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DENSITY OF ACCESSIBILITY FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH ONE-DIMENSIONAL CENTER

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For Yahsa Pesin, on his sixtieth birthday

Abstract. It is shown that stable accessibility property is C^r-dense among partially hyperbolic diffeomorphisms with one-dimensional center bundle, for $r \geq 2$, volume preserving or not. This establishes a conjecture by Pugh and Shub for these systems.

1. **Introduction.** Partially hyperbolic systems are diffeomorphisms $f\colon M \to M$ with a Tf-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that Tf is contracting on E^s, expanding on E^u, and has an intermediate behavior on E^c. For more details, see §2.

Accessibility is a concept arising from control theory (see for instance [9] and [18]). A pair of distributions X and Y on a manifold has the *accessibility property* if one can join any two points in the manifold by a path which is piecewise tangent to either X or Y. See also [13] for an account of this. *Essential accessibility* is the weaker property that if A and B are measurable sets with positive measure, then some point of A must be joined to some point of B by such a path.

It was Brin and Pesin [1] (see also Sacksteder [17]) who first suggested in 1974 that accessibility for the pair of distributions E^s and E^u should be relevant in the context of ergodic theory, more precisely, to study ergodic properties of partially hyperbolic systems. A partially hyperbolic diffeomorphism is said to be accessible or essentially accessible if the associated distributions E^s and E^u have the corresponding property. These properties are crucial in efforts to use the Hopf method to prove ergodicity of a partially hyperbolic diffeomorphism.

Around 1995, Pugh and Shub developed a program to obtain ergodicity for (at least) a C^1-open and C^r-dense set of partially hyperbolic systems [12, 13]. More precisely they formulated the following:

Conjecture 1. *Stable ergodicity is C^r-dense among partially hyperbolic diffeomorphisms for $r \geq 2$.*

A stably ergodic diffeomorphism is a C^2-diffeomorphism such that all C^1-perturbations among C^2-volume preserving diffeomorphisms are ergodic.

Pugh and Shub divided this conjecture into two:

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Conjecture 2. *Essential accessibility implies ergodicity for a C^2-volume preserving partially hyperbolic diffeomorphism.*

Conjecture 3. *Stable accessibility is C^r-dense among partially hyperbolic diffeomorphisms, volume preserving or not.*

Conjectures 2 and 3 have been attacked by many authors, and there are now many partial results about them, an account of which may be found, for instance, in [6] or [16]. Let us recall some of these advances:

Conjecture 2 was proved by Brin and Pesin in [1] under the additional hypotheses of Lipschitzness of E^c, *dynamical coherence* (that is, unique integrability of the center bundle), and a technical condition on the rates of contraction/expansion of the invariant bundles, called *center bunching*, which requires the action of Tf on E^c to be close to conformal.

It took another 20 years until Grayson, Pugh and Shub [5] obtained the first result without using Lipschitzness of E^c, by proving Conjecture 2 for perturbations of the time-one map of the geodesic flow of a surface of constant negative curvature. This provided the first non-hyperbolic stably ergodic example.

Their result was been extended in several stages during the last decade [19, 13, 14]. The strongest result to date is that Conjecture 2 is true with one additional hypothesis, namely a mild form of center bunching, which we describe in the next section. It holds whenever $\dim E^c = 1$. This result was proved by Burns and Wilkinson [2], and, in the case of one dimensional center, by F. and M. Rodriguez Hertz and Ures [15].

Conjecture 3 was proved by Dolgopyat and Wilkinson with C^r-density weakened to C^1-density [4]. Many authors, starting with Brin and Pesin [1], have proved C^∞-density of stable accessibility within special classes of partially hyperbolic systems such as the time one maps of Anosov flows and extensions of Anosov diffeomorphisms by compact Lie groups. But general results about accessibility have mostly been restricted to the case of one dimensional center.

Didier showed that accessibility is C^1-open [3] when the center distribution is one-dimensional. This is still an open question in the case of higher dimensional center. In [11], Niţică and Török found a C^r-dense set of stably accessible diffeomorphisms among the following ones: r-normally hyperbolic diffeomorphisms with one-dimensional center distribution, having two close compact periodic leaves, volume preserving or not.

F. and M. Rodriguez Hertz and Ures [15] found a C^∞-dense set of stably accessible diffeomorphisms among the C^r-volume preserving partially hyperbolic diffeomorphisms with one-dimensional center distribution, proving the volume preserving part of Conjecture 3 for this case.

In this paper we extend the arguments in [15] to show that accessible diffeomorphisms are C^r-dense in the space of all C^r-partially hyperbolic diffeomorphisms with one-dimensional center, thereby completing the proof of Conjecture 3 for the case of one-dimensional center. The arguments in the present paper also apply to the volume preserving case, since the various perturbations that we consider can all be achieved while preserving volume.

2. **Preliminaries.** Let $f\colon M \to M$ be a diffeomorphism of a compact connected manifold M. We say that f is *partially hyperbolic* if the following holds. First, there is a nontrivial splitting of the tangent bundle, $TM = E^s \oplus E^c \oplus E^u$ that is invariant

under the derivative map Tf. Further, there is a Riemannian metric for which we can choose continuous positive functions ν, $\hat{\nu}$, γ and $\hat{\gamma}$ defined on M with

$$\nu, \hat{\nu} < 1 \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1} \tag{2.1}$$

such that, for any unit vector $v \in T_pM$,

$$\|Tfv\| < \nu(p), \qquad \text{if } v \in E^s(p), \tag{2.2}$$

$$\gamma(p) < \|Tfv\| < \hat{\gamma}(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{2.3}$$

$$\hat{\nu}(p)^{-1} < \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{2.4}$$

Denote by $PHD_1^r(M)$ the set of (not necessarily volume-preserving) C^r-partially hyperbolic diffeomorphisms of M with 1-dimensional center distribution. Recall that Conjecture 3 is already proved for 1-dimensional center distribution in the volume preserving setting [15]. Unless otherwise specified we give $PHD_1^r(M)$ the C^r-topology. It is convenient to let s, c and u denote the dimensions of E^s, E^c, and E^u, respectively. When necessary we use a subscript to indicate the dependence of the bundles on the diffeomorphism.

