Stable ergodicity of skew products
(Ergodicité stable des produits croisés)

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February 10, 1999

Abstract

Stable ergodicity is dense among compact Lie group extensions of Anosov diffeomorphisms of compact manifolds. Under the additional assumption that the base map acts on an infranilmanifold, an extension that is not stably ergodic must have a factor that has one of three special forms. A consequence is that stable ergodicity and stable ergodicity within skew products are equivalent in this case.

L'ergodicité stable est dense pour les extensions de difféomorphismes d'Anosov des variétés compactes par groupes de Lie compacts. Si, par ailleurs, le difféomorphisme d'Anosov agit sur une infranilvariété, alors toute extension qui n'est pas stablement ergodique possède un facteur qui prend une de trois formes particulières. Il s'ensuit que les notions de "ergodicité stable" et de "ergodicité stable entre les produits croisés" sont équivalentes.

Introduction

A diffeomorphism F of a compact manifold $\mathcal{M}$ is *partially hyperbolic* if the tangent bundle $T\mathcal{M}$ splits as a Whitney sum of TF-invariant subbundles:

$$T\mathcal{M} = E^u \oplus E^c \oplus E^s,$$

and there exist a Riemannian (or Finsler) metric on $\mathcal{M}$ and constants $\lambda < 1$ and $\mu > 1$ such that for every $p \in \mathcal{M}$,

$$m(T_pF|_{E^u}) > \mu > ||T_pF|_{E^c}|| \geq m(T_pF|_{E^c}) > \lambda > ||T_pF|_{E^s}|| > 0.$$

(The co-norm $m(A)$ of a linear operator A between Banach spaces is defined by $m(A) := \inf_{||v||=1} ||A(v)||$.) The bundles E^u, E^c and E^s are referred to as the *unstable*, *center* and *stable* bundles of F, respectively. A special case of a partially hyperbolic diffeomorphism is an Anosov diffeomorphism, for which $E^c = \{0\}$.

It has been known since the 1960's that volume preserving Anosov diffeomorphisms are *stably ergodic*. By a volume preserving diffeomorphism, we mean a diffeomorphism that is at least $C^{1+\alpha}$ for some $\alpha > 0$ and preserves a smooth measure on $\mathcal{M}$. Stable ergodicity of a volume preserving diffeomorphism F means that any volume preserving diffeomorphism which preserves the same smooth measure as F and is close enough to F in the C^1 topology is ergodic.

Starting with the seminal work of Grayson, Pugh and Shub [GPS], a series of recent papers ([Wi], [PS1], [PS2], [KK], [BPW]) has given many examples of (non Anosov) partially hyperbolic, volume preserving diffeomorphisms that are stably ergodic. Indeed, Pugh and Shub have conjectured [PS1] that stably ergodic diffeomorphisms should form an open dense subset of the volume-preserving, partially hyperbolic diffeomorphisms of a compact manifold. This is really a conjecture that the set is dense, since openness of the set is an immediate consequence of its definition.

The present paper addresses this conjecture in the special case of skew products. Let $f : M \to M$ be a diffeomorphism of a compact manifold M and let G be a compact Lie group. A function $\varphi : M \to G$ (assumed to be at least $C^{1+\alpha}$) defines a *skew-product* $f_\varphi : M \times G \to M \times G$ by

$$f_\varphi(x, g) = (f(x), \varphi(x)g).$$

A skew product is also called a G-*extension* of f. If the base diffeomorphism f is Anosov, then f_φ is partially hyperbolic. Since Haar measure on G is translation-invariant, f_φ will be volume preserving if f is volume preserving (and f_φ will preserve the product of Haar measure with the volume preserved by f).

We are able to verify Pugh and Shub's conjecture in the case of skew products. If f is a volume preserving Anosov diffeomorphism of a compact manifold M and G is a compact Lie group, then f_φ is stably ergodic for a dense set of functions $\varphi : M \to G$ (see Theorem A in Section 4). Under the further assumption that M is an infranilmanifold, we can show (see Theorem B in Section 4) that a skew product f_φ which fails to be stably ergodic must have a factor that has one of three specific forms:

1. $f \times Id_Y$, where Y is a nontrivial quotient of G;
2. $f \times R_\alpha$, where R_α is a rotation of a circle;
3. f_ψ, where $\psi : M \to \mathbf{T}^d$ is homotopic to a constant map and $\psi(M)$ lies in a coset of a lower dimensional Lie subgroup of $\mathbf{T}^d$.

