SHARKOVSKY'S THEOREM

KEITH BURNS AND BORIS HASSELBLATT

1. INTRODUCTION

In this note f is a continuous function from an interval into $\mathbb{R}$; although this is usually assumed in the literature, we do not need to the interval to be closed. f^n denotes the n-fold composition of f with itself. A point p is a periodic point for f with period m if $f^m(p) = p$, and it has *least period* m if in addition $f^k(p) \neq p$ for $1 \leq k < m$, that is, the orbit $\mathcal{O} := \{f^k(p) \mid k \in \mathbb{N}\}$ has cardinality m. A *fixed point* is a periodic point of period 1. If f has a periodic point of least period m, then m is called a least period for f.

1.1. **The Sharkovsky Theorem.** Sharkovsky's theorem involves the following ordering of the set $\mathbb{N}$ of positive integers, which is known as the Sharkovsky ordering:

$$3,\ 5,\ 7,\ \dots,\ 2\cdot 3,\ 2\cdot 5,\ 2\cdot 7,\ \dots,\ 2^2\cdot 3,\ 2^2\cdot 5,\ 2^2\cdot 7,\ \dots,\ 2^3,\ 2^2,\ 2,\ 1. \tag{1}$$

We think of this is as a descending list, and write $l \lhd m$ or $m \rhd l$ if l is to the right of m.[1] In the Sharkovsky ordering, the odd numbers greater than 1 appear in decreasing order from the left end of the list and the number 1 appears at the right end. The rest of $\mathbb{N}$ is included by successively doubling these end pieces, and inserting these doubled strings inward:

$$\text{odds},\ 2\cdot\text{odds},\ 2^2\cdot\text{odds},\ 2^3\cdot\text{odds},\dots,\ 2^3\cdot 1,\ 2^2\cdot 1,\ 2\cdot 1,\ 1. \tag{2}$$

Sharkovsky showed that this ordering describes which numbers can be least periods for a continous map of an interval.

Theorem 1.1. *If m is a least period for f and $m \rhd l$, then l is also a least period for f.*

We say that the presence of a period-m orbit *forces* the presence of a period-l orbit if every continuous interval map for which m is a least period also has l as a least period. Theorem 1.1 tells us that the presence

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[1] The pertinent literature in the early 1980s uses the reverse ordering and hence talks about minimal orbits where we talk about maximal ones. We prefer our order because Sharkovsky-larger numbers have greater implications, and in particular, larger minimal periods correspond to larger values of the topological entropy of a map.

of a period-m orbit forces the presence of a period-l orbits if $m \triangleright l$. If m is the leftmost number in (1) that is a least period of a map f, then we say that m is *Sharkovsky-maximal for* f.

Theorem 1.1 characterizes the subsets of $\mathbb{N}$ that are the set of least periods for a continuous interval mapping. We call $\mathscr{T} \subset \mathbb{N}$ a *tail* of the Sharkovsky order if $s \triangleleft t$ for all $s \notin \mathscr{T}$ and all $t \in \mathscr{T}$. In other words $\mathscr{T}$ is $\varnothing$ or $\mathbb{N}$ or the set of all numbers that come after some comma in (1). Theorem 1.1 can be restated as follows:

Theorem 1.2 (Sharkovsky Characterization [S])**.** *The set of least periods for a continuous interval map is a tail of the Sharkovsky order.*

The following complementary result is sometimes called the converse to Sharkovsky's Theorem, but is contained in Sharkovsky's original papers.

Theorem 1.3 (Sharkovsky Realization [S])**.** *Every tail of the Sharkovsky order is the set of least periods for a continous map of an interval into itself.*

Sharkovsky's Theorem is the union of Theorem 1.2 and Theorem 1.3: A subset of $\mathbb{N}$ is the set of least periods for a continuous map of an interval to itself if and only if the set is a tail of the Sharkovsky order. We give a proof of the Realization Theorem at the end of this note. Our main aim is to present a proof of the Characterization Theorem.

1.2. **History.** A capsule history of the Sharkovsky Theorem is in [M1], and much context is provided in [ALM]. The first result in this direction was obtained by Coppel [C1] in the 1950s: every point converges to a fixed point under iteration of a continuous map of a closed interval if and only if the map has no periodic points of least period 2; it is an easy corollary that 2 is the penultimate number in the Sharkovsky ordering.

Sharkovsky obtained the results described above and also reproved Coppel's theorem in a series of papers published in the 1960s [S]. He appears to have been unaware of Coppel's paper.