We say that f is *center bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\max\{\nu, \hat{\nu}\} < \gamma\hat{\gamma}. \tag{2.5}$$

Center bunching means that the hyperbolicity of f dominates the nonconformality of Tf on the center. Inequality (2.5) always holds when $Tf|_{E^c}$ is conformal. For then we have $\|T_pfv\| = \|T_pf|_{E^c(p)}\|$ for any unit vector $v \in E^c(p)$, and hence we can choose $\gamma(p)$ slightly smaller and $\hat{\gamma}(p)^{-1}$ slightly bigger than

$$\|T_pf|_{E^c(p)}\|.$$

By doing this we may make the ratio $\gamma(p)/\hat{\gamma}(p)^{-1} = \gamma(p)\hat{\gamma}(p)$ arbitrarily close to 1, and hence larger than both $\nu(p)$ and $\hat{\nu}(p)$. In particular, center bunching holds whenever E^c is one-dimensional.

The bundles E^u and E^s are uniquely integrable. As usual $\mathcal{W}^u$ and $\mathcal{W}^s$ will denote the foliations to which they are tangent. There are partially hyperbolic diffeomorphisms for which E^c is not integrable, but none of the known examples has one dimensional center. The question of whether the center distribution must be uniquely integrable if it is one dimensional is still open, even for partially hyperbolic diffeomorphisms of three dimensional manifolds.

We assume that we have a Riemannian metric on M adapted to f so that the inequalities at the beginning of this section hold. Distance with respect to this metric will be denoted by $d(\cdot, \cdot)$.

If $\mathcal{W}$ is a foliation of M, $\mathcal{W}_\rho(x)$ will denote the set of points that can be reached from x by a C^1-path of length less than ρ tangent to the foliation; this set is a disc for small enough ρ. We define $\mathcal{W}_{loc}(x)$ to be $\mathcal{W}_R(x)$ for a suitably small R. The radii such as ϵ and δ considered in the paper are, of course, much smaller than R.

3. **Remarks on accessibility.** In this section we give a brief survey of the basic properties of accessibility. Most of the results are taken from Didier's paper [3]. A partially hyperbolic diffeomorphism has the accessibility property if any two points are joined by a *us-path*. A us-path from x to y is a finite sequence of points $z_0, \ldots, z_m$ such that $z_0 = x$, $z_m = y$ and $z_i \in \mathcal{W}^u(z_{i-1}) \cup \mathcal{W}^s(z_{i-1})$ for $1 \le i \le m$.

Given a partially hyperbolic diffeomorphism f, the accessibility class $AC(x, f)$ of a point X is the set of all points that can be joined to x by us-paths. Being joined by a us-path is an equivalence relation on the points of the manifold, so the accessibility classes are pairwise disjoint and partition the manifold. The diffeomorphism has the accessibility property if and only if there is just one equivalence class, which is the entire manifold.

Accessibility classes are either open, or have empty interior. To see this, suppose x is in the interior of the accessibility class A. Then there is an open set U such that $x \in U \subset A$. The union of the leaves of a foliation that pass through an open set is open. Thus $U' = \bigcup_{y \in U} W^u(y)$ and $U'' = \bigcup_{y \in U} W^s(y)$ are open sets. They lie inside A by the definition of an accessibility class. Similarly $\bigcup_{x \in U'} W^s(z)$ and $\bigcup_{z \in U''} W^u(z)$ are open sets that lie inside A. An inductive argument shows that the set of all points that are joined to a point of U by a us-path is open and lies in A. But this open set contains $AC(x, f)$, which is the whole of A.

We say that the foliations W^s and W^u are *jointly integrable* at a point x if there is $\delta > 0$ such that $W^s_{loc}(y) \cap W^u_{loc}(z) \neq \emptyset$ for all $y \in W^s_\delta(x)$ and all $z \in W^u_\delta(x)$. See Figure 1.

Lemma 3.1. [3, Lemma 5] *Suppose W^s and W^u are jointly integrable at x for a C^r-partially hyperbolic diffeomorphism f. Let D be the set of all points of the form $W^u_{loc}(y) \cap W^s_{loc}(z)$ where $y \in W^s_\delta(x)$ and $z \in W^u_\delta(x)$. Then D is a C^1 immersed disc that is everywhere tangent to $E^s \oplus E^u$.*

Proof. By introducing suitable coordinates, we can reduce to the situation where W^s and W^u are continuous foliations of $\mathbb{R}^{c+s+u}$ with C^r leaves, tangent to distributions E^s and E^u. We may assume:

1. E^s and E^u are close to the $\mathbb{R}^s$ and $\mathbb{R}^u$ coordinate distributions respectively.
2. $W^s(0,0,0) = \{0\} \times \mathbb{R}^s \times \{0\}$ and $W^u(0,0,0) = \{(0,0)\} \times \mathbb{R}^u$.
3. $W^s_{loc}(0, y, 0) \cap W^u_{loc}(0, 0, z)$ is a single point for any $y \in \mathbb{R}^s$ and any $z \in \mathbb{R}^u$.

Define $\eta : \mathbb{R}^s \times \mathbb{R}^u \to \mathbb{R}^{c+s+u}$ so that $\eta(y, z)$ is the point of intersection of $W^s_{loc}(0, y, 0)$ and $W^u_{loc}(0, 0, z)$. It is easily shown that η is a continuous function whose image is the graph of a function $h : \mathbb{R}^s \times \mathbb{R}^u \to \mathbb{R}^{c+s+u}$. For each $(y, z) \in \mathbb{R}^s \times \mathbb{R}^u$, let $V^s(y, z)$ and $V^u(y, z)$ be the projections of $W^s(h(y, z))$ and $W^u(h(y, z))$ to $\{0\} \times \mathbb{R}^s \times \mathbb{R}^u$, which we identify with $\mathbb{R}^s \times \mathbb{R}^u$ in the obvious way. The leaves $W^s(h(y, z))$ and $W^u(h(y, z))$ are C^r and lie in the image of η; hence the restrictions of h to $V^s(y, z)$ and $V^u(y, z)$ are C^r for any $(y, z) \in \mathbb{R}^s \times \mathbb{R}^u$.

We can now apply Journé's theorem [7] to see that the function h is at least C^1. The statement in Journé's paper assumes that V^s and V^u are foliations, but all that is required is that $V^s(y, z)$ and $V^u(y, z)$ be uniformly transverse and depend continuously on (y, z). Journé assumes that the restrictions of h to the manifolds V^s and V^u are $C^{r,\alpha}$, where r is a positive integer and $0 < \alpha < 1$, and concludes that h is $C^{r,\alpha}$. The same conclusion holds when $r = 1$ and $\alpha = 0$; the proof in this case is similar to the proof that a function with continuous first order partial derivatives is C^1. In either case, we see that h is at least C^1. □

It follows from the previous lemma that if W^s and W^u are jointly integrable at every point in an accessibility class, then the class is an immersed submanifold of dimension $s + u$ tangent to $E^s \oplus E^u$ [3, Lemma 6].