A corollary of this result is that in the case when G is semisimple, a skew product is stably ergodic if and only if it is ergodic.

Our results are based on a recent theorem of Pugh and Shub (see Section 1) that provides a comprehensive criterion for a partially hyperbolic system to be stably ergodic. We also use ideas and results from Brin's work on skew products [B1, B2]. Brin introduced a sequence of Lie subgroups of the fiber of the skew product, and used these subgroups to characterize the ergodic skew products as well as those that are K-systems. He also proved an analogue of our density result: the ergodic skew products form an open and dense subset of the set of all skew products.

It follows from our results that a skew product over an Anosov diffeomorphism of an infranilmanifold is stably ergodic (among all volume preserving diffeomorphisms) if and only if it is stably ergodic within skew products.

Stable ergodicity within skew products has been the focus of a series of recent papers of Adler, Kitchens and Shub [AKS], Parry and Pollicott [PP], Field and Parry [FP], and Walkden [Wa]. The main interest has been in the case of Hölder continuous skew products with a subshift of finite type as the base.

A detailed formulation of the above results will be found in Section 4.

We would like to thank Mike Field, John Franks and Charles Pugh for several helpful discussions and Viorel Niţică for introducing us to Brin's papers. The first author was partially supported by NSF Grant DMS 9504760.

1 Pugh and Shub's theorem

A function is C^{k-} if its partial derivatives of order j exist and satisfy a Hölder condition with exponent $k-j$, where j is the greatest integer less than k. By convention, we let $C^{\infty-}=C^{\infty}$. Unless otherwise stated, we assume throughout this paper that $k\in(1,\infty]$. For these k the composition of two C^{k-} functions is again C^{k-}, and so the set $\mathrm{Diff}^{k-}(\mathcal{M})$ of C^{k-} diffeomorphisms of a manifold $\mathcal{M}$ is a group. We endow this group with the C^j topology, where j is again the greatest integer less than k; $\mathrm{Diff}^{k-}(\mathcal{M})$ is then a closed subgroup of $\mathrm{Diff}^{j}(\mathcal{M})$.

Let μ be a smooth volume element of $\mathcal{M}$. The μ-preserving C^{k-} diffeomorphisms also form a group, denoted by $\mathrm{Diff}^{k-}_{\mu}(\mathcal{M})$. We again use the C^j topology, where j is the greatest integer less than k. A diffeomorphism $F\in\mathrm{Diff}^{k-}_{\mu}(\mathcal{M})$ is *stably ergodic* if every $F'\in\mathrm{Diff}^{k-}_{\mu}(\mathcal{M})$ that is close enough to F in the C^1 topology is ergodic (with respect to μ). We similarly define stably weak mixing, stably mixing, and stably Kolmogorov (stably K).

If F is C^{k-} and partially hyperbolic, then its stable and unstable bundles are uniquely integrable and are tangent to foliations $\mathcal{W}^u_F$ and $\mathcal{W}^s_F$, whose leaves are C^{k-}. A partially-hyperbolic diffeomorphism is said to have the *accessibility property* if, for every pair of points $p,q\in\mathcal{M}$, there is a piecewise C^1 path $\gamma:[0,1]\to\mathcal{M}$ such that:

- $\gamma(0)=p$ and $\gamma(1)=q$;
- there exist $0=t_0<t_1<\cdots<t_n=1$ such that $\gamma([t_i,t_{i+1}])\subseteq\mathcal{W}^{a_i}_F(\gamma(t_i))$, where $a_i=u$ or s, for $i=0,\ldots,n-1$.

The path γ is called a $\mathcal{W}^{u,s}(F)$-path with n legs.

Partial hyperbolicity is an open property in the C^1 topology on diffeomorphisms of $\mathcal{M}$, and so any diffeomorphism F' of $\mathcal{M}$ that is sufficiently C^1-close to the partially hyperbolic diffeomorphism F has stable and unstable foliations $\mathcal{W}^u_{F'}$ and $\mathcal{W}^s_{F'}$. We say that F has the *stable accessibility property* if every F' sufficiently C^1-close to F has the accessibility property.

Pugh and Shub have shown that partial hyperbolicity and stable accessibility imply stable ergodicity under some relatively mild technical hypotheses. This theorem is the cornerstone of many of our arguments.