Sharkovsky's work did not become known outside eastern Europe until the second half of the 1970s. In 1975 this Monthly published a famous paper *Period three implies chaos* [LY] by Li and Yorke with the result that the presence of a periodic point of period 3 implies the presence of periodic points of all other periods, which amounts to 3 being the maximal number in the Sharkovsky order. Some time after the publication of [LY], Yorke attended a conference in East Berlin, during which a Ukrainian participant approached him. Although they had no language in common, Sharkovsky (for it was he) managed to convey that unbeknownst to Li and Yorke (and most of western mathematics) he had proved his results about periodic points of interval mappings well before [LY].

Besides introducing the idea of chaos to a wide audience, Li and Yorke's paper was to lead to global recognition of Sharkovsky's work[2]. Within a few years of [LY] new proofs of Sharkovsky's Characterisation Theorem appeared, one due to Štefan [Š], and a later one, which is now viewed as the "standard" proof, due to Block, Guckenheimer, Misiurewicz and Young [BGMY][3], Burkart [B], Ho and Morris [HM] and Straffin [St]. Nitecki's paper [N] provides a lovely survey from that time. Alsedà, Llibre and Misiurewicz improved this standard proof [ALM] and also gave a beautiful proof of the realization theorem.

The result has also been popular with contributors to the Monthly. We mention here a short proof of one step in the standard proof [BB] and several papers by Du [D]. Reading the papers by Du inspired the work that resulted in this article.

In the early 1980s three papers [C2, ALS, H] completely characterized the orbits of a continuous interval map with the property that their least period comes earlier in the Sharkovsky sequence than any other least periods for periodic points of that map. Results from their work and ideas from the proof in [ALM] are crucial in this paper.

1.3. **Aims of this article.** The standard proof of Sharkovsky's Characterization Theorem (Theorem 1.2) begins by studying orbits of odd least period with the property that their least period comes earlier in the Sharkovsky sequence than any other least periods for periodic points of that map. It shows firstly that such orbits must be of a special type, known as a Štefan cycle, and secondly that a Štefan cycle of length m forces the presence of periodic orbits with least period l for all $l \lhd m$. We use the results of [C2, ALS, H] and ideas from [ALM] to expand this two-step strategy so that it applies even when the least period of the map that appears earliest in the Sharkovsky order is even. The standard proof has to resort to other arguments in these cases. In our proof, the only case that needs special treatment is when all of the least periods of the map are powers of 2 and there is no least period that comes earliest in the Sharkovsky sequence.

1.4. **Outline.** After some observations about the doubling structure in (2), Section 3 introduces the basic consequences of the Intermediate-Value Theorem on which the Sharkovsky Theorem rests, and we show how these are used. Specifically, we show how knowledge of a periodic orbit translates into information about how the intervals between points of the orbit are mapped, and how this in turn provides information about

[2]It should not be forgotten that Li and Yorke's work contains more than a special case of Sharkovsky's: "chaos" is not just "periods of all orders".

[3]This citation is often pronounced "bigamy".

other periodic points. The proof of Sharkovsky's Characterization Theorem (Theorem 1.2) has two main parts. Section 4 shows that Štefan cycles force the presence of all Sharkovsky-smaller periods. Section 5 shows the converse, that orbits with Sharkovsky-maximal period are Štefan cycles.

In a separate paper we give a more elementary treatment of the characterization in [C2, ALS, H] than is presently available in the literature.

We hope that the proof presented here will convey insight into why Sharkovsky's Characterization Theorem is true. To this end we have made all arguments fully explicit.

2. The doubling structure

Our proof proceeds by induction on the number of factors of 2 in the least period of the map that appears earliest in the Sharkovsky order. It uses the doubling structure (2) in the Sharkovsky ordering.

Remark 2.1. The doubling structure in (2) is equivalent to the property that if $2n \rhd l$, then $l = 1$ or $l = 2k$ with $n \rhd k$. It implies that $n \rhd k$ if and only if $2n \rhd 2k$.

We show here how this doubling arises.

Lemma 2.2. *A point is periodic for f if and only if it is periodic for f^2. If p is a periodic point whose least period with respect to f^2 is k then the least period of p with respect to f is $2k$ if k is even and k or $2k$ if k is odd.*

Proof. If the least f-period is odd the orbits of the point under f and f^2 are the same, hence have the same length. Otherwise the f^2-orbit consists of every other point of the f-orbit and has half the length. □

Proposition 2.3. *If $m = 2n$ is Sharkovsky-maximal for f then n is Sharkovsky-maximal for f^2.*

Proof. If there is a periodic point p for f^2 with least f^2-period $k \rhd n$ then $2k \rhd 2n = m$ by Remark 2.1, and $k \neq 1$ because 1 is the smallest number in the Sharkovsky order. If k is odd, the least f-period of p is k or $2k$ by Lemma 2.2, and $k \rhd 2k \rhd m$ since k is odd and $k \neq 1$. If k is even then p has least f-period $2k \rhd m$ by Lemma 2.2. Either way, m is not Sharkovsky-maximal for f. □

3. Cycles, intervals and covering relations

Sharkovsky's Theorem draws its conclusion from knowledge about periodic points but otherwise independently of a particular map. Accordingly, we introduce a way of looking at a periodic orbit independently of a continuous map.

