected for the analogous versions of Theorem 3.5 and Theorem 3.4. For instance, the group $G = \mathrm{Sp}(2n, \mathbb{R})$, the group of $(2n)\times(2n)$ symplectic matrices over $\mathbb{R}$, has rank n. Theorem 11.1′ implies that for any lattice $\Gamma \subset G$ and any compact manifold M with $\dim(M) \leqslant n - 1$, any action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ preserves a Borel probability measure. However, the main result of [20] shows for a lattice Γ in $\mathrm{Sp}(2n, \mathbb{R})$ that any action $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ preserves a Borel probability measure when $\dim(M) \leqslant 2n-2$. To obtain the optimal critical dimensions, it is necessary to use a more advanced invariance principle developed in [20] and based on key ideas from [21].

Recall that we interpret roots $\beta^{i,j}\colon A \to \mathbb{R}$ as the nonzero Lyapunov exponents for the action of $A \simeq \mathbb{R}^{n-1}$ on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ (for any A-invariant measure on G/Γ.) Each root $\beta^{i,j}$ has a corresponding root subgroup $U^{i,j} \subset \mathrm{SL}(n,\mathbb{R})$. Given an ergodic, A-invariant measure μ on M^α we also have fiberwise Lyapunov exponents $\lambda^F_{1,\mu}, \lambda^F_{2,\mu}, \dots, \lambda^F_{p,\mu}\colon A \to \mathbb{R}$ for the restriction of the derivative of the A-action on (M^α, μ) to the fiberwise tangent bundle $F \subset TM^\alpha$ in M^α. Then, the roots $\beta^{i,j}$ and fiberwise Lyapunov exponents $\lambda^F_{i,\mu}$ are linear functions on the common vector space $A \simeq \mathbb{R}^{n-1}$. We say that a root $\beta^{i,j}$ is **resonant** with a fiberwise Lyapunov exponent $\lambda^F_{i,\mu}$ of μ if they are positively proportional; that is $\beta^{i,j}$ is resonant with $\lambda^F_{i,\mu}$ if there is a $c>0$ with

$$\beta^{i,j} = c\lambda^F_{i,\mu}.$$

Otherwise we say that $\beta^{i,j}$ is **not resonant with** $\lambda^F_{i,\mu}$. We say that a root $\beta^{i,j}$ of G is **nonresonant** if it is not resonant with any fiberwise Lyapunov exponent $\lambda^F_{i,\mu}$ for the ergodic, A-invariant measure μ.

The following is the key proposition from [20].

Proposition 11.5 ([20, Proposition 5.1])**.** *Suppose μ is an ergodic, A-invariant measure on M^α projecting to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ under the projection $\pi\colon M^\alpha \to \mathrm{SL}(n,\mathbb{R})/\Gamma$.*

Then, for every nonresonant root $\beta^{i,j}$, the measure μ is $U^{i,j}$-invariant.

Remark 11.6. Since each root $\beta^{i,j}$ is a nonzero functional on A, if a fiberwise exponent $\lambda^F_{i,\mu}$ is zero, then every root $\beta^{i,j}$ is not resonant with $\lambda^F_{i,\mu}$. Since no roots of $\mathrm{SL}(n,\mathbb{R})$ are positively proportional, if there are p fiberwise Lyapunov exponents $\{\lambda^F_{i,\mu}, 1 \leqslant i \leqslant p\}$ or, more generally, $p' \leqslant p$ coarse fiberwise Lyapunov exponents $\{\chi^F_{i,\mu}, 1 \leqslant i \leqslant p'\}$ then Proposition 11.5 implies that μ is invariant under all-but-p' root subgroups $U^{i,j}$. Moreover, if every fiberwise Lyapunov exponent $\lambda^F_{i,\mu}$ is in general position with respect to every root $\beta^{i,j}$ then from Proposition 11.5, μ is automatically G-invariant.

$\ast$11.3. **Coarse-Lyapunov Abramov–Rohlin and Proof of Proposition 11.5.** The proof of Proposition 11.5 follows from a version of the Abramov–Rohlin Theorem (see equation (7.2), page 32) for entropies subordinated to coarse-Lyapunov foliations. We outline these ideas and the proof of Proposition 11.5 in this section.

Each root $\beta^{i,j}$ of $\mathrm{SL}(n,\mathbb{R})$ is a Lyapunov exponent for the A-action on (M^α,μ) (corresponding to vectors tangent to $U^{i,j}$ orbits in M^α.) Let $\chi^{i,j}$ denote the **coarse Lyapunov exponent** for the A-action on (M^α,μ) containing $\beta^{i,j}$; that is, $\chi^{i,j}$ is the equivalence class of all Lyapunov exponents for the A-action on (M^α,μ) that are positively proportional to $\beta^{i,j}$. Let $\{\lambda^F_{i,\mu}, 1 \leqslant i \leqslant p\}$ denote the collection of fiberwise Lyapunov exponents. We have that

$$\chi^{i,j} = \{\beta^{i,j}\} \text{ if } \beta^{i,j} \text{ is not resonant with any } \lambda^F_{i,\mu}.$$

Otherwise, $\chi^{i,j}$ contains $\beta^{i,j}$ and all fiberwise Lyapunov exponents $\lambda^F_{i,\mu}\colon A \to \mathbb{R}$ that are positively proportional to $\beta^{i,j}$.

For μ-a.e. $x \in M^\alpha$ there is a coarse Lyapunov manifold $W^{\chi^{i,j}}(x)$ through x (see Section 6.4.3). If $\chi^{i,j} = \{\beta^{i,j}\}$ then for $x \in M^\alpha$, $W^{\chi^{i,j}}(x)$ is simply the $U^{i,j}$-orbit of x. Otherwise, $W^{\chi^{i,j}}(x)$ is a higher-dimensional manifold which intersects the fibers of M^α

nontrivially. The partition of (M^α, μ) into $W^{\chi^{i,j}}$-manifolds forms an A-invariant partition $\mathcal{F}^{\chi^{i,j}}$ with $C^{1+\beta}$-leaves.

If $\beta^{i,j}$ is resonant with some fiberwise Lyapunov exponent, let $\chi^{i,j,F}$ denote the corresponding coarse fiberwise Lyapunov exponent; that is, $\chi^{i,j,F}$ is the equivalence class of fiberwise Lyapunov exponents that are positively proportional to $\beta^{i,j}$. If $\beta^{i,j}$ is not resonant with any fiberwise Lyapunov exponent, let $\chi^{i,j,F}$ denote the zero functional. If $\chi^{i,j,F}$ is nonzero, for μ-a.e. $x \in M^\alpha$ there is a **coarse fiberwise Lyapunov manifold** $W^{\chi^{i,j,F}}(x)$ through x. (To construct fiberwise coarse Lyapunov manifolds $W^{\chi^{i,j,F}}(x)$, recall that the fibers of M^α are permuted by the dynamics of A; all constructions in Section 6 may be carried out fiberwise in the setting of a skew-product of diffeomorphisms over a measurable base if the $C^{1+\beta}$ norms of the fibers are uniformly bounded.) If $\chi^{i,j,F}$ is zero, we simply define $W^{\chi^{i,j,F}}(x) = \{x\}$. We have that $W^{\chi^{i,j,F}}(x)$ is contained in the fiber through x and that $W^{\chi^{i,j}}(x)$ is the $U^{i,j}$-orbit of $W^{\chi^{i,j,F}}(x)$.

For each $\chi^{i,j}$ and $a \in A$ with $\beta^{i,j}(a) > 0$ we define a conditional entropy of a conditioned on $\chi^{i,j}$-manifolds, denoted by $h_\mu(a \mid \chi^{i,j})$ as in Section 7.5. Similarly, we can define a conditional entropy of a conditioned on the fiberwise coarse Lyapunov manifolds associated to $\chi^{i,j,F}$, denoted by $h_\mu(a \mid \chi^{i,j,F})$. In this setting, we have the following "coarse-Lyapunov Abramov–Rohlin formula."

Theorem 11.7. *Let μ be an ergodic, A-invariant measure on M^α that projects to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$. For any $a \in A$ with $\beta^{i,j}(a) > 0$.*

$$h_\mu(a \mid \chi^{i,j}) = h_{\mathrm{Haar}}(a \mid \beta^{i,j}) + h_\mu(a \mid \chi^{i,j,F}). \tag{11.2}$$

Above,

$$h_{\mathrm{Haar}}(a \mid \beta^{i,j})$$

denotes the conditional entropy of translation by a in $\mathrm{SL}(n,\mathbb{R})/\Gamma$ conditioned along $U^{i,j}$-orbits in $\mathrm{SL}(n,\mathbb{R})/\Gamma$.

Proof of Theorem 11.7. We first show the upper bound

$$h_\mu(a \mid \chi^{i,j}) \leqslant h_{\mathrm{Haar}}(a \mid \beta^{i,j}) + h_\mu(a \mid \chi^{i,j,F}). \tag{11.3}$$

This is a standard estimate in abstract ergodic theory whose proof we include for completeness.