In the case when E^c is one dimensional, such an accessibility class has codimension one. Furthermore, in this case an argument which goes back to the 1970s — the Brin quadrilateral argument — shows that $AC(x, f)$ is open if W^s and W^u are not jointly integrable at a point x. Thus, in the case (considered in this paper) of

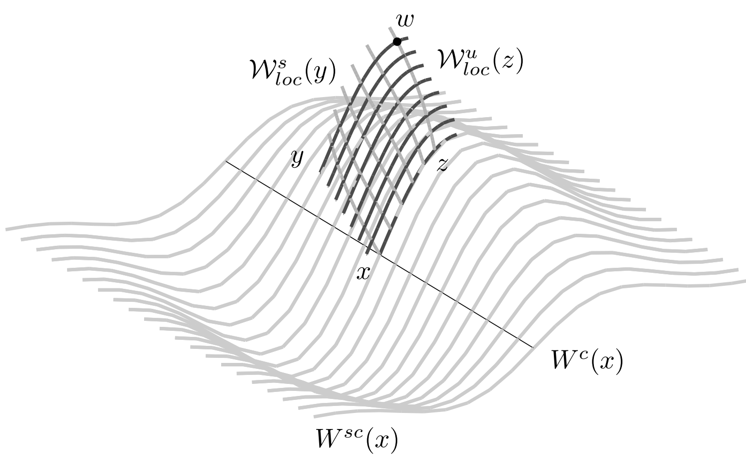

FIGURE 1. W^s_f and W^u_f are jointly integrable at x

one dimensional center, there are precisely two types of accessibility classes: open sets and immersed codimension one submanifolds tangent to $E^s \oplus E^u$. Each open accessibility class contains a point at which W^s and W^u are not jointly integrable; the other accessibility classes do not.

Proposition 3.2. [3, Proposition 5] *Let f be a partially hyperbolic diffeomorphism with one dimensional center. If $AC(x, f)$ is open, then there exist a C^1-neighborhood $\mathcal{U}$ of f and $\epsilon > 0$ such that $B(x, \epsilon) \subset AC(x, g)$ for all $g \in \mathcal{U}$.*

Proof. Since $AC(x, f)$ is open, there is a point $y \in AC(x, f)$ at which the foliations W^s_f and W^u_f are not jointly integrable. The local stable and unstable manifolds depend continuously on the point and the diffeomorphism. Hence there are neighborhoods U of y and $\mathcal{U}_1$ of f in the C^1-topology such that if $g \in \mathcal{U}_1$, then W^s_g and W^u_g are not jointly integrable at any point $z \in U$ for any $g \in \mathcal{U}_1$. Now we can choose another neighborhood $\mathcal{U}_2$ of f in the C^1-topology such that x is joined to some point in U by a us-path for every $g \in \mathcal{U}_2$. Finally we take $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$. □

We denote the union of all the non-open accessibility classes by $\Gamma(f)$. Observe that $\Gamma(f)$ is a compact set, since it lies in a compact manifold and its complement is open. It is laminated by codimension one submanifolds tangent to $E^s \oplus E^u$. These laminae are accessibility classes. If $x \in \Gamma(f)$, we denote by $\Gamma(x, f)$ the lamina of $\Gamma(f)$ that contains x. Note that if $x \in \Gamma(f)$, then W^s_f and W^u_f are jointly integrable at x and $\Gamma(x, f) = AC(x, f)$. The accessibility property for f is equivalent to $\Gamma(f) = \emptyset$.

Remark 3.3. Didier's result in [3] that the set of diffeomorphisms in $PHD^r_1(M)$ with the accessibility property is open follows from Proposition 3.2. Indeed, if this set were not open, we could find a diffeomorphism $f \in PHD^r_1(M)$ that is accessible and a sequence $f_n \to f$ in $PHD^r_1(M)$ such that $\Gamma(f_n) \neq \emptyset$ for each n. Choose a point $x_n \in \Gamma(f_n)$ for each n. Since M is compact, we may assume (by passing to a subsequence if necessary) that the sequence x_n converges to a point x. Since

$AC(x,f)$ is open (in fact the whole of M) it follows from Proposition 3.2 that $AC(x_n,f_n)$ is open for all large n, which is a contradiction.

4. **The main result and an outline of its proof.** Denote by $\mathcal{A}$ the set of all diffeomorphisms in $PHD_1^r(M)$ with the accessibility property. Didier [3] showed that $\mathcal{A}$ is a C^1-open subset of $PHD_1^r(M)$. Note that the assumption of one dimensional center is crucial in Didier's work. It is not known whether accessibility is an open property when the center is higher dimensional.

In this paper we prove the following result.

Theorem 4.1. *$\mathcal{A}$ is C^r-dense in $PHD_1^r(M)$.*

We extend the arguments in [15] where the analogous result is proved for the subspace of volume preserving diffeomorphisms in $PHD_1^r(M)$. The proof in this paper can also be adapted to the volume preserving case; all of the perturbations that we need can be made in a volume preserving way. Together with Didier's result, Theorem 4.1 and its analogue in [15] establish the conjecture of Pugh and Shub about the density of accessibility (Conjecture 3) in the case when the center bundle E^c is one dimensional.

Denote by $\mathcal{K}(M)$ the space of compact subsets of M with Hausdorff distance. We say that a function $\Phi : PHD_1^r(M) \to \mathcal{K}(M)$ is upper-semicontinuous with respect to the C^r-topology if we have $x \in \Phi(f)$ whenever there are sequences $x_n \to x$ in M and $f_n \to f$ in the C^r-topology such that $x_n \in \Phi(f_n)$ for all n.