Theorem 1.1 [PS2] *Let $F \in \mathrm{Diff}^{k-}_\mu(\mathcal{M})$ where $\mathcal{M}$ is compact and $k > 1$. If F is center-bunched, partially hyperbolic, stably dynamically coherent, and stably accessible, then F is stably ergodic.*

Before proceeding, we discuss the technical conditions "center-bunched" and "stably dynamically coherent".

A partially hyperbolic diffeomorphism F is *center-bunched* if, for every $p \in \mathcal{M}$, the quantity

$$\mu_c = \|T_pF|_{E^c_F}\|/m(T_pF|_{E^c_F})$$

is close to one. The details can be found in section 4 of [PS2]. This property is C^1-open, and immediately satisfied when $\mu_c = 1$.

A partially hyperbolic diffeomorphism F is *dynamically coherent* if the distributions E^c_F, $E^c_F \oplus E^u_F$, and $E^c_F \oplus E^s_F$ are integrable, and everywhere tangent to foliations $\mathcal{W}^c_F$, $\mathcal{W}^{cu}_F$, and $\mathcal{W}^{cs}_F$, called the *center*, *center-unstable*, and *center-stable* foliations, respectively. Stably dynamically coherent means that all diffeomorphisms close enough to F in the C^1 topology are dynamically coherent. If F is dynamically coherent and $\mathcal{W}^c_F$ is a C^1 foliation, then F is stably dynamically coherent. For in this case $\mathcal{W}^c_F$ is normally hyperbolic and plaque-expansive. Then the theory developed in [HPS] can be applied to show that dynamical coherence is C^1-open.

Pugh and Shub's theorem can be strengthened slightly.

Corollary 1.2 *Let F satisfy the hypotheses of Theorem 1.1. Then F is stably K.*

Recall that a volume-preserving diffeomorphism $F : \mathcal{M} \to \mathcal{M}$ has the *Kolmogorov property* if there is a sub-σ-algebra $\mathcal{A}$ of the Borel σ-algebra $\mathcal{B}$ such that $F^{-1}\mathcal{A} \subseteq \mathcal{A}$, $\bigcup_{n=-\infty}^{\infty} F^{-n}\mathcal{A}$ generates $\mathcal{B}$, and $\bigcap_{n=0}^{\infty} F^{-n}\mathcal{A}$ is the trivial σ-algebra. A diffeomorphism F that has the Kolmogorov property is called a *K-system.* K-systems are ergodic and have no nontrivial factors of zero entropy (see [P]).

Proof of Corollary 1.2: A set $A \subset \mathcal{M}$ is called essentially u,s-saturated if there are sets A_u and A_s that are respectively a union of entire unstable leaves and a union of entire stable leaves respectively and satisfy $\mu(A \bigtriangleup A_u) = 0 =$

$\mu(A \triangle A_s)$. The argument in section 9 of [PS2] shows that, under the hypotheses of Theorem 1.1, every μ-preserving diffeomorphism that is close enough to F in the C^1-topology has the property that all essentially u, s-saturated sets have measure 0 or full measure. Proposition 5.2 of [BP] says that this latter property implies the K-property. $\square$

2 Skew products

Let $f \in \mathrm{Diff}^{k-}_\nu(M)$, let G be a Lie group and let μ be the product of the volume ν on M and Haar measure on G. A C^{k-} function $\varphi : M \to G$ defines a *skew-product over* f, $f_\varphi : M \times G \to M \times G$, by

$$f_\varphi(x, g) = (f(x), \varphi(x)g).$$

Since Haar measure is translation-invariant, $f_\varphi \in \mathrm{Diff}^{k-}_\mu(M \times G)$. Skew products over f are also called *G-extensions of f*. The set of all C^{k-} skew products over f, denoted

$$\mathrm{Ext}^{k-}(f, G) = \{f_\varphi \,|\, \varphi \in C^{k-}(M, G)\},$$

is a closed subset of $\mathrm{Diff}^{k-}_\mu(M \times G)$; its connected components correspond to homotopy classes of φ.

In this paper, we consider C^{k-} skew products f_φ for which f is an Anosov diffeomorphism of a compact manifold and φ maps into a connected Lie group with a Riemannian metric that is bi-invariant, i.e. invariant under both left and right translations. The bi-invariance of the metric means that both left and right translations are isometries. When we refer to "skew products", we implicitly make these assumptions.