3.1. **From cycles to covering relations.** A *cycle* is a finite subset $\mathcal{O}$ of $\mathbb{R}$ together with a *cyclic* permutation π of $\mathcal{O}$. Its *(least) period* or *length* is the number of points in $\mathcal{O}$. Two cycles $(\mathcal{O},\pi)$ and $(\mathcal{O}',\pi')$ are considered equivalent if there is an order-preserving bijection from $\mathcal{O}$ to $\mathcal{O}'$ which carries the action of π to the action of π'. Thus, a cycle can be thought of as an abstract finite ordered set with a cyclic permutation.

Definition 3.1. An interval whose endpoints are in $\mathcal{O}$ is called an $\mathcal{O}$-*interval.* If it contains only two points of $\mathcal{O}$ then it is called a *basic* $\mathcal{O}$-interval, and these two points are said to be *adjacent.* Unless otherwise specified, I, J, K and L will denote $\mathcal{O}$-intervals.

We say that an $\mathcal{O}$-interval I *covers* an $\mathcal{O}$-interval J and write $I \to J$ if J lies between points of $\pi(I\cap\mathcal{O})$, in other words if J is contained in the convex hull[4] of $\pi(I\cap\mathcal{O})$.

So far the discussion has not involved a continuous map of the line. A continuous map f is said to *realize* the cycle $(\mathcal{O},\pi)$ if $\mathcal{O}$ is a periodic orbit for f and π is the restriction of f to $\mathcal{O}$.[5] A simple but important observation is that $I \to J$ implies that $J \subset f(I)$ for *any* map f that realizes $(\mathcal{O},\pi)$, so knowledge about a periodic orbit engenders knowledge about how special intervals are moved around by *any* map that realizes the periodic orbit. This is one of the main ingredients to the arguments.

3.2. **From covering relations to periodic points.** The other main idea is that this knowledge of how intervals are moved around in turn produces information about the presence of other periodic points. We show this in the next two lemmas, which, together with Lemma 3.7, are the basic ingredients of the proof. The results of this section hold for any closed bounded intervals, although we only apply them to $\mathcal{O}$-intervals.

Lemma 3.2. *If $I \subset f(I)$ for a closed bounded interval I, then f has a fixed point in I.*

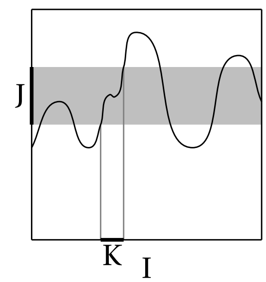
Proof. Let $I = [a_1,a_2]$. Then there are $b_1,b_2 \in [a_1,a_2]$ with $f(b_i) = a_i$. Since $f(b_1)-b_1 \le 0$ and $f(b_2)-b_2 \ge 0$, it follows from the Intermediate-Value Theorem that $f(x)-x=0$ for some x between b_1 and b_2. □

Lemma 3.3 (Itinerary Lemma)**.** *Let $J_0,\dots,J_{n-1}$ be intervals such that $J_i \subset f(J_{i-1})$ for $1 \le i < n$ and $J_0 \subset f(J_{n-1})$ (this is called a* cycle of intervals*). Then there is a fixed point p of f^n such that $f^i(p) \in J_i$ for $0 \le i < n$. (We say that p* follows the cycle*.)*

[4] The convex hull of a subset of $\mathbb{R}$ is the smallest interval in which it lies.

[5] A useful example is the map on $[\min\mathcal{O},\max\mathcal{O}]$ that agrees with π on $\mathcal{O}$ and is linear between any two points of $\mathcal{O}$.

Proof. We write $I \rightarrowtail J$ if $f(I) = J$. If $f(I) \supset J$, there is an interval $K \subset I$ such that $K \rightarrowtail J$ because the intersection of the graph of f with the rectangle $I \times J$ contains an arc that joins the top and bottom sides of the rectangle. We can choose K to be the projection to I of such an arc.