Fix $a \in A$ with $\beta^{i,j}(a) > 0$. Let $\hat\eta$ be an increasing measurable partition of G/Γ subordinate to the partition into $U^{i,j}$-orbits. Let $\pi\colon M^\alpha \to G/\Gamma$ be the natural projection and let $\eta = \pi^{-1}\hat\eta$. Let $\xi > \eta$ be an increasing measurable partition of (M^α, μ) subordinate to the partition into $W^{\chi^{i,j}}$-manifolds. Let ζ be the partition of (M^α, μ) into the level sets of $\pi\colon M^\alpha \to G/\Gamma$; that is, ζ is the partition of M^α into fibers of the fibration $\pi\colon M^\alpha \to G/\Gamma$. Let $\xi^F := \xi \vee \zeta$ be the join of ξ and ζ. The partitions $\hat\eta$, ξ, and ξ^F satisfy

(1) $h_\mu(a, \eta) = h_{\mathrm{Haar}}(a, \hat\eta) = h_{\mathrm{Haar}}(a \mid \beta^{i,j})$,
(2) $h_\mu(a, \xi) = h_\mu(a \mid \chi^{i,j})$, and
(3) $h_\mu(a, \xi^F) = h_\mu(a \mid \chi^{i,j,F})$.

We have the following computation (see for example [63, Lemma 6.1]):

$$\begin{aligned}
h_\mu(a \mid \chi^{i,j}) &:= h_\mu(a,\xi) \\
&= h_\mu(a, \eta \vee \xi) \\
&\leqslant h_\mu(a,\eta) + h_\mu\left(a, \xi \vee \bigvee_{n\in\mathbb{Z}} a^n(\eta)\right) \\
&= h_{\mathrm{Haar}}(a,\hat\eta) + h_\mu(a, \xi \vee \zeta) \\
&= h_{\mathrm{Haar}}(a \mid \beta^{i,j}) + h_\mu(a \mid \chi^{i,j,F})
\end{aligned}$$

and (11.3) follows.

On the other hand, summing over all roots β with $\beta(a) > 0$ we have from the classical Abramov–Rohlin theorem (7.2), the product structure of entropy in Theorem 7.8, and an analogous version of Theorem 7.8 for the fiberwise entropy $h_\mu(a \mid \zeta)$ appearing in (7.2) that

$$\begin{aligned}
h_\mu(a) &= \sum_{\chi(a)>0} h_\mu(a \mid \chi) \\
&= \sum_{\beta^{i,j}(a)>0} h_\mu(a \mid \chi^{i,j}) + \sum_{\substack{\chi^F \text{ nonres.} \\ \chi^F(a)>0}} h_\mu(a \mid \chi^F) \\
&\leqslant \sum_{\beta^{i,j}(a)>0} \big(h_{\mathrm{Haar}}(a \mid \beta^{i,j}) + h_\mu(a \mid \chi^{i,j,F})\big) + \sum_{\substack{\chi^F \text{ nonres.} \\ \chi^F(a)>0}} h_\mu(a \mid \chi^F) \\
&= \sum_{\beta^{i,j}(a)>0} h_{\mathrm{Haar}}(a \mid \beta^{i,j}) + \sum_{\chi^F(a)>0} h_\mu(a \mid \chi^F) \\
&= h_{\mathrm{Haar}}(a) + h_\mu(a \mid \zeta) \\
&= h_\mu(a).
\end{aligned}$$

In the second and third lines, the second sum is over all fiberwise coarse Lyapunov exponents that are not resonant with any root β of G. Since entropies are non-negative quantities, it follows that

$$h_\mu(a \mid \chi^{i,j}) = h_{\mathrm{Haar}}(a \mid \beta^{i,j}) + h_\mu(a \mid \chi^{i,j,F})$$

for all $\beta^{i,j}$ with $\beta^{i,j}(a) > 0$. □

Remark 11.8. A more general version of Theorem 11.7 appears in [21, Theorem 13.6] where the factor map π is allowed to be measurable and the measure $\pi_*(\mu)$ on the factor system is an arbitrary ergodic, A-invariant measure.

The proof of Proposition 11.5 is a straightforward consequence of Theorem 11.7.

Proof of Proposition 11.5. Given a root $\beta^{i,j}$ and $a \in A$ such that $\beta^{i,j}(a) > 0$ we have defined the a conditional entropy $h_\mu(a \mid \beta^{i,j})$ for the entropy of translation by a conditioned on $U^{i,j}$-orbits in M^α. From an appropriate version of the Margulis-Ruelle inequality (see Theorem 7.6(1) and (8.3)), for $a \in A$ with $\beta^{i,j}(a) > 0$ we have that

$$h_\mu(a \mid \beta^{i,j}) \leqslant \beta^{i,j}(a). \tag{11.4}$$

On the other hand, if $\beta^{i,j}$ is nonresonant then $\chi^{i,j,F}$ is the zero functional whence the coarse Lyapunov manifold $W^{\chi^{i,j}}(x)$ associated to $\chi^{i,j}$ is simply the $U^{i,j}$-orbit of x for every $x \in M^\alpha$ and the term $h_\mu(a \mid \chi^{i,j,F})$ in (11.2) of Theorem 11.7 vanishes. Hence, by

Theorem 11.7,

$$h_\mu(a \mid \beta^{i,j}) = h_\mu(a \mid \chi^{i,j}) = h_{\mathrm{Haar}}(a \mid \beta^{i,j}) + 0 = \beta^{i,j}(a). \tag{11.5}$$

From (11.4) and (11.5), we have that the conditional entropy $h_\mu(a \mid \beta^{i,j})$ attains is maximal possible value. In particular, from the invariance principle in Theorem 8.5(5), it follows that μ is $U^{i,j}$-invariant. □

$*$11.4. **Proof of Theorem 3.5 using the advanced invariance principle.** We outline another proof of Theorem 3.5 based on Proposition 11.5.

Proof of Theorem 3.5 using Proposition 11.5. From Claim 11.2, it is sufficient to construct a G-invariant probability measure on M^α. Note that $A \simeq \mathbb{R}^{n-1}$ is abelian (and in particular amenable, see Remark 13.2) and that the space of probability measures on M^α projecting to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ is nonempty, A-invariant, and weak-$*$ compact. The Krylov-Bogolyubov theorem thus gives an A-invariant probability measure μ on M^α projecting to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$. Moreover, since the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ is A-ergodic, we may assume μ is A-ergodic.

Let $\dim(M) = d \leqslant n-2$. The fiberwise tangent bundle F of M^α is d-dimensional and therefore there are at most d Fiberwise Lyapunov exponents

$$\lambda^F_{1,\mu}, \cdots, \lambda^F_{k,\mu}, \quad k \leqslant d.$$

As no pair of roots of $\mathrm{SL}(n,\mathbb{R})$ is positively proportional, there are at most d roots that are resonant with the fiberwise Lyapunov exponent $\lambda^F_{j,\mu}$. All other roots $\beta^{i,j}$ are nonresonant. By Proposition 11.5, if $\beta^{i,j}$ is not resonant with any $\lambda^F_{j,\mu}$, then μ is $U^{i,j}$-invariant.

Let $H \subset \mathrm{SL}(n,\mathbb{R})$ be the subgroup that preserves μ. We claim $H = G$ completing the proof. As $d \leqslant n-2$, μ is invariant under A and all-but-at-most-$(n-2)$ root subgroups $U^{i,j}$. Then H has codimension at most $(n-2)$. From [16, Lemma 2.5], we have that H is parabolic; that is, H is conjugate to a group of block-upper-triangular matrices (see Remark 2.8). However, the proper closed parabolic subgroups of $\mathrm{SL}(n,\mathbb{R})$ of maximal codimension are conjugate to the codimension $(n-1)$ subgroup

$$\left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \right\}. \tag{11.6}$$

(See Section VII.7, especially Proposition 7.76 of [65] for discussion on the structure of parabolic subgroups.) As H has codimension at most $n-2$, it thus follows that $H = G$ as there are no proper parabolic subgroups of G with codimension less than $(n-1)$. □

Remark 11.9. The above proof has the advantage that it generalizes to give invariance of measures in the optimal critical dimension for actions by lattices in other Lie groups including $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(n,n)$, or $\mathrm{SO}(n,n+1)$ on manifolds of the optimal dimension. As discussed in Section 11.2 for a lattice Γ in a group such as $G = \mathrm{Sp}(2n,\mathbb{R})$, the proof in Section 11.1 yields that any $C^{1+\beta}$ action of Γ on a manifold of dimension at most $\mathrm{rank}(G) - 1$, any A-invariant measure on M^α that projects to Haar on G/Γ is G-invariant. However, the above proof establishes this result for manifolds M where the critical dimension is $r(G)$, the number in the last column of Table 1 (page 14) defined in [16, 20] (see also Footnote 2.) For $\mathbb{R}$-split groups G we have $r(G) = d_0(G)$. In particular, the above proof can be adapted to show the following:

Theorem 11.10. *Let G be a higher-rank simple Lie group G with finite center, let Γ be a lattice in G, let M be a closed manifold, and let $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be an action. Then*

(1) if $\dim(M) \leqslant r(G) - 1$, every A-invariant probability measure on M^α that projects to the Haar measure on G/Γ is G-invariant;
(2) if $\dim(M) \leqslant r(G)$ and α is volume-preserving, every A-invariant probability measure on M^α that projects to the Haar measure on G/Γ is G-invariant.

In particular, if $\dim(M) \leqslant r(G) - 1$, every action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ preserves a Borel probability measure.

12. Proof outline of Theorem 5.2

To establish Theorem 3.4, from the discussion in Section 5 it is enough to establish Theorem 5.2: the action α has uniform subexponential growth of derivatives. We outline the proof of Theorem 5.2.