We think of $M \times G$ as a bundle over M, so that M is the *base* and the *fibers* are the sets $\{x\} \times G$ for $x \in M$. Let $\pi_M : M \times G \to M$ and $\pi_G : M \times G \to G$ be the projections of $M \times G$ onto M and G, respectively. We equip $M \times G$ with a product metric. On M we use a Riemannian metric adapted to the Anosov diffeomorphism f. On G we use the chosen bi-invariant metric.

We denote left and right translations by $h \in G$, by L_h and R_h, respectively. By abuse of notation, we shall also denote by L_h and R_h the maps on $M \times G$ that send (x, g) to (x, hg) and to (x, gh) respectively.

Brin and Pesin showed that skew products of the type we are considering are partially hyperbolic; see Theorem 2.2 in [BP]. The next proposition describes the partially hyperbolic splitting for f_φ and the associated foliations. Let $TM = E^u_f \oplus E^s_f$ be the Anosov splitting for f, and let $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$ be the stable and unstable foliations for f.

Proposition 2.1 [BP] *The skew product f_φ is partially hyperbolic, dynamically coherent, and center-bunched. The leaves of the center foliation $\mathcal{W}^c_{f_\varphi}$ are the* *fibers of* $M \times G$*. For* $a = u$ *or* s*, each leaf of* $\mathcal{W}^{ac}_{f_\varphi}$ *is the product of* G *with a leaf of* $\mathcal{W}^a_f$*. Each* $\mathcal{W}^a_{f_\varphi}$ *leaf is the graph of a* C^{k-} *function from a leaf of* $\mathcal{W}^a_f$ *to* G*. For any* $g \in G$*, the right translation* R_g *carries* $\mathcal{W}^a_{f_\varphi}$ *leaves to* $\mathcal{W}^a_{f_\varphi}$ *leaves.*

An immediate consequence of Theorem 1.1, Corollary 1.2 and the previous proposition is:

Corollary 2.2 *Let* G *be compact. Then* $f_\varphi : M \times G \to M \times G$ *is stably ergodic and stably* K *if it has the stable accessibility property.*

3 Algebraic factors of skew products

Skew products commute with the projection π_M to the base manifold and with all right translations R_g of $M \times G$. It is natural to consider factors of skew products that share the same properties. This motivates the following definition.

We shall say that the C^{k-} dynamical system $h : M \times Y \to M \times Y$ is an *algebraic factor* of the C^{k-} skew product f_φ if $Y = H\backslash G$, where H is a closed subgroup of G, and there exists a C^{k-} function $\Phi : M \to G/H$ for which we have a commutative diagram

$$\begin{array}{ccc} M \times G & \xrightarrow{\quad f_\varphi \quad} & M \times G \\ \pi_\Phi \big\downarrow & & \pi_\Phi \big\downarrow \\ M \times H\backslash G & \xrightarrow{\quad h \quad} & M \times H\backslash G \end{array}$$

in which $\pi_\Phi(x, g) = (x, \Phi(x)^{-1}g)$. By $\Phi(x)^{-1}$ we mean $\{g^{-1} \,|\, g \in \Phi(x)\}$, which is an element of $H\backslash G$.

Any function $\Phi : M \to G/H$ induces a partition $\mathcal{P}$ of $M \times G$ that is invariant under any right translation R_g, $g \in G$, in the sense that R_g permutes elements of the partition. The elements of $\mathcal{P}$ are the set $P = \bigcup_{x \in M}\{x\} \times \Phi(x)$ and its translates by the maps R_g, $g \in G$. The map π_Φ sends elements of $\mathcal{P}$ to constant sections of $H\backslash G$. The existence of the quotient map h is equivalent to the partition $\mathcal{P}$ being f_φ-invariant. Conversely, a partition of $M \times G$ each of whose elements meets every fiber of $M \times G$ induces an algebraic factor if it is invariant under f_φ and right translations; cf. Section 6.

Note that an algebraic factor preserves the product of volume on M with the projection of Haar measure to $H\backslash G$. Thus an algebraic factor is a factor in the ergodic-theory sense. All ergodic properties, such as ergodicity, weak mixing, mixing, and the Kolmogorov property are inherited by factors. Hence if a skew product has an algebraic factor that does not possess one of these properties, then the skew product itself does not have the property.