Thus, there is an interval $K_{n-1} \subset J_{n-1}$ such that $K_{n-1} \rightarrowtail J_0$. Then $K_{n-1} \subset f(J_{n-2})$, and there is an interval $K_{n-2} \subset J_{n-2}$ such that $K_{n-2} \rightarrowtail K_{n-1}$. Inductively, there are intervals $K_i \subset J_i$, $0 \le i < n$, such that

$$K_0 \rightarrowtail K_1 \rightarrowtail \cdots \rightarrowtail K_{n-1} \rightarrowtail J_0.$$

Any $x \in K_0$ satisfies $f^i(x) \in K_i \subset J_i$ for $0 \le i < n$ and $f^n(x) \in J_0$. Since $K_0 \subset J_0 = f^n(K_0)$, Lemma 3.2 implies that f^n has a fixed point in K_0. □

3.3. **Getting the right period.** The least period of p in Lemma 3.3 is a factor of n, and it may differ from n:

Example 3.4. If $J_0 = [-1,0]$, $J_1 = [0,1]$ and $f(x) = -2x$, one obtains the periodic point 0, which is fixed instead of having least period 2. What is wrong here is that 0 is the common endpoint of the intervals.

Definition 3.5. We say that a cycle of n intervals $J_0,\dots,J_{n-1}$ such that $J_i \subset f(J_{i-1})$ for $1 \le i < n$ and $J_0 \subset f(J_{n-1})$ is *period-forcing* if every p with $f^n(p) = p$ and $f^i(p) \in J_i$ for $0 \le i < n$ has least period n.

Remark 3.6. An interval $I \subset f(I)$ is a period-forcing cycle (of length 1).

A sufficient condition for a cycle of intervals to be period-forcing is pivotal here and in [BGMY]:

Lemma 3.7. *A cycle of basic $\mathcal{O}$-intervals that does not consist of repetitions of a shorter sequence and is not followed by a point of $\mathcal{O}$ is period-forcing.*

Proof. Denote the cycle by $J_0,\dots,J_{n-1}$. If $f^n(p) = p$ and p has least period n_0 then $n_0|n$. If p follows this cycle then $p \notin \mathcal{O}$, so $f^i(p) \in \text{Int}(J_i)$ for $0 \le i < n$. Since the interiors are pairwise disjoint, the sequence

$$J_0,\dots,J_{n-1},J_0,\dots,J_{n-1},\dots$$

has n_0 as a period. Then $n|n_0$, because n is the least period of the cycle. It follows that $n_0 = n$. □

3.4. **Period 3 implies all periods.** To show how the basic tools are used we prove the most celebrated special case of the Sharkovsky Theorem.

Up to left-right symmetry a period-3 orbit is $\mathcal{O} = \bullet \rightleftarrows \bullet \to \bullet$. If we denote the left and right basic $\mathcal{O}$-intervals by I and J, respectively, then $I \rightleftarrows J \circlearrowleft$. Since $J \circlearrowleft$, it follows from Lemma 3.2 that J contains a fixed point for f. The points of $\mathcal{O}$ do not follow the cycle $I \rightleftarrows J$ because they have least period 3 whereas the least period of a point that follows this cycle divides 2. By Lemma 3.7, f has an orbit with least period 2. Points of $\mathcal{O}$ do not stay in the interval J for more than two consecutive iterates of f, so the loop

$$I \rightleftarrows \overbrace{J \to J \to \cdots \to J}^{l-1\ J\text{'s}}$$

is period-forcing if $l > 3$ by Lemma 3.7, and f has a periodic point of least period l for each $l > 3$.

Thus, the presence of a period-3 orbit causes every positive integer to be a least period. Indeed, the same argument applied to the two rightmost $\mathcal{O}$-intervals shows that $\mathcal{O} := \bullet \rightleftarrows \bullet \to \cdots \to \bullet \to \bullet$ forces all periods. [M2] and [ALM] discuss related topics, and Proposition 4.2 extends this argument to all Štefan cycles with odd period.

4. Štefan cycles force Sharkovsky-lesser periods

Our proof of the Sharkovsky Theorem 1.2 has two main stages. In this section we show that, for any m, the presence of a Štefan cycle of least period m implies the presence of periodic orbits of least period l for each l such that $m \triangleright l$.

4.1. Štefan cycles.