12.1. **Setup for proof.** For $n \geqslant 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a cocompact lattice. Let M be a compact manifold and let $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ an action. As assume either that $\dim(M) \leqslant n-2$ or that $\dim(M) \leqslant n-1$ and that α preserves a volume form. We recall the following constructions from the proof of Theorem 3.5:

(1) The manifold $M^\alpha = (\mathrm{SL}(n,\mathbb{R}) \times M)/\Gamma$ is the suspension space introduced in Section 10.1. M^α is fiber bundle over $\mathrm{SL}(n,\mathbb{R})/\Gamma$ with fibers diffeomorphic to M. Moreover, M^α and $\mathrm{SL}(n,\mathbb{R})/\Gamma$ have natural (left) $\mathrm{SL}(n,\mathbb{R})$-actions and the projection $\pi\colon M^\alpha \to \mathrm{SL}(n,\mathbb{R})/\Gamma$ intertwines these G-actions.
(2) $A \subset \mathrm{SL}(n,\mathbb{R})$ denotes the subgroup of diagonal matrices with positive entries. We have $A \simeq \mathbb{R}^{n-1}$ which is a higher-rank, free abelian group if $n \geqslant 3$.
(3) Given an ergodic, A-invariant Borel probability measure μ on M^α we have **fiberwise Lyapunov exponents**
$$\lambda^F_{1,\mu}, \dots, \lambda^F_{p,\mu}\colon A \to \mathbb{R}$$
for the restriction of the derivative of the A-action on M^α to the fibers of M^α introduced in Section 10.2.
(4) $\beta^{i,j}\colon A \to \mathbb{R}$ are the **roots** of $\mathrm{SL}(n,\mathbb{R})$ and $U^{i,j}$ are the corresponding **root subgroups** introduced in Section 9.1.

12.2. **Two key propositions.** The proof of Theorem 5.2 is by contradiction and follows from the following two propositions. Our first key proposition is an analogue of Proposition 6.3.

Proposition 12.1. *Suppose that $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ fails to have uniform subexponential growth of derivatives. Then there exists a Borel probability measure μ' on M^α such that*

(1) μ' is A-invariant and ergodic;
(2) there exists a nonzero fiberwise Lyapunov exponent $\lambda^F_{j,\mu'}\colon A \to \mathbb{R}$.

The proof of Proposition 12.1 is very similar to the proof of Proposition 6.3 with some minor modifications and notational differences. We include an outline of the proof in Section 13.3; see also [16, Section 4] for complete details.

The measure μ' in Proposition 12.1 projects to an ergodic, A-invariant measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$. If μ' projected to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ then, from Theorem 11.1 and the bounds on the dimension M, the measure μ' would be G-invariant and, as explained below, the proof of Theorem 5.2 would be complete. However, there may exist ergodic A-invariant measures on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ that are not the Haar measure.[6]

By carefully averaging the measure μ' along root subgroups $U^{i,j}$ and applying Ratner's measure classification theorem [103] to the projected measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ we obtain the following.

Proposition 12.2. *Let $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ be an action. Suppose there exists an ergodic, A-invariant measure μ' on the suspension space M^α with a nonzero fiberwise Lyapunov exponent $\lambda^F_{j',\mu'}\colon A \to \mathbb{R}$. Then there exists a Borel probability measure μ on M^α such that*

(1) μ is A-invariant and ergodic;
(2) there exists a nonzero fiberwise Lyapunov exponent $\lambda^F_{j,\mu}\colon A \to \mathbb{R}$;
(3) μ projects to the Haar measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$.

Remark 12.3.

(1) Propositions 12.1 and 12.2 hold in full generality; they do not depend on the comparison between the dimension of M and the rank of $\mathrm{SL}(n,\mathbb{R})$. The constraint on the dimension of M is used to obtain a contradiction in the proof of Theorem 5.2 by applying Theorem 11.1 and Zimmer's cocycle superrigidity to the fiberwise derivative cocycle.
(2) Propositions 12.1 and 12.2 heavily use the fact that Γ is cocompact in $\mathrm{SL}(n,\mathbb{R})$ so that the manifold M^α is compact. For instance, if M^α is not compact then the proof of Proposition 12.1 (compare with proof of Proposition 6.3) fails as there may be escape of mass into the cusp of G/Γ. Thus, more subtle arguments are required to establish the analogue of Theorem 5.2 in the case that Γ is nonuniform. In the case that $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, such arguments appear in [17].
(3) Both Proposition 12.1 and Proposition 12.2 holds for C^1 actions. The $C^{1+\beta}$ hypotheses is later used (along with the dimension bounds) to conclude that the A-invariant measure μ obtained in Proposition 12.2 is, in fact, G-invariant by applying Theorem 11.1.

12.3. **Proof of Theorem 5.2.** We deduce Theorem 5.2 from Proposition 12.1, Proposition 12.2, Theorem 11.1, and Theorem 4.2.

Proof of Theorem 5.2 . Let $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ be as in Theorem 5.2. For the sake of contradiction, assume that

$$\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$$

fails to have uniform subexponential growth of derivatives. Let μ' be the measure guaranteed by Proposition 12.1. We then apply Proposition 12.2 to obtain an ergodic, A-invariant Borel probability measure μ on M^α that projects to the Haar measure on G/Γ and has a non-zero fiberwise Lyapunov exponent. In either case considered in Theorem 5.2, it follows from Theorem 11.1 that μ is G-invariant.

Recall that we write $\pi\colon M^\alpha \to \mathrm{SL}(n,\mathbb{R})/\Gamma$ for the natural projection and let F be the fiberwise tangent bundle; that is, F is sub-vector-bundle of TM^α given by $F = \ker D\pi$. As F is G-invariant, we may apply Zimmer's cocycle superrigidity theorem, Theorem 4.2, to the fiberwise derivative cocycle $\mathcal{A}(g,x) = D_x g\restriction_{F(x)}$ of the μ-preserving $\mathrm{SL}(n,\mathbb{R})$-action on M^α. Since the fibers have dimension at most $n-1$ and since there are no non-trivial representations $\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$ for $d < n$, it follows from Theorem 4.2

[6] In fact, for certain lattices Γ there exist ergodic A-invariant measures on $\mathrm{SL}(n,\mathbb{R})/\Gamma$ that have positive entropy for some element of A as shown by M. Rees; see [32, Section 8].

that the fiberwise derivative cocycle $\mathcal{A}(g,x) = D_x g\restriction_{F(x)}$ is cohomologous to a compact-valued cocycle: there is a compact group $K \subset \mathrm{SL}(d,\mathbb{R})$ and measurable $\Phi\colon M^\alpha \to \mathrm{GL}(d,\mathbb{R})$ such that

$$\Phi(g\cdot x) D_x g\restriction_{F(x)} \Phi(x)^{-1} \in K.$$

By Poincaré recurrence to sets on which the norm and conorm of Φ are bounded, it follow for any $g \in G$ and $\epsilon > 0$ that the set of $x \in M^\alpha$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|D_x g^n\restriction_{F(x)}\| \geqslant \epsilon$$

has μ-measure zero. This contradicts the existence of nonzero fiberwise Lyapunov exponent for μ. This contradiction completes the proof of Theorem 5.2. ☐

13. Discussion of the proof of Propositions 12.1 and 12.2

We outline the main steps in the proof of Propositions 12.1 and 12.2.

13.1. Averaging measures on M^α.

Let $H = \{h^t : t \in \mathbb{R}\}$ be a 1-parameter root subgroup of $\mathrm{SL}(n,\mathbb{R})$. Given a measure μ on M^α and $T \geqslant 0$ we define

$$H^T * \mu := \frac{1}{T}\int_0^T (h^t)_*\mu \, dt$$

to be the measure obtained by averaging the translates of μ over the interval $[0,T]$.

Let $s \in A$. Given any s-invariant measure μ on M^α, the **average top fiberwise Lyapunov exponent** of s with respect to μ is

$$\lambda^F_{\mathrm{top}}(s,\mu) = \inf_{n\to\infty} \frac{1}{n} \int \log \|D_x(s^n)\restriction_F\| \, d\mu(x). \tag{13.1}$$

Note that if μ is moreover A-invariant and A-ergodic with fiberwise Lyapunov exponents $\lambda^F_{1,\mu},\dots,\lambda^F_{p,\mu}\colon A \to \mathbb{R}$ then

$$\lambda^F_{\mathrm{top}}(s,\mu) = \max_{1\leqslant i\leqslant p} \lambda^F_{i,\mu}(s).$$

We have the following facts which we invoke throughout our averaging procedures.