Theorem 4.2. *The map $\Gamma\colon PHD_1^r(M) \to \mathcal{K}(M)$ is upper-semicontinuous with respect to the C^r-topology on $PHD_1^r(M)$ for any $r \geq 1$.*

Proof of Theorem 4.2. We prove upper-semicontinuity of Γ with respect to the C^1-topology, since this implies upper-semicontinuity with respect to the C^r-topology for $r > 1$. It is enough to show that if $x \notin \Gamma(f)$, then there exist a C^1-neighborhood $\mathcal{V}$ of f in $PHD_1^r(M)$ and $\epsilon > 0$ such that $y \notin \Gamma(g)$ for all $y \in B(x,\epsilon)$ and all $g \in \mathcal{V}$. But this is an immediate consequence of Proposition 3.2 and the definition of Γ. □

It is a classical result that the set of continuity points of an upper-semicontinuous function such as Γ is residual, see e.g. §39.IV.2 in [8]. In particular, it is dense. Theorem 4.1 now follows immediately from the next result.

Theorem 4.3. *If f is a continuity point of Γ, then $\Gamma(f) = \emptyset$.*

The rest of this paper is dedicated to proving Theorem 4.3. Here is an outline of its proof:

In the first place, we show that there is a C^r-dense set of diffeomorphisms of $PHD_1^r(M)$ for which the accessibility class of every periodic point is open, this is, $\Gamma(g) \cap \mathrm{Per}(g) = \emptyset$ for a C^r-dense set of $g \in PHD_1^r(M)$ (Proposition 6.2). Recall that if a point x is in $\Gamma(g)$, then W_g^s and W_g^u are jointly integrable at x. So, in order to get this dense set we use an unweaving method (see Lemma 6.1), which allows us to break up the joint integrability of W^s and W^u on periodic orbits. In this way, we "open" the accessibility class of a periodic point by means of a C^r-small perturbation. The unweaving method, in turn, is based on the Keepaway Lemma (Lemma 5.2) which may be found in Section 5.

On the other hand, in Section 8, we assume there exists a continuity point f of Γ with $\Gamma(f) \neq \emptyset$. Under this hypothesis, we find an open set $\mathcal{N}$ in $PHD_1^r(M)$ such that every $h \in \mathcal{N}$ has a periodic point with nonopen accessibility class, that is, $\Gamma(h) \cap \mathrm{Per}(h) \neq \emptyset$ for every $h \in \mathcal{N}$ (Lemma 8.2). We therefore obtain a contradiction.

5. **The Keepaway Lemma.** Let f be a diffeomorphism preserving a foliation $\mathcal{W}$ tangent to a continuous sub-bundle E of TM. Denote by $\mathcal{W}(x)$ the leaf of $\mathcal{W}$ through x and by $\mathcal{W}_\epsilon(x)$ the set of points that are reached from x by a curve contained in $\mathcal{W}(x)$ of length less than ϵ.

The following lemma was already proved by Mañé [10, Lemma 5.2.] when the dimension of E is 1. The general case is presented in [15]. We reproduce the proof since it is quite short and the lemma is fundamental to this paper.

Given a (small) embedded manifold V transverse to $\mathcal{W}$ whose dimension equals the codimension of E and $\delta > 0$, define

$$B_\delta(V) = \bigcup_{y \in V} \mathcal{W}_\delta(y).$$

We will always assume that V and δ are chosen so that the discs $\mathcal{W}_{5\delta}(y)$ for $y \in V$ are pairwise disjoint. There is no need for V to be connected.

Definition 5.1. Let us say that the bundle E is μ-uniformly expanded by Tf if there is a constant $\mu > 1$ such that $\|Tf^{-1}|_E\| < \mu^{-1} < 1$.

Lemma 5.2 (Keepaway Lemma)**.** *Assume that the bundle E is μ-uniformly expanded by Tf. Let $N > 0$ be such that $\mu^N > 5$ and let V be a small manifold transverse to $\mathcal{W}$ whose dimension is complementary to that of the leaves of $\mathcal{W}$. Suppose that for some $\epsilon > 0$ we have*

$$f^n(B_{5\epsilon}(V)) \cap B_\epsilon(V) = \emptyset \qquad \text{for } n = 1, \dots, N.$$

Then for each $x \in M$ there is a point $z \in \mathcal{W}_\epsilon(x)$ such that $f^n(z) \notin B_\epsilon(V)$ for all $n \geq 1$.

Proof. We shall construct a sequence of closed discs $D_0, D_1, D_2, \dots$ starting from $D_0 = \overline{\mathcal{W}_\epsilon(x)}$ such that $f^{-1}(D_n) \subset D_{n-1}$ for all $n > 0$ and $D_i \cap B_\epsilon(V) = \emptyset$ for all $i \leq n$. Then z can be chosen to be any point in

$$\bigcap_{n=0}^{\infty} f^{-n}(D_n).$$

In fact this intersection will consist of a unique point in our construction.

Observe that μ-uniform expansion of E means that, for any given integer $k \geq 1$, any point $p \in M$ and any $\delta > 0$, we have

$$\overline{\mathcal{W}_\delta(f^k(p))} \subset \overline{\mathcal{W}_{\mu^k\delta}(f^k(p))} \subset f^k(\mathcal{W}_\delta(p)). \tag{5.6}$$

The construction is as follows:

0. Set $D_0 = \overline{\mathcal{W}_\epsilon(w_0)}$, where $w_0 = x$.
1. If $n < N$, put $D_n = f^n(D_0)$.
2. For the N^{th} iterate, we still have $f^N(D_0) \cap B_\epsilon(V) = \emptyset$. Since $\mu^N > 5$, we see from the right half of (5.6) with $p = w_0$, $k = N$ and $\delta = \epsilon$ that $f^N(D_0)$ contains the round ball of radius 5ϵ centered at $f^N(w_0)$, i.e.,

$$\overline{\mathcal{W}_{5\epsilon}(f^N(w_0))} \subset f^N(D_0).$$

We set $D_N = \overline{\mathcal{W}_{5\epsilon}(f^N(w_0))}$.
3. For $n > N$, we continue to set $D_n = \overline{\mathcal{W}_{5\epsilon}(f^n(w_0))}$ until we reach $n = n_1$, where n_1 is the first n such that

$$\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))} \cap B_\epsilon(V) \neq \emptyset.$$

We get $D_n \subset f(D_{n-1})$ for $N < n < n_1$ from (5.6) with $\delta = 5\epsilon$, $k = 1$ and $p = f^{n-1}(w_0)$.