Definition 4.1. A periodic orbit with odd least period m is a Štefan cycle if $m = 1$ or if $m \geq 3$ and the orbit contains a point x such that

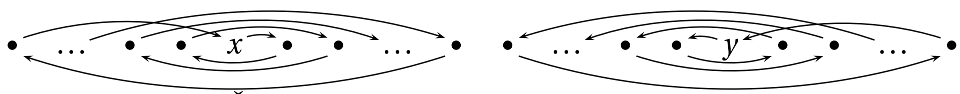
$$f^{m-1}(x) < f^{m-3}(x) < \cdots < f^2(x) < x < f(x) < f^3(x) < \cdots < f^{m-2}(x) \tag{3}$$

or a point y such that

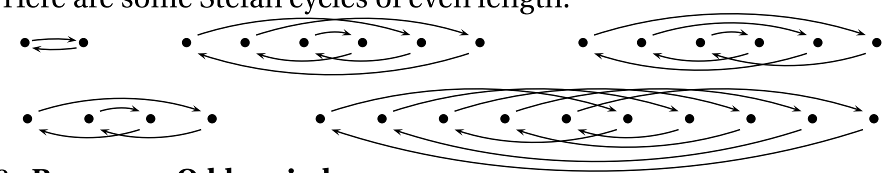
$$f^{m-2}(y) < f^{m-4}(y) < \cdots < f(y) < y < f^2(y) < f^4(y) < \cdots < f^{m-1}(y). \tag{4}$$

A periodic orbit for f with even least period is a Štefan cycle if its left and right halves are swapped by f and are both Štefan cycles for f^2, i.e., if $x_1 < \cdots < x_{2n}$ are the points of $\mathcal{O}$, then f is a bijection between $\mathcal{O}_L := \{x_1, \ldots, x_n\}$ and $\mathcal{O}_R := \{x_{n+1}, \ldots, x_{2n}\}$, and both $\mathcal{O}_L$ and $\mathcal{O}_R$ are Štefan cycles for f^2. (Since these have half the period of the original cycle, this definition eventually reduces to odd period.)

In other words, a Štefan cycle with odd least period is a fixed point or “spirals out” in one of the following fashions:



Here are some Štefan cycles of even length:

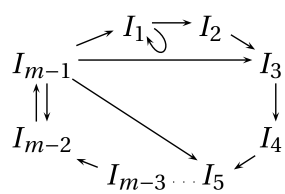


4.2. **Base case: Odd periods.**

Proposition 4.2. *If f has a Štefan cycle $\mathcal{O}$ with odd least period $m \rhd l$ then there is a period-forcing l-loop of basic $\mathcal{O}$-intervals.*

Proof. There is nothing to prove if $m = 1$, so we assume that $m \geq 3$. Let x be the middle point of $\mathcal{O}$ and define I_1 to be the interval between x and $f(x)$ and I_j to be the interval between $f^{j-2}(x)$ and $f^j(x)$ for $2 \leq j < m$. If $f(x) > x$, then (3) holds. If $f(x) < x$, then (4) holds. In either case, $I_1 \to I_1$ and $I_j \to I_{j+1}$ for $1 \leq j \leq m-2$. Also $I_{m-1} \to I_1 \cup I_3 \cup \cdots \cup I_{m-2}$. This is summarized by the following directed graph:

(5)



This graph contains a loop of least period l for any l such that $m \rhd l$:

- $l = 1$. The loop is $I \circlearrowleft$, which is period-forcing by Remark 3.6.
- $l < m$ even. The loop $I_{m-1} \to I_{m-l} \to I_{m-l+1} \to \cdots \to I_{m-2} \to I_{m-1}$ is period-forcing by Lemma 3.7 because points of $\mathcal{O}$ do not follow this loop—they have least period $m > l$.
- $l > m$. The loop $I_{m-1} \to \overbrace{I_1 \to \cdots \to I_1}^{l-m+2\ I_1\text{'s}} \to I_2 \to I_3 \to \cdots \to I_{m-2} \to I_{m-1}$ contains $l - m + 2 > 2$ consecutive I_1's. Points of $\mathcal{O}$ do not stay in I_1 for more than two consecutive iterates of f, so they cannot follow this loop. Again Lemma 3.7 tells us that this loop is period-forcing. □

4.3. **Induction: All periods.**

In terms of the Sharkovsky ordering (1), Proposition 4.2 covers all odd integers and hence the left and very right ends of the list of periods. We extend this result to the remainder of the list (1) by a doubling procedure that clarifies the doubling structure in the list.

Proposition 4.3. *Suppose f has a Štefan cycle $\mathcal{O}$ with least period $m \rhd l$. Then there is a period-forcing l-loop of $\mathcal{O}$-intervals.*

Proof. We use induction on the number of factors of 2 in m. If m is odd, the result is Proposition 4.2. If $m = 2n$ is such that the result is known for Štefan cycles with length n, we use the notations of Definition 4.1 and let $L = [x_1, x_n]$ and $R = [x_{n+1}, x_{2n}]$.

If $l \lhd m = 2n$ then by Remark 2.1 either $l = 1$ (and $[x_n, x_{n+1}] \circlearrowright$ is the desired 1-loop because f swaps $\mathcal{O}_L$ and $\mathcal{O}_R$) or $l = 2k$ with $k \lhd n$.