If we restrict further to the class of algebraic factors that are themselves skew products, then *skew-stable* ergodic properties are also preserved:

Lemma 3.1 *Let N be a normal subgroup of G, and let $h : M \times N\backslash G \to M \times N\backslash G$ be an algebraic factor of the C^{k-} skew product $f_\varphi : M \times G \to M \times G$.*

Then h is an $N\backslash G$-skew product, and if f_φ is stably ergodic/mixing/K among G-skew products then h is stably ergodic/mixing/K among $N\backslash G$-skew products.

Proof: Let $p : G \to N\backslash G$. Let h be an algebraic factor of f_φ induced by $\Phi : M \to G/N$. It follows from the definition of algebraic factor that $h(x,y) = (f(x), w(x,y))$, where, for all $g \in G$,

$$w(x, \Phi(x)^{-1}g) = \Phi(f(x))^{-1}\varphi(x)g = \Phi(f(x))^{-1} \cdot p(\varphi(x)) \cdot p(g).$$

Since N is normal and $\Phi(x) \in G/N$, we have $\Phi(x) \in N\backslash G$ and $p(g) = \Phi(x) \cdot \Phi(x)^{-1}g$ for all $g \in G$. Hence

$$w(x,y) = \Phi(f(x))^{-1} \cdot p(\varphi(x)) \cdot \Phi(x) \cdot y,$$

for all $y \in N\backslash G$ and $h = f_\psi$, where

$$\psi = (\Phi \circ f)^{-1} \cdot (p \circ \varphi) \cdot \Phi.$$

With respect to the bi-invariant metrics on G and $N\backslash G$, the canonical projection $p : G \to N\backslash G$ is a Riemannian submersion. Riemannian submersions have an isometric path-lifting property: given any geodesic arc $\gamma : [0,1] \to N\backslash G$ and any point $q \in p^{-1}\gamma(0)$, there is a unique geodesic arc $\tilde{\gamma} : [0,1] \to G$ such that $p \circ \tilde{\gamma} = \gamma$, $\tilde{\gamma}(0) = q$, and such that all vectors tangent to $\tilde{\gamma}$ are horizontal, i.e. orthogonal to the fibers of the projection p (see e.g. Prop 3.31 in [CE]). Using this property, we can lift a function $\psi_1 : M \to N\backslash G$, whenever $d_{C^0}(\psi_1, p \circ \varphi)$ is sufficiently small, to a function $\widetilde{\psi_1} : M \to G$, as follows. Suppose that the C^0 distance between ψ_1 and $p \circ \varphi$ is smaller than the injectivity radius of $N\backslash G$. Then for every $x \in M$, there is a unique horizontal geodesic γ with $\gamma(0) = p \circ \varphi(x)$ and $\gamma(1) = \psi_1(x)$. Let $\tilde{\gamma}$ be the geodesic lift of γ such that $\tilde{\gamma}(0) = \varphi(x)$, and let

$$\widetilde{\psi_1}(x) = \tilde{\gamma}(1).$$

It is immediate that $\widetilde{\psi_1}$ has the following properties:

1. $p \circ \widetilde{\psi_1} = \psi_1$;
2. if ψ_1 is C^{k-} then so is $\widetilde{\psi_1}$;
3. $d_{C^{k-}}(\widetilde{\psi_1}, \varphi) = d_{C^{k-}}(\psi_1, p \circ \varphi)$.

Now suppose that h is not stably ergodic among skew products. Then there exists a C^{k-} function $\psi_0 : M \to N\backslash G$ such that f_{ψ_0} is not ergodic, with ψ_0 arbitrarily C^{k-}-close to ψ. For ψ_0 sufficiently close to ψ, the function $\psi_1 = (\Phi \circ f) \cdot \psi_0 \cdot \Phi^{-1}$ lifts to a C^{k-} function $\widetilde{\psi_1}$ that is C^{k-}-close to φ. Notice that f_{ψ_1} is conjugate to f_{ψ_0} by the measure-preserving diffeomorphism $(x,y) \mapsto (x, \Phi(x)y)$, and so f_{ψ_1} is not ergodic.

Since it has f_{ψ_1} as a non-ergodic factor, $f_{\widetilde{\psi}_1}$ is not ergodic. As such a $\widetilde{\psi}_1$ can be found arbitrarily C^{k-}-close to φ, this implies that f_φ is not stably ergodic.