Claim 13.1. *Let $s \in A$ and let μ be an s-invariant measure on M^α. Let $H = \{h^t, t \in \mathbb{R}\}$ be a one-parameter group contained in the centralizer of s in $\mathrm{SL}(n,\mathbb{R})$.*

(1) The measure $H^T * \mu$ is s-invariant for every $T \geqslant 0$.
(2) Any weak-$*$ limit point of $\{H^T * \mu\}$ as $T \to \infty$ is s-invariant.
(3) Any weak-$*$ limit point of $\{H^T * \mu\}$ as $T \to \infty$ is H-invariant.
(4) $\lambda^F_{\mathrm{top}}(s, H^T * \mu) = \lambda^F_{\mathrm{top}}(s,\mu)$ for every $T \geqslant 0$.
(5) If μ' is a weak-$*$ limit point of $\{H^T * \mu\}$ as $T \to \infty$ then

$$\lambda^F_{\mathrm{top}}(s,\mu') \geqslant \lambda^F_{\mathrm{top}}(s,\mu).$$

(1) is clear from definition and (2) follows since the set of s-invariant measures is closed. (3) follows from (the proof of) the Krylov-Bogolyubov theorem (see Claim 6.4). (4) is a standard computation which follows from the compactness of M^α and hence boundedness

of the cocycle. Indeed we have

$$\begin{aligned}
\lambda^F_{\mathrm{top}}(s, H^T * \mu) &= \inf_{n\to\infty} \frac{1}{n} \int \log \|D_x(s^n)\restriction_F\| \, d(H^T * \mu)(x) \\
&= \inf_{n\to\infty} \frac{1}{n}\frac{1}{T} \int \int_{t=0}^{T} \log \|D_{h^t\cdot x}(s^n)\restriction_F\| \, dt \, d\mu(x) \\
&= \inf_{n\to\infty} \frac{1}{n}\frac{1}{T} \int \int_{t=0}^{T} \log \|D_{h^t\cdot x}(h^t s^n h^{-t})\restriction_F\| \, dt \, d\mu(x) \\
&\leqslant \inf_{n\to\infty} \frac{1}{n}\frac{1}{T} \int \int_{t=0}^{T} \log \|D_{h^t\cdot x}(h^{-t})\restriction_F\| \\
&\qquad\qquad + \log \|D_x(s^n)\restriction_F\| + \log \|D_{s\cdot x}(h^t)\restriction_F\| \, dt \, d\mu(x) \\
&\leqslant \inf_{n\to\infty} \frac{1}{n} \left(\int \log \|D_x(s^n)\restriction_F\| \, d\mu(x) + 2K \right)
\end{aligned}$$

where

$$K = \sup \left\{ \log \|D_x(h^t)\restriction_F\| : x \in M, t \in [-T, T] \right\}.$$

(5) follows from the well-known fact that the average top Lyapunov exponent is upper-semicontinuous on the set of s-invariant measures (see for example [111] or [16, Lemma 3.2(b)]). Indeed, in the weak-$*$ topology, for each n the function

$$\mu \mapsto \frac{1}{n} \int \log \|D_x(s^n)\restriction_F\| \, d\mu(x)$$

is continuous. The pointwise infimum of a family of continuous functions is upper-semicontinuous.

Remark 13.2. Recall that a **Følner sequence** in a Lie group H equipped with a left-Haar measure m_H is a sequence $\{F_n\}$ of Borel subsets $F_n \subset H$, with $0 < m_H(F_n) < \infty$, such that for every compact subset $Q \subset H$ one has

$$\lim_{n\to\infty} \sup_{h\in Q} \frac{m_H\big((h \cdot F_n) \triangle F_n\big)}{m_H(F_n)} = 0.$$

If H admits a Følner sequence then H is said to be **amenable**. When $H = \mathbb{R}$, a Følner sequence is given by $F_n = [0, n]$. Examples of amenable groups include abelian groups, nilpotent groups, solvable groups, and compact groups. See [5] for more details.

Consider H to be an amenable Lie subgroup of $G = \mathrm{SL}(n, \mathbb{R})$. Given a Borel probability measure μ on M^α and a Følner sequence $\{F_n\}$ in H we define

$$F_n * \mu := \frac{1}{m_H(F_n)} \int_{F_n} h_* \mu \, dm_H(h).$$

By a computation analogous to (6.3) in the proof of Claim 6.4, any weak-$*$ limit point $\hat{\mu}$ of the sequence $\{F_n * \mu\}$ as $n \to \infty$ is an H-invariant measure on M^α. Moreover, properties analogous to those in Claim 13.1 hold when averaging an s-invariant measure μ against a Følner sequence $\{F_n\}$ in an amenable subgroup H contained in the centralizer $C_G(s)$ of s. See [16, Lemma 3.2] for precise formulations.

13.2. **Averaging measures on** $\mathrm{SL}(n, \mathbb{R})/\Gamma$**.** When averaging probability measures on $\mathrm{SL}(n, \mathbb{R})/\Gamma$ along 1-parameter unipotent subgroups we obtain additional properties of the limiting measures. The results stated in the following proposition are consequences of Ratner's measure classification and equidistribution theorems for unipotent flows [101–103]. See also [118].

We do not formulate Ratner's theorems here but only the consequences we use in the remainder.

Proposition 13.3. *Let $\hat{\mu}$ be a Borel probability measure on $\mathrm{SL}(n,\mathbb{R})/\Gamma$. For each 1-parameter root subgroup $U^{i,j}$*

(1) the weak-$*$ limit
$$U^{i,j} * \hat{\mu} := \lim_{T\to\infty}\{(U^{i,j})^T * \hat{\mu} : T \geqslant 0\}$$
exists;
(2) if $\hat{\mu}$ is A-invariant, so is $U^{i,j} * \hat{\mu}$;
(3) if $\hat{\mu}$ is A-invariant and A-ergodic, the measure $U^{i,j} * \hat{\mu}$ is A-ergodic;
(4) if $\hat{\mu}$ is A-invariant and $U^{i,j}$-invariant then $\hat{\mu}$ is $U^{j,i}$-invariant.

Proposition 13.3(1) follows from Ratner's measure classification and equidistribution theorems for unipotent flows. When U is higher-dimensional, we use an analogue of Proposition 13.3(1) due to Shah [107, Corollary 1.3]. Proposition 13.3(2) follows from the fact that A normalizes $U^{i,j}$ and that the limit in Proposition 13.3(1) exists and is hence unique. Proposition 13.3(4) is a consequence of Theorem 9 in [103] or Proposition 2.1 in [101].

Proposition 13.3(3) is a short argument that uses the A-invariance of $\hat{\mu}$ and the pointwise ergodic theorem: Since there is $s \in A$ such that $U^{i,j}$-orbits are contracted by s, by the pointwise ergodic theorem, the measurable hull of the partition into $U^{i,j}$-orbits refines the ergodic decomposition for A. Let η be the measurable hull of the partition into $U^{i,j}$-orbits and let $\{\hat{\mu}^\eta_x\}$ be a family of conditional measures of $\hat{\mu}$ for this partition. (Note that from Ratner's equidistribution theorem, we have that $\hat{\mu}^\eta_x$ is a homogeneous measure on a closed homogeneous submanifold.) If ϕ is a bounded, A-invariant measurable function then for $\hat{\mu}$-a.e. x, ϕ is constant $\hat{\mu}^\eta_x$-almost surely; in particular,
$$\phi(x) = \int \phi \, d\hat{\mu}^\eta_x$$
for $\hat{\mu}$-a.e. x. But $x \mapsto \int \phi \, d\hat{\mu}^\eta_x$ is a μ-almost everywhere defined, A-invariant function. In particular, $x \mapsto \int \phi \, d\hat{\mu}^\eta_x$ is constant μ a.s. by ergodicity of μ. It follows that ϕ is constant $\hat{\mu}$-a.s. and ergodicity follows.

13.3. **Proof of Proposition 12.1.** We outline the proof of Proposition 12.1. Recall the notation introduced in Section 10.1. In particular, $\pi\colon M^\alpha \to G/\Gamma$ is the canonical the projection and $F = \ker(D\pi)$ is the fiberwise tangent bundle of M^α. We write the derivative of translation by g in M^α as Dg and the restriction to the fiber of F through $x \in M^\alpha$ by $D_x g\restriction_{F(x)}$. Equip M^α with any Riemannian metric and write
$$\|Dg\restriction_F\| = \sup_{x\in M^\alpha} \|D_x g\restriction_{F(x)}\|.$$

Let $K = \mathrm{SO}(n)$. We equip with G with a right-invariant, left-K-invariant metric and induced distance function $d(\cdot,\cdot)$. We have the following elementary claim which allows us to transfer exponential growth properties between the Γ-action on M and the G-action on the fibers of M^α.

Claim 13.4. *If $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ is cocompact and if M is compact, then any action*
$$\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$$

has uniform subexponential growth of derivatives if and only if for every $\epsilon > 0$ there is a C such that for all $g \in \mathrm{SL}(n,\mathbb{R})$,

$$\|Dg\restriction_F\| \leqslant Ce^{\epsilon d(e,g)}.$$

With the above claim, we outline to main steps in the proof of Proposition 12.1.