Thus, by the inductive hypothesis there is a period-forcing k-loop

$$J_0 \rightleftarrows J_1 \to J_2 \to \cdots \to J_{k-1} \tag{6}$$

for f^2 in L. Let $J'_i \subset R$ be the convex hull of $f(J_i \cap \mathcal{O})$ to get a cycle

$$J_0 \rightleftarrows J'_0 \to J_1 \to J'_1 \to \cdots \to J_{k-1} \to J'_{k-1} \tag{7}$$

of $2k = l$ intervals. A periodic point p for f that follows the cycle (7) is a periodic point for f^2 that follows the period-forcing cycle (6) and hence has least period k with respect to f^2. Since the intervals in the cycle (7) are alternately in L and R, so are the iterates of p under f. Hence p has least period $l = 2k$ with respect to f. □

This proves that Štefan cycles force all Sharkovsky-lesser periods. The next section proves the converse — a periodic orbit whose least period is Sharkovsky-maximal is a Štefan cycle.

5. Maximal periods arise from Štefan cycles

We first show that for a periodic orbit whose least period is Sharkovsky-maximal one obtains the "swapping" of sides seen in Štefan cycles. We follow [ALM]; our Lemma 5.1 (which will be used many times) and Proposition 5.2 are essentially the two parts of [ALM, Lemma 2.1.6].

5.1. **Swapping sides.** Let $\mathcal{O}$ be a cycle with least period $m \geq 2$. Let x be the rightmost point of $\mathcal{O}$ such that $f(x) > x$ and let y be the point of $\mathcal{O}$ immediately to the right of x. Then $f(x) \geq y$ and $f(y) \leq x$ and hence $J_1 := [x, y] \subset f([x, y])$. For $i > 1$ take J_i to be the convex hull of $f(\mathcal{O} \cap J_{i-1})$. Then $J_1 \to J_1 \to J_2 \to \cdots$ and recursively $J_1 \subset J_1 \subset J_2 \subset \cdots$.

Each set $\mathcal{O}_i := \mathcal{O} \cap J_i$ is an "interval" in $\mathcal{O}$, i.e., $\mathcal{O}_i$ contains all points of $\mathcal{O}$ between the leftmost and rightmost points of $\mathcal{O}_i$. Since $J_i \subset J_{i+1}$, we have $\mathcal{O}_i \subset \mathcal{O}_{i+1}$ for $i \geq 1$. Furthermore $\mathcal{O}_i = \mathcal{O}_{i+1}$ if and only if $\mathcal{O}_i = \mathcal{O}$, since otherwise $\mathcal{O}_i$ would be a proper invariant subset of the cycle $\mathcal{O}$.

Lemma 5.1 (Swapping Lemma)**.** *If every periodic point of f with odd period less than $k+2$ is a fixed point then f maps every point of $\mathcal{O}_k$ to the opposite side of (x, y): if $z \in \mathcal{O}_k$ then $z \leq x \Rightarrow f(z) \geq y$ and $z \geq y \Rightarrow f(z) \leq x$.*

Proof. If there is a z for which the conclusion fails take $K \subset J_k$ to be either $[z,x]$ or $[y,z]$ depending on whether $z < x$ or $z > y$. Since $J_{k-1} \to J_k$ we also have $J_{k-1} \to K$. Furthermore, $K \to [x,y] = J_1$ because x and y both map to the opposite side of (x,y) while z does not.

This means there are cycles of k and $k+1$ intervals:

$$J_1 \to J_2 \to \cdots \to J_{k-1} \to K \quad \text{and} \quad J_1 \to J_2 \to \cdots \to J_{k-1} \to K \to J_1.$$

Applying Lemma 3.3 to whichever of these cycles has odd length gives a periodic point $p \in J_1$ with $f^{k-1}(p) \in K$ and with odd period less than $k+2$. By hypothesis, p is a fixed point, so also $p \in K \cap J_1 \subset \{x,y\}$. This is impossible since neither x nor y is a fixed point. □

5.2. **Base case: Odd periods.** The following result is a central portion of the standard proof of the Sharkovsky Theorem.

Proposition 5.2. *Suppose $m \geq 3$ is odd and f has no periodic points other than fixed points whose periods are odd and less than m. Then any orbit with least period m is a Štefan cycle.*

Proof. Let $\mathcal{O}$ be an orbit with least period m and x, y $\mathcal{O}_i$ as in Subsection 5.1. We cannot have both $f(x) = y$ and $f(y) = x$ because $m \neq 2$. We assume $f(y) < x$ and prove (3); if $f(x) > y$ a similar argument gives (4).