The same proof works when "ergodic" is replaced by "mixing" or "K". □

4 Statement of results

Our first result is that the conjecture of Pugh and Shub holds for skew products. Recall that we have given $C^{k-}(M,G)$ the C^j topology, where j is the greatest integer less than k.

Theorem A: [Density of stable ergodicity] *Let $f : M \to M$ be a C^{k-} volume-preserving Anosov diffeomorphism of a compact manifold, and let G be a compact, connected Lie group. There is a subset $\mathcal{E}$ of $C^{k-}(M,G)$ such that $\mathcal{E}$ is dense in the C^{k-} topology and the skew product f_φ is stably ergodic, and in fact, stably a K-system for every $\varphi \in \mathcal{E}$.*

Our second theorem characterizes the skew products that are not stably ergodic. In this result and its corollaries we assume that M is an infranilmanifold, although we actually use only the (possibly weaker) conditions that $\pi_1(M)$ is virtually nilpotent and the action of $f^* - I$ on $H^1(M, \mathbf{R})$ is invertible. These conditions are also satisfied, for example, if M has finite fundamental group, but since there are no known Anosov diffeomorphisms other than those on infranilmanifolds, we will not dwell on these conditions.

Theorem B: [Characterization of stable ergodicity] *Let $f : M \to M$ be a C^{k-} volume-preserving Anosov diffeomorphism of an infranilmanifold, let G be a compact, connected Lie group and let $\varphi : M \to G$ be C^{k-}.*

If f_φ is not *stably ergodic, then it has an algebraic factor $h : M \times \mathcal{H}\backslash G \to M \times \mathcal{H}\backslash G$, where one of the following holds:*

1. *$\mathcal{H} \neq G$, and h is the product of f with $Id_{\mathcal{H}\backslash G}$;*
2. *$\mathcal{H}$ is normal, $\mathcal{H}\backslash G$ is a circle, and h is the product of f with a rotation;*
3. *$\mathcal{H}$ is normal, $\mathcal{H}\backslash G$ is a d-torus, and $h = f_\psi$, where ψ is homotopic to a constant and maps M into a coset of a lower dimensional Lie subgroup of $\mathbf{T}^d$.*

If f_φ has an algebraic factor of type (1), it is not ergodic; if f_φ has an algebraic factor of type (2), but none of type (1), then it is ergodic, but not weak mixing; otherwise f_φ is Bernoulli.

The classification of skew products in Theorem B is based on a sequence of Lie subgroups of G that are determined by the behavior of $\mathcal{W}^{u,s}(f_\varphi)$-paths. This sequence was first studied by Brin in [B2] and is described in detail in Section 8. In our notation,[1] the sequence is:

$$H^0 \subset H \subset \overline{H} \subset K \subset G.$$

Brin proved the following (Propositions 1, 2, and 3 in [B2]):

(i) *there is a* C^{k-}*-dense set of* φ *for which* $H^0 = G$ (see Theorem 9.8);

(ii) f_φ *is ergodic if and only if* $K = G$;

(iii) f_φ *is a* K*-system if and only if* $\overline{H} = G$.

It follows from (iii) and work of Rudolph [R] that:

(iv) f_φ *is Bernoulli if and only if* $\overline{H} = G$.

We show (Theorem 9.1 and Propositions 12.2, 12.3, and 12.7):

(A) *if* $H^0 = G$*, then* f_φ *is stably accessible*;

(B1) *if* $K \neq G$*, then* f_φ *has an algebraic factor of type* (1);

(B2) *if* $\overline{H} \neq G$ *and* $K = G$*, then* f_φ *has an algebraic factor of type* (2);

(B3) *if* $H^0 \neq G$ *and* $\overline{H} = G$*, then* f_φ *has an algebraic factor of type* (3).

Case (B3) requires the assumption that the base M is an infranilmanifold; the other cases do not.

Theorem A is an immediate consequence of (A) and Corollary 2.2. It is through this corollary that we apply the theorem of Pugh and Shub.