4. For the n_1^{th} iterate, we cannot take $\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$, since this disc intersects $B_\epsilon(V)$. But there is a point $w_{n_1} \in \overline{\mathcal{W}_{4\epsilon}(f^{n_1}(w_0))}$ such that $\overline{\mathcal{W}_\epsilon(w_{n_1})} \subset B_{5\epsilon}(V) \setminus B_\epsilon(V)$. Indeed, let $y_{n_1} \in V$ be the center of the leaf in $B_\epsilon(V)$ that intersects $\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$. If y_{n_1} lies outside $\overline{\mathcal{W}_{2\epsilon}(f^{n_1}(w_0))}$, then we can take $w_{n_1} = f^{n_1}(w_0)$. If y_{n_1} lies inside $\overline{\mathcal{W}_{2\epsilon}(f^{n_1}(w_0))}$, then w_{n_1} can be any point in $\overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$ whose distance from $f^{n_1}(w_0))$ is 4ϵ.

   Choose $D_{n_1} = \overline{\mathcal{W}_\epsilon(w_{n_1})}$. We get $D_{n_1} \subset f(D_{n_1-1})$ using (5.6) with $\delta = 5\epsilon$, $k = 1$ and $p = f^{n_1-1}(w_0)$ and the fact that $D_{n_1} \subset \overline{\mathcal{W}_{5\epsilon}(f^{n_1}(w_0))}$.
5. Now, go to Step 1, replace D_0 by D_{n_1}, and continue the construction with the obvious modifications.

This algorithm gives the desired sequence of discs, and then the point z, proving the lemma. □

The Keepaway Lemma gives us an abundance of nonrecurrent points.

Corollary 1. *Suppose $f\colon M \to M$ has an invariant foliation $\mathcal{W}$ tangent to a bundle that is uniformly expanded by Tf. Then the set $\{z : z \notin \omega(z)\}$ of points that are nonrecurrent in the future is dense in every leaf of $\mathcal{W}$.*

Proof. Let y be a point in M. If y is not periodic, we can choose a transversal V to $\mathcal{W}$ that passes through y such that the hypothesis of the previous lemma is satisfied for any small enough $\epsilon > 0$; the lemma then gives us a point $z \in \mathcal{W}_\epsilon(y)$ that is not forward recurrent. If y is periodic, no other point of $\mathcal{W}(y)$ is periodic, so for any small $\epsilon > 0$ we can choose a nonperiodic point $y' \in \mathcal{W}_{\epsilon/2}(y)$ and then find a point $z \in \mathcal{W}_{\epsilon/2}(y')$ that is nonrecurrent in the future. □

6. **Unweaving.** The results of the previous section will allow us to break up the joint integrability of E^s and E^u along a minimal set. In [15, Lemma A.4.3] it is shown that if we have an f-periodic orbit K, then we can make a C^r-perturbation such that $g = f$ on K and there is no joint integrability of E^s and E^u along K. Here we extend this result to the case when K is a minimal set.

Lemma 6.1. *Let $K \subset \Gamma(f)$ be a minimal set for a diffeomorphism $f \in PHD_1^r(M)$. Then we can find $g \in PHD_1^r(M)$ as close to f in the C^r-topology as we wish such that $f|_K = g|_K$ and $AC(x, g)$ is open for some point $x \in K$.*

Proof. We construct g by perturbing f in the complement of the closed f-invariant set K. This ensures that K remains invariant under g.

The construction is an application of the Brin quadrilateral argument. We choose a closed us-quadrilateral with corners x, y, z, w such that $x \in K$ and $y \notin K$, as in Figure 1. The quadrilateral is constructed so that there are radii $\rho, \rho_1, \rho_2, \rho_3, \rho_4 > 0$ such that $B(y, \rho) \cap K = \emptyset$ and:

1. $w \in \overline{\mathcal{W}^s_{\rho_1}(y)}$ and $f^n(\overline{\mathcal{W}^s_{\rho_1}(y)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 1$;
2. $y \in \overline{\mathcal{W}^u_{\rho_2}(x)}$ and $f^{-n}(\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 1$;
3. $z \in \overline{\mathcal{W}^s_{\rho_3}(x)}$ and $f^n(\overline{\mathcal{W}^s_{\rho_3}(x)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 0$;
4. $w \in \overline{\mathcal{W}^u_{\rho_4}(z)}$ and $f^{-n}(\overline{\mathcal{W}^u_{\rho_4}(z)}) \cap B(y, \rho) = \emptyset$ for any $n \geq 0$.

A perturbation that changes f only inside $B(y, \rho)$ leaves x, z and w joined by a us-path. It is easy to break the us-connection from x to w through y by composing f with a "push" in the central direction that is restricted to $B(y, \rho)$ (see Figure 2).

In order to create the desired quadrilateral, we first choose $x_0 \in K$. We can then apply Corollary 1 with $\mathcal{W} = \mathcal{W}^u$ to find a point $y \in \mathcal{W}^u_{loc}(x_0)$ that is not forward recurrent and is as close to x_0 as we wish. We make sure that $y \in \mathcal{W}^u_\beta(x_0)$, where β is very small compared to the radius R of the local stable and unstable manifolds. Since y is not forward recurrent, it does not belong to the minimal set K. Choose $\delta > 0$ small enough so that $K \cap B(y, \delta) = \emptyset$ and $f^n(y) \notin B(y, \delta)$ for $n \geq 1$.

We now choose the point x. It must belong to $\mathcal{W}^u_{loc}(y) \cap K$ and have the property that no other point of $\mathcal{W}^u_{loc}(y) \cap K$ is closer to y. Note that $f^n(x) \notin B(y, \delta)$ for all n. This is because $f^n(B(y, \delta)) \cap K = \emptyset$ for all n.

We now apply Corollary 1 with f replaced by f^{-1} to choose a point $z \in \mathcal{W}^s_{loc}(x)$ very close to x that is not backward recurrent. We choose V to be a disc transverse to $\mathcal{W}^s$ that contains $\mathcal{W}^u_{loc}(x)$ and $\epsilon \ll \delta$ small enough so that the stable discs of radius 5ϵ centered at points of V are pairwise disjoint. We may assume that V was chosen so that $B_\epsilon(V)$ contains $\mathcal{W}^u_{2\beta}(y')$ for all points y' close enough to y. Lemma 5.2 gives us a point $z \in \mathcal{W}^s_\epsilon(x)$ whose backward orbit under f avoids $B_\epsilon(V)$.

Since $x \in K \subset \Gamma(f)$, x is a point of joint integrability of $E^s \oplus E^u$, and the points y and z are in the immersed codimension one submanifold $\Gamma(x, f)$. Hence $\mathcal{W}^s_{loc}(y)$ and $\mathcal{W}^u_{loc}(z)$ intersect in a unique point w. We may assume that ϵ was chosen small enough so that $w \in \mathcal{W}^u_{2\beta}(z)$. We now verify properties (1)–(4) above.