By Lemma 5.1, f maps every point of $\mathcal{O}_{m-2}$ to the opposite side of (x,y). This is not the case for $\mathcal{O}$, because then f would would be a bijection between $\mathcal{O}_L := \mathcal{O} \cap [\min \mathcal{O}, x]$ and $\mathcal{O}_R := \mathcal{O} \cap [y, \max \mathcal{O}]$, whereas these sets have different cardinalities since $\mathcal{O}$ has an odd number of points. Thus $\mathcal{O}_{m-2} \neq \mathcal{O}$, and the inclusions

$$\{x,y\} = \mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset \mathcal{O}_{m-2} \subset \mathcal{O}_{m-1}$$

are strict because $\mathcal{O}_i = \mathcal{O}_{i+1}$ only if $\mathcal{O}_i = \mathcal{O}$. Since $\mathcal{O}_{m-1}$ can contain at most m points, we see that $\mathcal{O}_i$ consists of $i+1$ points for $1 \leq i < m$ and $\mathcal{O}_{m-1} = \mathcal{O}$.

In particular $\mathcal{O}_2$ has three points. Since x, y, $f(x)$, and $f(y)$ are all in $\mathcal{O}_2$, two of these points are the same. We know $x < y$ and have assumed that $f(y) < x$, so the only possibility is $f(x) = y$. We now have $\mathcal{O}_1 = \{x,y\} = \{x, f(x)\}$ and $\mathcal{O}_2 = \{x, f(x), f^2(x)\}$. An inductive argument shows that $\mathcal{O}_i = \mathcal{O}_{i-1} \cup \{f^i(x)\}$ for $1 \leq i \leq m-1$.

Points of $\mathcal{O}_{m-2}$ are mapped to the opposite side of (x,y), so x, $f(x), \dots,$ $f^{m-1}(x)$ are alternately in $\mathcal{O}_L$ and $\mathcal{O}_R$, and $\{x, f^2(x), \dots, f^{m-1}(x)\} \subset \mathcal{O}_L$. Since $\mathcal{O}_i = \mathcal{O}_{i-1} \cup \{f^i(x)\}$ and these sets are "intervals" in $\mathcal{O}$, it follows that $f^i(x)$ is adjacent to an endpoint of $\mathcal{O}_{i-1}$. Hence

$$f^{m-1}(x) < \cdots < f^2(x) < x < y = f(x).$$

Likewise, $\{f(x), f^3(x), \dots, f^{m-2}(x)\} \subset \mathcal{O}_R$ and

$$f(x) < f^3(x) < \cdots < f^{m-2}(x).$$

Combining these sets of inequalities gives (3). $\square$

5.3. **Induction: All periods.** To generalize Proposition 5.2 to arbitrary Sharkovsky-maximal periods, we again use induction on the number of factors of 2 in the least period of the cycle, or, rather the number of doubling steps as one works inward from the ends of the list (1). Here, Proposition 5.2 serves as the base case for the induction.

Proposition 5.3. *Suppose m is the largest (i.e., leftmost) number in the Sharkovsky order that is a least period for f, and $\mathcal{O}$ is a periodic orbit of least period m for f. Then $\mathcal{O}$ is a Štefan cycle.*

Proof. We use induction on the number of factors of 2 in m. The result is trivial if $m = 1$ and is Proposition 5.2 if m is odd and $m > 1$. Thus the claim holds when m is odd.

If $m = 2n$ and the claim is true for n consider an orbit $\mathcal{O}$ with least period m. As before, choose x to be the rightmost point in $\mathcal{O}$ such that $f(x) > x$ and let y be the point of $\mathcal{O}$ that is immediately to the right of x. Since m is even and Sharkovsky-maximal, no odd number other than 1 is a least period for f. The Swapping Lemma 5.1 gives $z \in \mathcal{O}_L := \mathcal{O} \cap [\min \mathcal{O}, x] \Rightarrow f(z) \in \mathcal{O}_R := \mathcal{O} \cap [y, \max \mathcal{O}]$ and $z \in \mathcal{O}_R \Rightarrow f(z) \in \mathcal{O}_L$. Applying this with $z = x$, $z = f(x)$ and so on shows that the even iterates of x are in $\mathcal{O}_L$ and the odd ones are in $\mathcal{O}_R$.