Since the cases (A), (B1), (B2), and (B3) are mutually exclusive, we see that Theorem B will follow from the above statements, provided one can show that f_φ is not stably ergodic if it has an algebraic factor of types (1), (2), or (3). A factor of type (1) is nonergodic, so f_φ itself is nonergodic if it has a factor of type (1). We shall see that factors of types (2) and (3) are not stably ergodic; since the group $\mathcal{H}$ is normal in these cases, it follows from Lemma 3.1 that f_φ is not stably ergodic if it has a factor of types (2) or (3). Case (2) can be approximated by the product of f with a rational rotation, which is not ergodic. In case (3), since ψ is homotopic to a constant, we can make a small change to ψ to give us a function $\psi_0 : M \to \mathbf{T}^d$ that takes values in a coset of a closed subgroup $\mathcal{H}_0$ of $\mathbf{T}^d$. The skew product f_{ψ_0} belongs to case (1); it has $f \times Id_{\mathcal{H}_0 \backslash \mathbf{T}^d}$ as a factor, and so is not ergodic.

[1] In Remark 2 of [B2], the sequence appears as

$$H_{loc} \subset H_C \subset H_{\overline{C}} \subset H \subset G.$$

Brin's H_{loc} is a subgroup of our H^0; the other terms are the same except for notation.

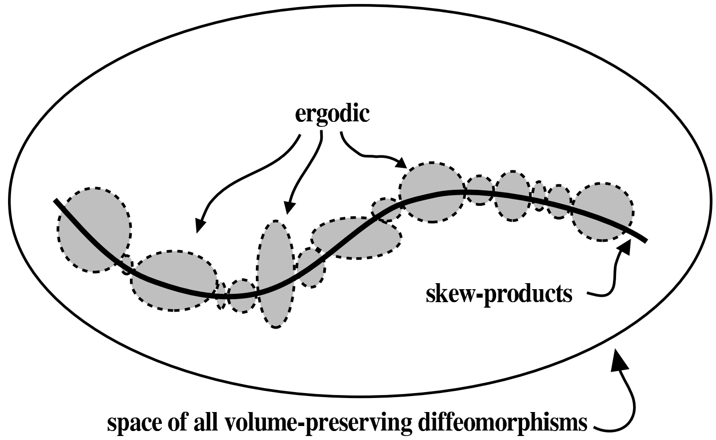

Figure 1: Stable ergodicity and stable ergodicity within skew products are the same.

Furthermore, we see that if f_φ has a factor of type (1), (2), or (3), then either f_φ is itself not ergodic or it can be perturbed to a nonergodic skew product. Thus f_φ is not stably ergodic within skew products if it has a factor of types (1), (2) or (3). On the other hand, if f_φ has no factors of these types, then f_φ is stably ergodic and even stably K (with perturbations allowed among all volume preserving diffeomorphisms).

We obtain

Corollary B1: *Let f_φ be a compact group extension of a volume preserving Anosov diffeomorphism f of a compact infranilmanifold. The following are equivalent:*

1. *f_φ is stably ergodic within skew products;*
2. *f_φ is stably ergodic;*
3. *f_φ is stably a K-system.*

The picture is summarized in Figure 1.

Stable ergodicity within skew products was first studied by Brin ([B1], [B2]), who proved the analogue of Theorem A: the set of C^{k-} skew products that are stably ergodic within skew products is C^{k-} dense and C^1 open (see Remark 2.1 and Proposition 2.3 in [B1]).

Since a semisimple Lie group does not have abelian quotients, semisimple skew products cannot have algebraic factors of type (2) or (3). This gives

Corollary B2: *Let f_φ be as in Theorem B and suppose in addition that the group G is semisimple. Then f_φ is stably ergodic if and only if it is ergodic.*

For circle extensions, we have the following:

Corollary B3: *Let f_φ be as in Theorem B and suppose in addition that the group G is the circle $\mathbf{T}$. Let m be the index of $(f^* - Id)H^1(M, \mathbf{Z})$ in $H^1(M, \mathbf{Z})$. Then f_φ is stably ergodic if and only if there are no functions $\Phi \in C^{k-}(M, \mathbf{T})$ that satisfy an equation of the form*

$$m\varphi = \Phi \circ f - \Phi + c,$$

where $c \in \mathbf{T}$ is a constant.

This corollary is proved at the end of Section 12.

5 Stable accessibility

As far as we know, this property was first studied systematically in the context of partially hyperbolic systems by Brin and Pesin [BP] in 1974. They used accessibility to show ergodicity of partially hyperbolic systems where the center foliation is Lipschitz.

The following discussion is based on [BPW], although it differs in the details. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be a pair of topological foliations on a compact connected manifold $\mathcal{M}$ with dimension n.