Let ρ_3 be the distance in $\mathcal{W}^s_{loc}(x)$ from x to z. Then $\rho_3 \leq \epsilon \ll \delta$. The stable manifold $\mathcal{W}^s_{loc}(x)$ contracts under forward iteration of f, so we have $d(f^n(x), f^n(z)) \ll \delta$ for all $n \geq 0$. Since $f^n(x) \notin B(y, \delta)$ for $n \geq 0$, we see that (3) holds as long as $\rho < \delta/2$.

The proof of (1) is similar. Let ρ_1 be the distance in $\mathcal{W}^s_{loc}(y)$ from y to w. We may assume that ϵ was chosen small enough so that $\rho_1 \ll \delta$. Since iteration of f contracts $\mathcal{W}^s_{loc}(y)$ and $f^n(y) \notin B(y, \delta)$ for $n \geq 1$, we see as before that (1) will hold if $\rho < \delta/2$.

To prove (4), let ρ_4 be the distance in $\mathcal{W}^u_{loc}(z)$ from z to w. Then $\rho_4 < 2\beta$. Since $\mathcal{W}^u_{loc}(z)$ contracts under iteration of f^{-1}, we see from the choices made above that (4) will hold if ρ is small enough.

Finally let ρ_2 be the distance in $\mathcal{W}^u_{loc}(x)$ from x to y. Then $\beta \geq \rho_2 > \delta$. Iteration of f^{-1} contracts $\mathcal{W}^u_{\rho_2}(x)$. Choose n_0 so that the diameter of $f^{-n}(\mathcal{W}^u_{\rho_2}(x))$ is less than $\delta/2$ for $n \geq n_0$. Then $f^{-n}(\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap B_{\delta/2}(y) = \emptyset$ for $n \geq n_0$, since otherwise $f^{-n}(x)$ would be a point of K in $B_\delta(y)$.

On the other hand, there is $\rho_2' < \rho$ such that such that $f^{-n}(\overline{\mathcal{W}^u_{\rho_2}(x)}) \subset \mathcal{W}^u_{\rho_2'}(y)$ for all $n \geq 1$. This means that $f^{-n}(\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap \overline{\mathcal{W}^u_{\rho_2 - \rho_2'}(y)} = \emptyset$, for otherwise $f^{-n}(x)$ would be a point of $K \cap \mathcal{W}^u_{loc}(y)$ closer to y than x. It now follows from a compactness argument that we can choose a positive $\rho < \delta/2$ such that $f^{-n}(\overline{\mathcal{W}^u_{\rho_2}(x)}) \cap B_\rho(y) = \emptyset$ for $1 \leq n \leq n_0$. Property (2) holds for any such $\rho > 0$.

Let us consider a C^r-perturbation of f of the form $g = f \circ h$, where $\text{supp}(h) \subset B(y, \rho)$ see Figure 2. Recall that $B(y, \rho) \cap K = \emptyset$, so this implies that $f = g$ on K. We produce a push so that

$$W^s_{g,loc}(w) \cap W^u_{f,loc}(y) = \emptyset \tag{6.7}$$

See Figure 2. Now, Properties (1)–(4) above imply that $\mathcal{W}^u_{g,loc}(y) = \mathcal{W}^u_{f,loc}(y)$, $\mathcal{W}^s_{g,loc}(z) = \mathcal{W}^s_{f,loc}(z)$ and $\mathcal{W}^u_{g,loc}(z) = \mathcal{W}^s_{g,loc}(z)$, so we do not change the fact that there is a us-path from x to z to w, but we do change the local stable disc of w. Now, we have $x \in \mathcal{W}^s_{g,\epsilon}(z)$ and $w \in \mathcal{W}^u_{g,\epsilon}(z)$. If z belonged to $\Gamma(g)$, we would have

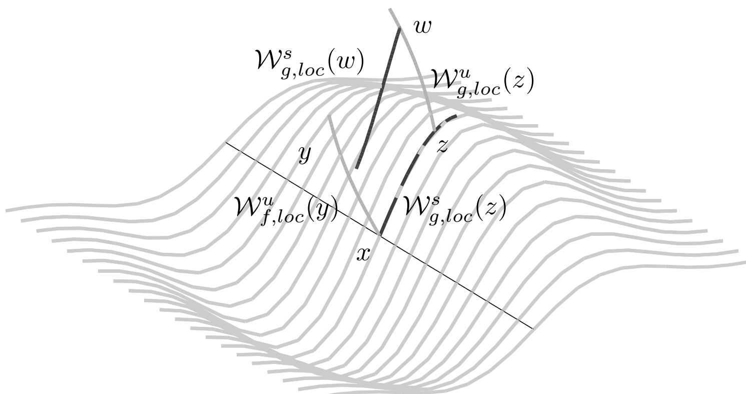

FIGURE 2. Lemma 6.1: Opening the accessibility class of x

$\mathcal{W}^s_{g,loc}(w) \cap \mathcal{W}^u_{g,loc}(x) \neq \emptyset$, due to joint integrability. But this this would contradict (6.7), since $\mathcal{W}^u_{g,loc}(y) = \mathcal{W}^u_{f,loc}(y)$. Hence $z \notin \Gamma(g)$ and $AC(z,g)$ is open. But $AC(z,g) = AC(x,g)$ since $z \in \mathcal{W}^s_\epsilon(x)$. Thus $AC(x,g)$ is open, as desired. □

Proposition 6.2. *$PHD^r_1(M)$ contains a C^r-dense set of diffeomorphisms with the property that the accessibility class of every periodic point is open.*

Proof. For $k \geq 1$ let $\mathcal{U}_k$ denote the set of all diffeomorphisms in $PHD^r_1(M)$ with the property that the periodic points of period k are all hyperbolic. Each $\mathcal{U}_k$ is open and C^r-dense by the Kupka-Smale theorem. The number of periodic points of period k is finite and constant on each component of $\mathcal{U}_k$. It is immediate from the previous lemma that $\mathcal{U}_k$ has a C^r-dense subset $\mathcal{U}'_k$ such that the accessibility class of every periodic point with period k for every diffeomorphism in $\mathcal{U}'_k$ is open. The set $\mathcal{U}'_k$ is C^1-open, by Proposition 3.2. The diffeomorphisms in $\bigcap_{k\geq 1} \mathcal{U}'_k$ have the property that the accessibility class of every period point is open. This set is residual by Baire's theorem, in particular, it is dense. □

7. Preliminary lemmas.

Henceforth we consider a fixed diffeomorphism $f \in PHD^r_1(M)$. Here we present two lemmas which will be used in the next section to show that if f is a continuity point of Γ, then $\Gamma(f) = \emptyset$. The lemmas apply to all diffeomorphisms close enough to f in the C^r-topology.