Proof of Proposition 12.1. We assume $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ fails to have uniform subexponential growth of derivatives. Then, by Claim 13.4, there exist $\epsilon > 0$, integers $m_n \in \mathbb{N}$ with $m_n \to \infty$, elements $g_{m_n} \in G$ with $d(g_{m_n}, e) = m_n$, points $x_{m_n} \in M_\alpha$, and unit vectors $v_{m_n} \in T_{x_{m_n}} M_\alpha$ such that

$$\|D_{x_{m_n}} g_{m_n}(v_{m_n})\| \geqslant e^{\epsilon m_n}.$$

Let UF denote the unit sphere bundle in F and, given $g \in G$, let UDg denote the induced action on UF: given $x \in M^\alpha$ and $v \in UF(x)$ write

$$UD_x g(v) = \frac{D_x g(v)}{\|D_x g(v)\|}$$

and

$$UDg(x,v) = (g\cdot x, UD_x g(v))\,.$$

By the singular value decomposition of matrices, the group $G = \mathrm{SL}(n,\mathbb{R})$ can be written as $G = KAK$ where $K = \mathrm{SO}(n)$. (For general simple Lie groups G we use the Cartan decomposition). We can thus write each $g_{m_n} \in G$ as

$$g_{m_n} = k_n a_n k_n'$$

where $k_n, k_n' \in K$ and $a_n \in A$. Write

$$x_n' = k_n' \cdot x_{m_n}, \quad x_n'' = a_n k_n' \cdot x_{m_n},$$
$$v_n' = UD_{x_{m_n}} k_n'(v_{m_n}), \quad v_n'' = UD_{x_{m_n}}(a_n k_n')(v_{m_n}).$$

Then

$$\|D_{x_{m_n}} g_{m_n}(v_{m_n})\| = \|D_{x_n''} k_n(v_n'')\| \cdot \|D_{x_n'} a_n(v_n')\| \cdot \|D_{x_{m_n}} k_n'(v_{m_n})\|$$

and so

$$\epsilon \leqslant \lim_{n\to\infty} \frac{1}{m_n} \log \|D_{x_{m_n}} g_{m_n}(v_{m_n})\| = \lim_{n\to\infty} \frac{1}{m_n} \log \|D_{x_n'} a_n(v_n')\|$$

as $\|D_x k\restriction_F\|$ is uniformly bounded over all $k \in K$ and $x \in M^\alpha$.

Note that

$$|m_n - d(a_n, e)| = |d(g_{m_n}, e) - d(a_n, e)| \leqslant d(k_n, e) + d(k_n', e)$$

is uniformly bounded in n. Thus ${m_n}^{-1} d(a_n, e) \to 1$. As $A \simeq \mathbb{R}^{n-1}$, for each n there is a unique $\tilde{a}_n$ with $a_n = (\tilde{a}_n)^{m_n}$; moreover, as A is geodesically embedded in G, we have $d(\tilde{a}_n, e) \to 1$.

For each n, let ν_n be the empirical measure on UF given by

$$\nu_n = \frac{1}{m_n} \sum_{j=0}^{m_n-1} (\tilde{a}_n)^j_* \delta_{(x_n', v_n')}.$$

Taking a subsequence $\{n_j\}$, we may assume that ν_{n_j} converges to some ν_∞ and that $\tilde{a}_{n_j}$ converges to some $s \in A$. Note that $d(s,e) = 1$. Let $\overline{\mu}$ denote the image of ν under the natural projection $UF \to M^\alpha$. Adapting the proofs of Claim 6.4 and Proposition 6.3 one can show that

(1) ν_∞ is UDs-invariant whence $\overline{\mu}$ is s-invariant;

(2) $\lambda^F_{\mathrm{top}}(s,\overline{\mu}) \geqslant \epsilon > 0$.

Take a Følner sequence $\{F_n\}$ in A and let $\tilde{\mu}$ be any weak-$*$ limit point of $\{F_n * \overline{\mu}\}$ as $n \to \infty$. Then, from analogues of the properties in Claim 13.1 for averaging over Følner sequences, we have that

(1) $\tilde{\mu}$ is A-invariant;
(2) $\lambda^F_{\mathrm{top}}(s,\tilde{\mu}) \geqslant \epsilon > 0$.

We take μ' to be an A-ergodic component of $\tilde{\mu}$ with $\lambda^F_{\mathrm{top}}(s,\mu') \geqslant \epsilon > 0$. □

13.4. **Proof of Proposition 12.2 for $\mathrm{SL}(3,\mathbb{R})$.** To simplify ideas, we outline the proof of Proposition 12.2 assuming Γ is a cocompact lattice in $\mathrm{SL}(3,\mathbb{R})$. We perform two averaging procedures on the measure μ' from the hypotheses of Proposition 12.2 to obtain the measure μ in the conclusion of Proposition 12.2.

Proof of Proposition 12.2 for $\Gamma \subset \mathrm{SL}(3,\mathbb{R})$. Take $\mu_0 = \mu'$ to be the ergodic, A-invariant probability measure in the hypotheses of Proposition 12.2 with nonzero fiberwise exponent

$$\lambda^F_{j,\mu_0}\colon A \to \mathbb{R}, \qquad \lambda^F_{j,\mu_0} \neq 0.$$

First averaging. Consider the elements

$$s = \mathrm{diag}(\tfrac{1}{4},2,2) \quad \text{and} \quad \overline{s} = \mathrm{diag}(2,2,\tfrac{1}{4})$$

of $A \subset \mathrm{SL}(3,\mathbb{R})$. Note that s and $\overline{s}$ are linearly independent and hence form a basis for $A \simeq \mathbb{R}^2$. As the linear functional λ^F_{j,μ_0} is nonzero, either

$$\lambda^F_{j,\mu_0}(s) \neq 0 \quad \text{or} \quad \lambda^F_{j,\mu_0}(\overline{s}) \neq 0.$$

Without loss of generality we may assume that

$$\lambda^F_{j,\mu_0}(s) \neq 0.$$

Take s_0 to be either s or s^{-1} so that $\lambda^F_{j,\mu_0}(s_0) > 0$.

Consider the 1-parameter subgroup

$$U^{2,3} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Note that $U^{2,3}$ commutes with s_0. Let μ_1 be any weak-$*$ limit point of $\{(U^{2,3})^T * \mu\}$ as $T \to \infty$. From Claim 13.1, μ_1 is s_0-invariant and $\lambda^F_{\mathrm{top}}(s_0,\mu_1) \geqslant \lambda^F_{\mathrm{top}}(s_0,\mu_0)$.

We now average μ_1 over a Følner sequence in A: identifying A with $\mathbb{R}^2$ let $A^T = [0,T] \times [0,T]$ define a Følner sequence $\{A^T\}$ in A. Then

$$A^T * \mu_1 := \frac{1}{T^2} \int_0^T \int_0^T (t_1,t_2)_* \mu_1 \, d(t_1,t_2).$$

Let μ_2 be any weak-$*$ limit point of $\{A^T * \mu_1\}$ as $T \to \infty$. Then, from facts analogous to those in Claim 13.1, μ_2 is A-invariant and

$$\lambda^F_{\mathrm{top}}(s_0,\mu_2) \geqslant \lambda^F_{\mathrm{top}}(s_0,\mu_1) > 0.$$

Note that μ_2 might no longer be $U^{2,3}$-invariant.

We investigate properties of the projection of each measure μ_0, μ_1, and μ_2 to $\mathrm{SL}(3,\mathbb{R})/\Gamma$. For each j, we denote by $\hat{\mu}_j = \pi_*(\mu_j)$ the image of μ_j under the projection $\pi\colon M^\alpha \to \mathrm{SL}(3,\mathbb{R})/\Gamma$.

Observe that $\hat\mu_1 = U^{2,3} * \hat\mu_0$ is $U^{2,3}$-invariant. Since $\hat\mu_0$ was A-invariant, from Proposition 13.3(2) we have that $\hat\mu_1$ is A-invariant and it follows that $\hat\mu_1 = \hat\mu_2$ so $\hat\mu_2$ is $U^{2,3}$-invariant and A-invariant. From Proposition 13.3(4), $\hat\mu_2$ is invariant under the subgroup

$$\left\{\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\} \subset \mathrm{SL}(3,\mathbb{R})$$

generated by A, $U^{2,3}$ and $U^{3,2}$ in $\mathrm{SL}(3,\mathbb{R})$. Moreover, since $\hat\mu_0$ was A-ergodic, from Proposition 13.3(3) the measure $\hat\mu_1 = \hat\mu_2$ is A-ergodic.

Returning to M^α, as $\lambda^F_{\mathrm{top}}(s_0,\mu_2) > 0$ and as $\hat\mu_2$ is A-ergodic, we may replace μ_2 with an A-ergodic component μ_2' of μ_2 such that

(1) $\lambda^F_{\mathrm{top}}(s_0,\mu_2') > 0$, and
(2) the projection of μ_2' to $\mathrm{SL}(3,\mathbb{R})/\Gamma$ is $\hat\mu_2$.

Let $\lambda^F_{1,\mu_2'},\dots,\lambda^F_{p',\mu_2'}\colon A\to\mathbb{R}$ denote the fiberwise Lyapunov exponents for the A-invariant, A-ergodic measure μ_2'. Then $0<\lambda^F_{j',\mu_2'}(s_0) = \lambda^F_{\mathrm{top}}(s_0,\mu_2')$ for some $1\leqslant j'\leqslant p'$ whence some fiberwise Lyapunov exponent $\lambda^F_{j',\mu_2'}\colon A\to\mathbb{R}$ is a nonzero linear functional.