Thus, $\mathcal{O}_L$ and $\mathcal{O}_R$ are periodic orbits for f^2 with least period n, are swapped by f and are Štefan cycles for f^2 by our inductive hypothesis and Proposition 2.3 $\square$

Proposition 5.3 and Proposition 4.3 together yield most of Theorem 1.2. A nonempty subset of $\mathbb{N}$ contains a Sharkovsky-maximal element unless the set is infinite and consists entirely of powers of 2. Thus the only case of Theorem 1.2 that remains unsettled is when the set of least periods consists entirely of powers of 2 and there are infinitely many of these. In this case there is no Sharkovsky-maximal orbit, and we need to establish that no power of 2 is missing from the set of least periods.

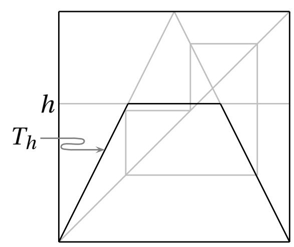
Proposition 5.4. *If all least periods of a continuous map f from an interval into $\mathbb{R}$ are powers of 2 then all periodic orbits are Štefan cycles.*

Proof. We show by induction on k that all periodic orbits of least period 2^k are Štefan cycles. This is trivial if $k = 0$. If the claim holds for k, consider a periodic orbit $\mathcal{O}$ with least period 2^{k+1}. Let $\mathcal{O}_L$ be the set consisting of the leftmost 2^k points of $\mathcal{O}$, and let $\mathcal{O}_R$ be the set consisting of the rightmost 2^k points of $\mathcal{O}$. Lemma 5.1 implies that f is a bijection between $\mathcal{O}_L$ and $\mathcal{O}_R$ and that both of these are periodic orbits for f^2. Since all least periods of f^2 are powers of 2, each of $\mathcal{O}_L$ and $\mathcal{O}_R$ is a Štefan cycle for f^2 by the inductive hypothesis. Hence $\mathcal{O}$ is a Štefan cycle for f. □

Sharkovsky's Characterization Theorem (Theorem 1.2) follows immediately from Proposition 5.4, Proposition 5.3 and Proposition 4.3.

6. Sharkovsky's Realization Theorem

An elegant proof of Sharkovsky's Realization Theorem 1.3 is given in [ALM]. They consider the family of truncated tent maps $T_h\colon [0,1] \to [0,1]$, $x \mapsto \min(h, 1-2|x-1/2|)$ for $0 \le h \le 1$.



The truncated tent maps have three key properties.

(a) T_0 has only one periodic point (the fixed point 0) while the tent map T_1 has a 3-cycle $\{2/7, 4/7, 6/7\}$ and hence has all natural numbers as least periods by the Sharkovsky Characterization Theorem 1.2.
(b) T_1 has a finite number of periodic points for each least period. [6]
(c) If $h \le k$, any periodic orbit of T_h is an orbit for T_k and any periodic orbit of T_k with maximum at most h is an orbit for T_h.

What makes the proof so elegant is that h plays three roles: as a parameter, as the maximum value of T_h, and as a point of an orbit.

For $m \in \mathbb{N}$, let $h(m) := \min\{\max \mathcal{O} \mid \mathcal{O} \text{ is an } m\text{-cycle of } T_1\}$. Two easy consequences of this definition and (a)–(c) are

(d) T_h has an m-cycle if and only if $h(m) \le h$.[7]
(e) If $m \ne l$, then $h(m) \ne h(l)$ since cycles of different lengths are disjoint.

Lemma 6.1. *$h(m) > h(l)$ if $m \rhd l$.*

Proof. $T_{h(m)}$ has an m-cycle by (d), hence an l-cycle by Theorem 1.2, so $h(m) \ge h(l)$ by (d). Also, $h(m) \ne h(l)$ by (e). □

[6] Inspection of the graph of T_1^n shows that T_1 has exactly 2^{n+1} points of (not necessarily least) period n for each n.

[7] $T_{h(m)}$ has a single m-cycle, namely the orbit of $h(m)$.

Together with (d) this implies that for any $m \in \mathbb{N}$ the set of least periods of $T_{h(m)}$ is the tail of the Sharkovsky order consisting of m and all $k \lhd m$.

The set of all powers of 2 is the only other tail of the Sharkovsky order (besides $\varnothing$, which is the set of least periods of the translation $x \mapsto x+1$ on $\mathbb{R}$). Let $h(2^\infty) := \sup_k h(2^k)$. Then $T_{h(2^\infty)}$ has periodic points of least period 2^k for all k by (d) above. On the other hand, if m is not a power of 2 then $m \rhd 2m \rhd 2^k$ for all k, and so $h(m) > h(2m) > h(2^k)$ by Lemma 6.1, and $h(m) > h(2^\infty)$. By (d) the only least periods of $T_{h(2^\infty)}$ are powers of 2.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208
E-mail address: `burns@math.northwestern.edu`

DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MA 02155
E-mail address: `bhasselb@tufts.edu`