The first lemma is an application of the Anosov Closing Lemma. If $h \in PHD^r_1(M)$ and $x \in \Gamma(h)$, let us denote by $\Gamma_\rho(x,h)$ the set of points in the lamina $\Gamma(x,h)$ that can be reached from x by a C^1-path of length less than ρ. It follows from Theorem 4.2 and the continuous dependence of the stable and unstable bundles on the diffeomorphism that if $h_n \to h$ in the C^1 topology on $PHD^r_1(M)$, $x_n \to x$ in M and $x_n \in \Gamma_\rho(x_n, h_n)$ for each n, then $x \in \Gamma(h)$ and $\overline{\Gamma_\rho(x_n,h_n)} \to \overline{\Gamma_\rho(x,h)}$ in the Hausdorff topology.

Lemma 7.1. *There are a neighborhood $\mathcal{N}_1$ of f in $PHD^r_1(M)$, an integer $n_0 > 0$, and a radius $\rho > 0$ such that the following property holds for any $h \in \mathcal{N}_1$: If there is a point $y \in \Gamma(h)$ with $h^n(y) \in \Gamma_\rho(y,h)$ for some $n \geq n_0$, then there is a periodic point of h in $\Gamma(y,h)$ with period n.*

Proof. This follows from the Anosov Closing Lemma and the remarks preceding the lemma. □

We define a *central curve* to be a C^1-curve with unit speed that is tangent to E^c at all times. The following lemma states that if a (short) central curve σ_0 hits the disc $\Gamma_\rho(y,h)$, then any central curve close enough to σ_0 also hits $\Gamma_\rho(y,h)$. The length of the central curves and the proximity of their origins are uniform over a neighborhood of f in $PHD^r_1(M)$. The lemma involves an orientation for the one dimensional bundle E^c. This bundle may not be globally orientable, but all that is needed in the lemma is a local orientation in the neighborhood of a point.

Lemma 7.2. *For each $\rho > 0$, there are a neighborhood $\mathcal{N}_2$ of f in $PHD^r_1(M)$, and $\Delta > 0$ such that the following holds for any h in $\mathcal{N}_2$: Suppose that $x_0 \notin \Gamma(h)$ and $\sigma_0 : [0,\Delta] \to M$ is a central curve with $\sigma_0(0) = x_0$ and $y_0 = \sigma_0(t_0) \in \Gamma(h)$ for some $t_0 \in (0,\Delta]$. Suppose $\sigma : [0, 2\Delta] \to M$ is a central curve such that $d(\sigma(0), x_0) < d(x,\Gamma(h))$ and $\dot\sigma(0)$ is oriented in the same direction as $\dot\sigma_0(0)$. Then σ intersects $\Gamma_\rho(y_0,h)$ in a unique point y. Moreover if y_0 is the first point of σ_0 that is in $\Gamma(h)$, then y is the first point of σ that is in $\Gamma(h)$. See Figure 3.*

Proof. This is a consequence of the continuity of $h \mapsto E^c_h, E^s_h, E^u_h$ and of the transversality of central curves and the laminae of the set $\Gamma(h)$. □

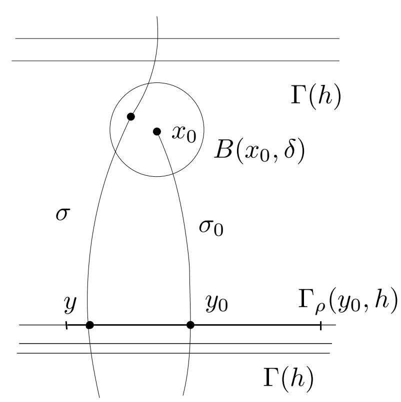

FIGURE 3. Lemma 7.2: Central curves hitting $\Gamma(y,h)$

8. **Creating a periodic point with non open accessibility class.** This section is devoted to proving Theorem 4.3. What we wish to show is that if $f \in PHD^r_1(M)$ is a continuity point of the function $\Gamma : PHD^r_1(M) \to \mathcal{K}(M)$, then we have $\Gamma(f) = \emptyset$. In order to do this, we assume that $\Gamma(f) \neq \emptyset$, and show that in that case there is an open set of diffeomorphisms with a periodic point whose accessibility class is not open (Lemma 8.2). This contradicts Proposition 6.2 above.

By continuity of Γ at f there is a neighborhood $\mathcal{N}_3$ of f in the C^r-topology on $PHD^r_1(M)$ such that for any $g \in \mathcal{N}_3$, we have

$$\sup\{dist(x,\Gamma(g)) : x \in \Gamma(f)\} < \Delta,$$

where Δ is the constant of Lemma 7.2.

Now, the set $\Gamma(f)$ is closed and invariant, hence it contains a minimal set K. Applying Lemma 6.1, we can choose g in $\mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the neighborhoods of f defined in the previous section, such that $AC(x_0, g)$ is open for some $x_0 \in K \subset \Gamma(f)$. Recall that $dist(x_0, \Gamma(g)) < \Delta$.

Let n_0, ρ and Δ be the numbers defined in Lemma 7.2 and Lemma 7.1, and choose an orientation for the one dimensional bundle E^c on the ball $B(x_0, \Delta)$.

Lemma 8.1. *There is $n > n_0$ such that $g^n(x_0) \in B(x_0, \delta/2)$ and $Tg^n(x_0)$ preserves orientation of E^c.*

Proof. The point x_0 is recurrent because $x_0 \in K$ and $f|_K = g|_K$. Hence there is an integer $n_1 > n_0$ such that $g^{n_1}(x_0) \in B(x_0, \delta/2)$. If $Tg^{n_1}(x_0)$ preserves orientation, we can take $n = n_1$.