Second averaging. Consider now the elements $s = (2,2,\frac14)$ and $\overline{s} = (2,\frac14,2)$ in A. Again, either

$$\lambda^F_{j',\mu_2'}(s)\neq 0 \quad \text{or} \quad \lambda^F_{j',\mu_2'}(\overline{s})\neq 0.$$

Case 1: $\lambda^F_{j',\mu_2'}(s)\neq 0$. Take $s_1 = s$ or $s_1 = s^{-1}$ so that $\lambda^F_{j',\mu_2'}(s_1)>0$. Consider the one-parameter group $U^{1,2}$ which commutes with s_1. As above, any weak-$*$ limit point μ_3 of $\{(U^{1,2})^T * \mu_2'\}$ as $T\to\infty$ is s_1-invariant, with

$$\lambda^F_{\mathrm{top}}(s_1,\mu_3) = \lambda^F_{\mathrm{top}}(s_1,\mu_3) \geqslant \lambda^F_{\mathrm{top}}(s_1,\mu_2') > 0.$$

Let μ_4 be any weak-$*$ limit point of $\{A^T * \mu_3\}$ as $T\to\infty$ (where $A^T * \mu_3$ is as in the first averaging). Then μ_4 is A-invariant and

$$\lambda^F_{\mathrm{top}}(s_1,\mu_4) \geqslant \lambda^F_{\mathrm{top}}(s_1,\mu_3) > 0.$$

We claim that the projection $\hat\mu_4$ of μ_4 to $\mathrm{SL}(3,\mathbb{R})/\Gamma$ is the Haar measure. Since the groups $U^{1,2}$ and $U^{3,2}$ commute and since $\hat\mu_2$ was $U^{3,2}$-invariant, it follows that $\hat\mu_3 = U^{1,2} * \hat\mu_2$ is $U^{3,2}$-invariant. Also, since $\hat\mu_2$ was A-invariant, Proposition 13.3(2) shows that $\hat\mu_3$ is A-invariant. Thus $\hat\mu_3 = \hat\mu_4$ and $\hat\mu_4$ is also invariant under the actions of A, $U^{1,2}$, and $U^{3,2}$. By Proposition 13.3(4) it follows that $\hat\mu_4$ is invariant under the groups $U^{2,1}$ and $U^{2,3}$; in particular $\hat\mu_4$ is invariant under the following subgroups of $\mathrm{SL}(3,\mathbb{R})$:

$$\left\{\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}\right\}.$$

These two groups generate all of $\mathrm{SL}(3,\mathbb{R})$, and hence $\hat\mu_4$ is the Haar measures.

Case 2: $\lambda^F_{j',\mu_2'}(\overline{s})\neq 0$. Take $s_1 = \overline{s}$ or $s_1 = \overline{s}^{-1}$ so that $\lambda^F_{j',\mu_2'}(s_1)>0$. Consider the one-parameter group $U^{1,3}$ which commutes with s_1. As above, any weak-$*$ limit point μ_3 of $\{(U^{1,3})^T * \mu_2'\}$ as $T\to\infty$ is s_1-invariant, with

$$\lambda^F_{\mathrm{top}}(s_1,\mu_3) \geqslant \lambda^F_{\mathrm{top}}(s_1,\mu_2') > 0.$$

Let μ_4 be any weak-$*$ limit point of $\{A^T * \mu_3\}$ as $T\to\infty$. Then μ_4 is A-invariant and

$$\lambda^F_{\mathrm{top}}(s_1,\mu_4) \geqslant \lambda^F_{\mathrm{top}}(s_1,\mu_3) > 0.$$

Again, we claim that $\hat{\mu}_4 = U^{1,3} * \hat{\mu}_2$ is the Haar measure. Since the groups $U^{1,3}$ and $U^{2,3}$ commute, it follows that $\hat{\mu}_3$ is $U^{2,3}$-invariant. Also, since $\hat{\mu}_2$ was A-invariant, Proposition 13.3(2) shows that $\hat{\mu}_3$ is A-invariant. Thus $\hat{\mu}_3 = \hat{\mu}_4$ and $\hat{\mu}_4$ is also invariant under the actions of A, $U^{1,3}$ and $U^{2,3}$. By Proposition 13.3(4) it follows that $\hat{\mu}_4$ is invariant under the following subgroups of $\mathrm{SL}(3,\mathbb{R})$:

$$\left\{\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}\right\}.$$

Again, these two groups generate all of $\mathrm{SL}(3,\mathbb{R})$, and hence $\hat{\mu}_4$ is the Haar measure.

Completion of proof. In either Case 1 or Case 2, since the Haar measure $\hat{\mu}_4$ is A-ergodic, we may take an A-ergodic component μ_4' of μ_4 projecting to the Haar measure with

$$\lambda^F_{\mathrm{top}}(s_1, \mu_4') > 0.$$

If $\lambda^F_{1,\mu_4'}, \dots, \lambda^F_{p'',\mu_4'} \colon A \to \mathbb{R}$ denote the fiberwise Lyapunov exponents for the A-invariant, A-ergodic measure μ_4' then $0 < \lambda^F_{j'',\mu_4'}(s_1) = \lambda^F_{\mathrm{top}}(s_1, \mu_4')$ for some $1 \leqslant j'' \leqslant p''$ whence some fiberwise Lyapunov exponent $\lambda^F_{j'',\mu_4'} \colon A \to \mathbb{R}$ is a nonzero linear functional.

This completes the proof of Proposition 12.2. □

13.5. **Modifications to proof of Proposition 12.2 in** $\mathrm{SL}(n,\mathbb{R})$**.** When Γ is a cocompact lattice in $\mathrm{SL}(n,\mathbb{R})$ we replace the first averaging step with a more complicated averaging.

First averaging. We again take $\mu_0 = \mu'$ to be the A-invariant measure in Proposition 12.2 with nonzero fiberwise exponent

$$\lambda^F_{j,\mu_0} \colon A \to \mathbb{R}, \qquad \lambda^F_{j,\mu_0} \neq 0.$$

Without loss of generality (by conjugating by a permutation matrix) we may assume that for the element

$$s = \mathrm{diag}(\tfrac{1}{2^{n-1}}, 2, \dots, 2)$$

of $A \subset \mathrm{SL}(n,\mathbb{R})$, we have

$$\lambda^F_{j,\mu_0}(s) \neq 0.$$

Take s_0 to be either s, or s^{-1} so that $\lambda^F_{j,\mu_0}(s_0) > 0$.

Consider the unipotent subgroup $U \subset \mathrm{SL}(n,\mathbb{R})$ of matrices of the form

$$U = \left\{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & * & \cdots & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\right\}.$$

Note that U commutes with s_0.

Let $\{F_n\}$ be a Følner sequence in U and let μ_1 be any weak-$*$ limit point of $\{F_n * \mu_0\}$ as $n \to \infty$ where

$$F_n * \mu_0 = \frac{1}{m_U(F_n)} \int_{F_n} u_* \mu_0 \, du.$$

From facts analogous to those in Claim 13.1, we have that μ_1 is s_0-invariant and $\lambda^F_{\mathrm{top}}(s_0, \mu_1) \geqslant \lambda^F_{\mathrm{top}}(s_0, \mu_0) > 0$. Moreover, as U is higher-dimensional, we use [107, Corollary 1.3] rather than Proposition 13.3(1) to conclude (as least for certain Følner sequences $\{F_n\}$ in U with nice geometry) that the projection $\hat{\mu}_1$ of μ_1 to G/Γ is the limit

$$\hat{\mu}_1 = \lim F_n * \hat{\mu}_0$$

and is A-invariant, ergodic, and U-invariant.

We again average μ_1 over a Følner sequence for the form

$$A^T = [0,T] \times \cdots \times [0,T]$$

in A (identified with $\mathbb{R}^{n-1}$) and let μ_2 be any weak-$*$ limit point of $\{A^T * \mu_1\}$ as $T \to \infty$. Then μ_2 is A-invariant and

$$\lambda^F_{\mathrm{top}}(s_0, \mu_2) \geqslant \lambda^F_{\mathrm{top}}(s_0, \mu_1) > 0.$$

Again, we have equality of the projected measures $\hat{\mu}_1 = \hat{\mu}_2$ so $\hat{\mu}_2$ is U-invariant and A-invariant. From Proposition 13.3(4), $\hat{\mu}_2$ is also invariant under the subgroup

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & & * \\ 0 & * & * & & * \\ \vdots & & & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix} \right\}.$$

As $\hat{\mu}_2$ is A-ergodic, we may replace μ_2 with an A-ergodic component μ_2' of μ_2 such that

(1) $\lambda^F_{\mathrm{top}}(s_0, \mu_2') > 0$, and
(2) the projection of μ_2' to $\mathrm{SL}(n,\mathbb{R})/\Gamma$ is $\hat{\mu}_2$.

Then, if $\lambda^F_{1,\mu_2'}, \ldots, \lambda^F_{p',\mu_2'} \colon A \to \mathbb{R}$ denote the fiberwise Lyapunov exponents for μ_2', we have $0 < \lambda^F_{j',\mu_2'}(s_0) = \lambda^F_{\mathrm{top}}(s_0, \mu_2')$ for some $1 \leqslant j' \leqslant p'$.

Second averaging. Consider now the roots $\beta^{1,2}$ and $\beta^{1,n}$ of G. Since $\beta^{1,2}$ and $\beta^{1,n}$ are not proportional, at most one of $\beta^{1,2}$ and $\beta^{1,n}$ is proportional to $\lambda^F_{j',\mu_2'}$. In particular, we may find either s or $\overline{s}$ in A such that

(1) $\beta^{1,2}(s) = 0$ but $\lambda^F_{j',\mu_2'}(s) \neq 0$; or
(2) $\beta^{1,n}(\overline{s}) = 0$ but $\lambda^F_{j',\mu_2'}(\overline{s}) \neq 0$.

The two cases in the second averaging step of in Section 13.4 are then identical to the above, where we either average over the 1-parameter group $U^{1,2}$ in the case $\lambda^F_{j',\mu_2'}(s) \neq 0$ or $U^{1,n}$ in the case $\lambda^F_{j,\mu_2'}(\overline{s}) \neq 0$. The structure theory of $\mathrm{SL}(n,\mathbb{R})$ will then imply that the measure obtained after the second averaging projects to the Haar measure.

References

[1] L. M. Abramov and V. A. Rohlin, *Entropy of a skew product of mappings with invariant measure*, Vestnik Leningrad. Univ. **17** (1962), 5–13.

[2] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Proc. Steklov Inst. Math. **90** (1967), 1–235.

[3] A. Avila and M. Viana, *Extremal Lyapunov exponents: an invariance principle and applications*, Invent. Math. **181** (2010), 115–189.

[4] L. Barreira, Y. Pesin, and J. Schmeling, *Dimension and product structure of hyperbolic measures*, Ann. of Math. (2) **149** (1999), 755–783.

[5] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.

[6] Y. Benoist, *Five lectures on lattices in semisimple Lie groups*, Géométries à courbure négative ou nulle, groupes discrets et rigidités, 2009, pp. 117–176.

[7] E. J. Benveniste, *Rigidity and deformations of lattice actions preserving geometric structures*, ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–The University of Chicago.

[8] E. J. Benveniste, *Rigidity of isometric lattice actions on compact Riemannian manifolds*, Geom. Funct. Anal. **10** (2000), 516–542.

[9] E. J. Benveniste and D. Fisher, *Nonexistence of invariant rigid structures and invariant almost rigid structures*, Comm. Anal. Geom. **13** (2005), 89–111.

[10] T. Bogenschütz and H. Crauel, *The Abramov-Rokhlin formula*, Ergodic theory and related topics, III (Güstrow, 1990), 1992, pp. 32–35.

[11] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535.

[12] M. R. Bridson, F. Grunewald, and K. Vogtmann, *Actions of arithmetic groups on homology spheres and acyclic homology manifolds*, Math. Z. **276** (2014), 387–395.

[13] M. R. Bridson and K. Vogtmann, *Actions of automorphism groups of free groups on homology spheres and acyclic manifolds*, Comment. Math. Helv. **86** (2011), 73–90.

[14] A. Brown, *Smoothness of stable holonomies inside center-stable manifolds*, 2016. Preprint, arXiv:1608.05886.

[15] A. Brown, *Smooth ergodic theory of $\mathbb{Z}^d$-actions part 2: Entropy formulas for rank-1 systems*, 2016. Preprint, arXiv:1610.09997.

[16] A. Brown, D. Fisher, and S. Hurtado, *Zimmer's conjecture: Subexponential growth, measure rigidity, and strong property (T)*, 2016. Preprint, arXiv:1608.04995.

[17] A. Brown, D. Fisher, and S. Hurtado, *Zimmer's conjecture for actions of* $\mathrm{SL}(m,\mathbb{Z})$, 2017. Preprint, arXiv:1710.02735.

[18] A. Brown, D. Fisher, and S. Hurtado, *Zimmer's conjecture for non-uniform lattices*, 2018. In preparation.

[19] A. Brown and F. Rodriguez Hertz, *Smooth ergodic theory of $\mathbb{Z}^d$-actions part 1: Lyapunov exponents, dynamical charts, and coarse Lyapunov manifolds*, 2016. Preprint, arXiv:1610.09997.

[20] A. Brown, F. Rodriguez Hertz, and Z. Wang, *Invariant measures and measurable projective factors for actions of higher-rank lattices on manifolds*, 2016. Preprint, arXiv:1609.05565.

[21] A. Brown, F. Rodriguez Hertz, and Z. Wang, *Smooth ergodic theory of $\mathbb{Z}^d$-actions part 3: Product structure of entropy*, 2016. Preprint, arXiv:1610.09997.

[22] A. Brown, F. Rodriguez Hertz, and Z. Wang, *Global smooth and topological rigidity of hyperbolic lattice actions*, Ann. of Math. (2) **186** (2017), 913–972.

[23] M. Burger and N. Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12** (2002), 219–280.

[24] S. Cantat, *Version kählérienne d'une conjecture de Robert J. Zimmer*, Ann. Sci. École Norm. Sup. (4) **37** (2004), 759–768.

[25] S. Cantat, *Progrès récents concernant le programme de zimmer* (2017). Séminaire Bourbaki. Vol. 2017–2018. Exposés 1136.

[26] S. Cantat and J. Xie, *Algebraic actions of discrete groups: The p-adic method*, 2017. Preprint.

[27] S. Cantat and A. Zeghib, *Holomorphic actions, Kummer examples, and Zimmer program*, Ann. Sci. Éc. Norm. Supér. (4) **45** (2012), 447–489.

[28] V. Climenhaga and A. Katok, *Measure theory through dynamical eyes*, 2012. Preprint, arXiv:1208.4550.

[29] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. (2) **135** (1992), 165–182.

[30] M. de la Salle, *Strong (T) for higher rank lattices*, 2017. Preprint, 1711.01900.

[31] T. de Laat and M. de la Salle, *Strong property (T) for higher-rank simple Lie groups*, Proc. Lond. Math. Soc. (3) **111** (2015), 936–966.

[32] M. Einsiedler and A. Katok, *Invariant measures on G/Γ for split simple Lie groups G*, Comm. Pure Appl. Math. **56** (2003), 1184–1221.

[33] M. Einsiedler and A. Katok, *Rigidity of measures—the high entropy case and non-commuting foliations*, Israel J. Math. **148** (2005), 169–238.

[34] A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space*, Publ. Math. Inst. Hautes Études Sci. **127** (2018), 95–324.

[35] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.

[36] B. Farb and P. Shalen, *Real-analytic actions of lattices*, Invent. Math. **135** (1999), 273–296.

[37] R. Feres and F. Labourie, *Topological superrigidity and Anosov actions of lattices*, Ann. Sci. École Norm. Sup. (4) **31** (1998), 599–629.

[38] D. Fisher, *Local rigidity of group actions: past, present, future*, Dynamics, ergodic theory, and geometry, 2007, pp. 45–97.

[39] D. Fisher, *Deformations of group actions*, Trans. Amer. Math. Soc. **360** (2008), 491–505.

[40] D. Fisher, *Groups acting on manifolds: around the Zimmer program*, Geometry, rigidity, and group actions, 2011, pp. 72–157.
[41] D. Fisher, *Recent progress in the zimmer program*, 2017. Preprint, arXiv:1711.07089.
[42] D. Fisher and G. A. Margulis, *Local rigidity for cocycles*, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), 2003, pp. 191–234.
[43] D. Fisher and G. Margulis, *Almost isometric actions, property (T), and local rigidity*, Invent. Math. **162** (2005), 19–80.
[44] D. Fisher and G. Margulis, *Local rigidity of affine actions of higher rank groups and lattices*, Ann. of Math. (2) **170** (2009), 67–122.
[45] D. Fisher and K. Whyte, *Continuous quotients for lattice actions on compact spaces*, Geom. Dedicata **87** (2001), 181–189.
[46] J. Franks, *Anosov diffeomorphisms*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), 1970, pp. 61–93.
[47] J. Franks and M. Handel, *Area preserving group actions on surfaces*, Geom. Topol. **7** (2003), 757–771.
[48] J. Franks and M. Handel, *Distortion elements in group actions on surfaces*, Duke Math. J. **131** (2006), 441–468.
[49] H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Statist. **31** (1960), 457–469.
[50] É. Ghys, *Sur les groupes engendrés par des difféomorphismes proches de l'identité*, Bol. Soc. Brasil. Mat. (N.S.) **24** (1993), 137–178.
[51] É. Ghys, *Actions de réseaux sur le cercle*, Invent. Math. **137** (1999), 199–231.
[52] E. R. Goetze and R. J. Spatzier, *Smooth classification of Cartan actions of higher rank semisimple Lie groups and their lattices*, Ann. of Math. (2) **150** (1999), 743–773.
[53] H. Hu, *Some ergodic properties of commuting diffeomorphisms*, Ergodic Theory Dynam. Systems **13** (1993), 73–100.
[54] S. Hurder, *Rigidity for Anosov actions of higher rank lattices*, Ann. of Math. (2) **135** (1992), 361–410.
[55] S. Hurder, *Affine Anosov actions*, Michigan Math. J. **40** (1993), 561–575.
[56] S. Hurder, *Infinitesimal rigidity for hyperbolic actions*, J. Differential Geom. **41** (1995), 515–527.
[57] M. Kanai, *A new approach to the rigidity of discrete group actions*, Geom. Funct. Anal. **6** (1996), 943–1056.
[58] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.
[59] A. Katok and J. Lewis, *Local rigidity for certain groups of toral automorphisms*, Israel J. Math. **75** (1991), 203–241.
[60] A. Katok and J. Lewis, *Global rigidity results for lattice actions on tori and new examples of volume-preserving actions*, Israel J. Math. **93** (1996), 253–280.
[61] A. Katok, J. Lewis, and R. Zimmer, *Cocycle superrigidity and rigidity for lattice actions on tori*, Topology **35** (1996), 27–38.
[62] A. Katok and R. J. Spatzier, *Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions*, Tr. Mat. Inst. Steklova **216** (1997), 292–319.
[63] A. Katok and F. Rodriguez Hertz, *Measure and cocycle rigidity for certain nonuniformly hyperbolic actions of higher-rank abelian groups*, J. Mod. Dyn. **4** (2010), 487–515.
[64] J. F. C. Kingman, *The ergodic theory of subadditive stochastic processes*, J. Roy. Statist. Soc. Ser. B **30** (1968), 499–510.
[65] A. W. Knapp, *Lie groups beyond an introduction*, Second, Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002.
[66] F. Labourie, *Large groups actions on manifolds*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), 1998, pp. 371–380.
[67] V. Lafforgue, *Un renforcement de la propriété (T)*, Duke Math. J. **143** (2008), 559–602.
[68] F. Ledrappier, *Propriétés ergodiques des mesures de Sinaï*, Inst. Hautes Études Sci. Publ. Math. (1984), 163–188.
[69] F. Ledrappier, *Positivity of the exponent for stationary sequences of matrices*, Lyapunov exponents (Bremen, 1984), 1986, pp. 56–73.
[70] F. Ledrappier and J.-M. Strelcyn, *A proof of the estimation from below in Pesin's entropy formula*, Ergodic Theory Dynam. Systems **2** (1982), 203–219 (1983).
[71] F. Ledrappier and P. Walters, *A relativised variational principle for continuous transformations*, J. London Math. Soc. (2) **16** (1977), 568–576.
[72] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2) **122** (1985), 509–539.