If $Tg^{n_1}(x_0)$ reverses orientation, we then pick $n_2 > n_0$ such that $g^{n_2}(x_0)$ is in $B(x_0, \delta/2)$ and is close enough to x_0 so that $g^{n_1}(g^{n_2}(x_0)) \in B(x_0, \delta/2)$. If $Tg^{n_2}(x_0)$ preserves orientation, we can take $n = n_2$; if not we can take $n = n_1 + n_2$. Figure 4 illustrates the case in which $n = n_1 + n_2$. □

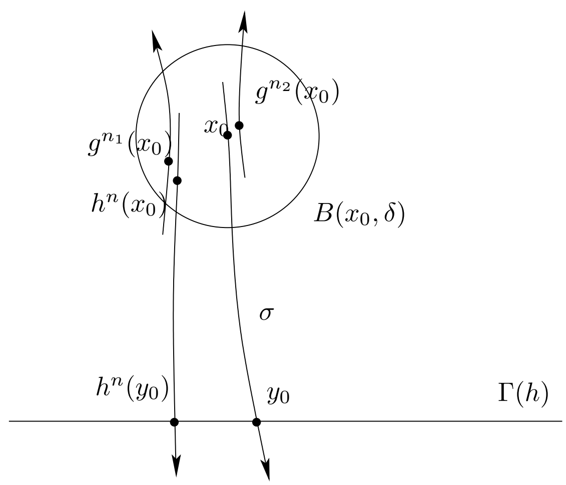

FIGURE 4. Lemmas 8.1 and 8.2

Lemma 8.2. *There is a C^r-open neighborhood $\mathcal{N}$ of g, such any $h \in \mathcal{N}$ has a periodic point in $\Gamma(h)$.*

Proof. Any h close enough to g in the C^r-topology satisfies the following properties:

1. $h \in \mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3$.
2. $B(x_0, \delta) \subset AC(x_0, h)$ for some $\delta < \Delta$.
3. $\Gamma(h) \cap B(x_0, \Delta) \neq \emptyset$.
4. There is $n > n_0$ such that $h^n(x_0) \in B(x_0, \delta)$ and Th^n preserves the orientation of E^c near x_0.

The first property holds because $g \in \mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3$ and this set is open. The second follows from Proposition 3.2. The third holds is a consequence of the choice of Δ and the fact that $h \in \mathcal{N}_3$. The fourth follows from Lemma 8.1.

We now show that any diffeomorphism h satisfying these three properties has a periodic point in $\Gamma(h)$. Let σ be a central arc connecting x_0 and $\Gamma(h)$. Choose a point $y_0 \in \sigma \cap \Gamma(h)$ such that σ contains no point of $\Gamma(h)$ except y_0, see Lemma 7.2. Choose $n > n_0$ such that $h^n(x_0) \in B(x_0, \delta)$ and $Th^n(x_0)$ preserves orientation of E^c.

Then the central arc $h^n(\sigma)$ connects $h^n(x_0)$ to a point $y \in \Gamma_\rho(y_0, h)$. Observe that there are no points of $\Gamma(h)$ on $h^n(\sigma)$ between $h^n(x_0)$ and y. Since the set $\Gamma(h)$ is invariant, the image under h^n of the point y_0 where the curve σ first hits $\Gamma(h)$ must be the point where $h^n(\sigma)$ first hits $\Gamma(h)$. Hence $h^n(y_0) = y$. We can now apply Lemma 7.1 to obtain a periodic point of h in $\Gamma(y_0, h)$. □

This contradicts Proposition 6.2, so for a continuity point f of Γ we must have $\Gamma(f) = \emptyset$, concluding the proof of Theorem 4.3.

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REFERENCES

[1] M. Brin and Ya Pesin, *Partially hyperbolic dynamical systems*, Math. USSR Izv., **8**, (1974), 177–218.

[2] K. Burns and A. Wilkinson, *On the ergodicity of partially hyperbolic systems*, to appear in Ann. Math.

[3] P. Didier, *Stability of accessibility*, Ergodic Theory Dynam. Systems, **23**, (2003), 1717–1731.

[4] D. Dolgopyat and A. Wilkinson, *Stable accessibility is C^1-dense*, Astérisque, **287** (2003), 33–60.

[5] M. Grayson, C. Pugh and M. Shub, *Stably ergodic diffeomorphisms*, Ann. Math, **140** (1994), 295–329.

[6] B. Hasselblatt and Y. Pesin, *Partially hyperbolic dynamical systems*, in "Handbook of Dynamical Systems," v.1B, Elsevier, 2006.

[7] J.-L. Journé, *A regularity lemma for functions of several variables*, Rev. Mat. Iberoamericana **4** (1988), 187–193.

[8] C. Kuratowski, "Topologie II," Monografie Matematyczne, vol. 21., publisher unknown, Warszawa-Wrocław, 1950.

[9] C. Lobry, *Une proprieté générique des champs de vecteurs*, Czechoslovak Math. J., **22** (1972), 230–237.

[10] R. Mañé, *Contributions to the stability conjecture*, Topology, **4**, (1978), 383–396.

[11] V. Niţică and A. Török, *An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one*, Topology, **40**, (2001), 259–278.

[12] C. Pugh and M. Shub, *Stable ergodicity and partial hyperbolicity*, in "International Conference on Dynamical Systems, Montevideo, Uruguay, 1995 — a tribute to Ricardo Mañé," (eds F. Ledrappier, J. Lewowicz and S. Newhouse), Harlow: Longman. Pitman Res. Notes Math. Ser., **362**, (1996), 182–187.

[13] C. Pugh and M. Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, J. of complexity, **13**, (1997), 125–179.

[14] C. Pugh and M. Shub, *Stable ergodicity and julienne quasi-conformality*, J. Eur. Math. Soc., **2**, (2000), 1–52.

[15] F. Rodriguez Hertz, M. Rodriguez Hertz and R. Ures, *Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1d-center bundle*, to appear in Invent. Math.

[16] F. Rodriguez Hertz, M. Rodriguez Hertz and R. Ures, *A survey of partially hyperbolic dynamics*, in "Partially Hyperbolic Dynamics, Laminations, and Teichmller Flow," (eds G. Forni, M. Lyubich, C. Pugh and Michael Shub), Fields Inst. Comm., AMS, (2007), 35-88.

[17] R. Sacksteder, *Strongly mixing transformations*,in " Global Analysis," Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968, pp. 245–252. Amer. Math. Soc., Providence, R. I., (1970), 245–252.

[18] E. Sontag, "Mathematical Control Theory. Deterministic Finite-Dimensional Systems," Second Edition. Springer N.Y, 1998.

[19] A. Wilkinson, *Stable ergodicity of the time-one map of a geodesic flow*, Ergodic Theory Dynam. Systems, **18** (1997), 1545–1587.

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