[73] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (1985), 540–574.
[74] J. W. Lewis, *Infinitesimal rigidity for the action of* $\mathrm{SL}(n, \mathbf{Z})$ *on* $\mathbf{T}^n$, Trans. Amer. Math. Soc. **324** (1991), 421–445.
[75] A. Lubotzky, S. Mozes, and M. S. Raghunathan, *The word and Riemannian metrics on lattices of semisimple groups*, Inst. Hautes Études Sci. Publ. Math. (2000), 5–53.
[76] A. Manning, *There are no new Anosov diffeomorphisms on tori*, Amer. J. Math. **96** (1974), 422–429.
[77] G. A. Margulis, *Non-uniform lattices in semisimple algebraic groups*, Lie groups and their representations (Proc. Summer School on Group Representations of the Bolyai János Math. Soc., Budapest, 1971), 1975, pp. 371–553.
[78] G. A. Margulis, *Non-uniform lattices in semisimple algebraic groups*, Lie groups and their representations (proc. summer school on group representations of the bolyai jános math. soc., budapest, 1971), 1975, pp. 371–553.
[79] G. A. Margulis, *Factor groups of discrete subgroups and measure theory*, Funktsional. Anal. i Prilozhen. **12** (1978), 64–76.
[80] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, Berlin, 1991.
[81] G. A. Margulis and N. Qian, *Rigidity of weakly hyperbolic actions of higher real rank semisimple Lie groups and their lattices*, Ergodic Theory Dynam. Systems **21** (2001), 121–164.
[82] G. A. Margulis and G. M. Tomanov, *Invariant measures for actions of unipotent groups over local fields on homogeneous spaces*, Invent. Math. **116** (1994), 347–392.
[83] Y. Matsushima and S. Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, Ann. of Math. (2) **78** (1963), 365–416.
[84] J. Moser, *On a theorem of Anosov*, J. Differential Equations **5** (1969), 411–440.
[85] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J., 1973. Annals of Mathematics Studies, No. 78.
[86] G. D. Mostow and T. Tamagawa, *On the compactness of arithmetically defined homogeneous spaces*, Ann. of Math. (2) **76** (1962), 446–463.
[87] A. Navas, *Actions de groupes de Kazhdan sur le cercle*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 5, 749–758 (French, with English and French summaries).
[88] A. Navas, *Groups of circle diffeomorphisms*, Spanish, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2011.
[89] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trans. Mosc. Math. Soc. **19** (1968), 197–221.
[90] K. Parwani, *Actions of* $\mathrm{SL}(n, \mathbb{Z})$ *on homology spheres*, Geom. Dedicata **112** (2005), 215–223.
[91] Ja. B. Pesin, *Families of invariant manifolds that correspond to nonzero characteristic exponents*, Math. of the USSR, Izvestija **10** (1978), 1261–1305.
[92] Ja. B. Pesin, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Russ. Math. Surveys **32** (1977), 55–114.
[93] L. Polterovich, *Growth of maps, distortion in groups and symplectic geometry*, Invent. Math. **150** (2002), 655–686.
[94] G. Prasad, *Strong rigidity of* $\mathbf{Q}$*-rank* 1 *lattices*, Invent. Math. **21** (1973), 255–286.
[95] N. T. Qian, *Anosov automorphisms for nilmanifolds and rigidity of group actions*, Ergodic Theory Dynam. Systems **15** (1995), 341–359.
[96] N. Qian, *Infinitesimal rigidity of higher rank lattice actions*, Comm. Anal. Geom. **4** (1996), 495–524.
[97] N. Qian, *Tangential flatness and global rigidity of higher rank lattice actions*, Trans. Amer. Math. Soc. **349** (1997), 657–673.
[98] N. Qian and C. Yue, *Local rigidity of Anosov higher-rank lattice actions*, Ergodic Theory Dynam. Systems **18** (1998), 687–702.
[99] M. S. Raghunathan, *On the first cohomology of discrete subgroups of semisimple Lie groups*, Amer. J. Math. **87** (1965), 103–139.
[100] M. S. Raghunathan, *A proof of Oseledec's multiplicative ergodic theorem*, Israel J. Math. **32** (1979), 356–362.
[101] M. Ratner, *Invariant measures for unipotent translations on homogeneous spaces*, Proc. Nat. Acad. Sci. U.S.A. **87** (1990), 4309–4311.
[102] M. Ratner, *On measure rigidity of unipotent subgroups of semisimple groups*, Acta Math. **165** (1990), 229–309.
[103] M. Ratner, *Invariant measures and orbit closures for unipotent actions on homogeneous spaces*, Geom. Funct. Anal. **4** (1994), 236–257.

[104] V. A. Rohlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Translation **10** (1952), 1–52.
[105] D. Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Brasil. Mat. **9** (1978), 83–87.
[106] A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. colloq. function theory, bombay, 1960), 1960, pp. 147–164.
[107] N. A. Shah, *Limit distributions of polynomial trajectories on homogeneous spaces*, Duke Math. J. **75** (1994), 711–732.
[108] Ja. G. Sinaĭ, *Classical dynamic systems with countably-multiple Lebesgue spectrum. II*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 15–68.
[109] Ya. G. Sinai, *Markov partitions and c-diffeomorphisms*, Funct. Anal. Appl. **2** (1968), 61–82.
[110] G. Stuck, *Minimal actions of semisimple groups*, Ergodic Theory Dynam. Systems **16** (1996), 821–831.
[111] M. Viana, *Lectures on Lyapunov exponents*, Cambridge Studies in Advanced Mathematics, vol. 145, Cambridge University Press, Cambridge, 2014.
[112] P. Walters, *A dynamical proof of the multiplicative ergodic theorem*, Trans. Amer. Math. Soc. **335** (1993), 245–257.
[113] A. Weil, *On discrete subgroups of Lie groups. II*, Ann. of Math. (2) **75** (1962), 578–602.
[114] A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. (2) **80** (1964), 149–157.
[115] S. Weinberger, $\mathrm{SL}(n,\mathbb{Z})$ *cannot act on small tori*, Geometric topology (Athens, GA, 1993), 1997, pp. 406–408.
[116] S. Weinberger, *Some remarks inspired by the* C^0 *Zimmer program*, Geometry, rigidity, and group actions, 2011, pp. 262–282.
[117] D. Witte, *Arithmetic groups of higher* $\mathbf{Q}$*-rank cannot act on* 1*-manifolds*, Proc. Amer. Math. Soc. **122** (1994), 333–340.
[118] D. Witte Morris, *Ratner's theorems on unipotent flows*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2005.
[119] D. Witte Morris, *Introduction to arithmetic groups*, Deductive Press, 2015.
[120] S. Ye, *Euler characteristics and actions of automorphism groups of free groups*, Algebr. Geom. Topol. **18** (2018), 1195–1204.
[121] S. Ye, *The action of matrix groups on aspherical manifolds*, Algebr. Geom. Topol. **18** (2018), 2875–2895.
[122] R. J. Zimmer, *Arithmetic groups acting on compact manifolds*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), 90–92.
[123] R. J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.
[124] R. J. Zimmer, *Volume preserving actions of lattices in semisimple groups on compact manifolds*, Inst. Hautes Études Sci. Publ. Math. **59** (1984), 5–33.
[125] R. J. Zimmer, *Actions of semisimple groups and discrete subgroups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 1987, pp. 1247–1258.
[126] R. J. Zimmer, *Lattices in semisimple groups and invariant geometric structures on compact manifolds*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), 1987, pp. 152–210.
[127] R. J. Zimmer, *Spectrum, entropy, and geometric structures for smooth actions of Kazhdan groups*, Israel J. Math. **75** (1991), 65–80.
[128] B. P. Zimmermann, $\mathrm{SL}(n,\mathbb{Z})$ *cannot act on small spheres*, Topology Appl. **156** (2009), 1167–1169.

UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA
Email address: `awb@uchicago.edu`