Definition: An $\mathcal{F}_1, \mathcal{F}_2$-*path* is a path $\psi : [0,1] \to \mathcal{M}$ consisting of a finite number of consecutive arcs — called *legs* — each of which is a curve that lies in a single leaf of one the two foliations[2]. The pair $\mathcal{F}_1, \mathcal{F}_2$ is *transitive* if any two points in $\mathcal{M}$ are joined by an $\mathcal{F}_1, \mathcal{F}_2$-path and is *stably transitive* if any pair of foliations sufficiently close to $\mathcal{F}_1, \mathcal{F}_2$ is transitive.

Recall that if Z is a compact connected orientable n-dimensional manifold with boundary and $q_0 \in \mathcal{M}$, we can define the degree of a continuous map $\psi : (Z, \partial Z) \to (\mathcal{M}, \mathcal{M} \setminus \{q_0\})$ to be the unique integer $l \geq 0$ such that there are generators ζ_Z for $H_n(Z, \partial Z)$ and $\zeta_{\mathcal{M}}$ for $H_n(\mathcal{M}, \mathcal{M} \setminus \{q_0\})$ with $\psi_*(\zeta_Z) = l\zeta_{\mathcal{M}}$, where ψ_* is the map induced by ψ on n-dimensional homology. Two properties of degree are important in the following.

- If $\widetilde{\psi} : (Z, \partial Z) \to (\mathcal{M}, \mathcal{M} \setminus \{q_0\})$ is close enough to ψ in the C^0 topology, then $\widetilde{\psi}$ and ψ have the same degree.
- If $\deg \psi \neq 0$, then $\psi^{-1}(q_0) \neq 0$.

Definition: A point q_0 can be $\mathcal{F}_1, \mathcal{F}_2$-*engulfed* from a point p_0 if there is a continuous map $\Psi : Z \times [0,1] \to \mathcal{M}$ such that:

[2] We make the convention that consecutive legs must be of opposite types, although it is permissible to have a leg of length 0. The initial leg may be of either type.

1. Z is a compact, connected, orientable, n-dimensional manifold with boundary;

2. for each $z \in Z$, the curve $\psi_z(\cdot) = \Psi(z, \cdot)$ is an $\mathcal{F}_1, \mathcal{F}_2$-path with $\psi_z(0) = p_0$;

3. there is a constant C such that every path ψ_z has at most C legs;

4. $\psi_z(1) \neq q_0$ for all $z \in \partial Z$.

5. the map $(Z, \partial Z) \to (\mathcal{M}, \mathcal{M} \setminus \{q_0\})$ defined by $z \mapsto \Psi(z, 1)$ has positive degree.

It is evident that q_0 can be reached from p_0 along an $\mathcal{F}_1, \mathcal{F}_2$-path if q_0 can be engulfed from p_0. Engulfing is stable under small perturbations of p_0, q_0 and the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proposition 5.1 *Suppose that q_0 can be $\mathcal{F}_1, \mathcal{F}_2$-*engulfed *from point p_0 and there is an $\mathcal{F}_1, \mathcal{F}_2$-path from q_0 to q_1. Then q_1 can be $\mathcal{F}_1, \mathcal{F}_2$-*engulfed *from p_0.*

Proof: The proposition follows easily from the special case in which q_0 and q_1 are joined by an $\mathcal{F}_1, \mathcal{F}_2$-path with one leg. We now consider that case and assume, after possibly renaming the foliations, that q_0 and q_1 lie in same leaf of $\mathcal{F}_1$. Let d be the dimension of the leaves of $\mathcal{F}_1$. It is possible to find an open set Q containing q_0 and q_1 and a homeomorphism $\rho : \mathbf{R}^{n-d} \times \mathbf{R}^d \to Q$ such that:

1. ρ sends each set $\{\text{const}\} \times \mathbf{R}^d$ into a single leaf of $\mathcal{F}_1$;

2. $q_0 = \rho(0, 0)$ and $q_1 \in \rho(\{0\} \times \mathbf{R}^d)$.

For $q \in Q$, let γ_q be the path that is the image under ρ of the line segment from $\rho^{-1}(q)$ to $\rho^{-1}(q) + \rho^{-1}(q_1)$. The map that sends q to the other end of γ_q is a homeomorphism of Q which maps a neighborhood of q_0 to a neighborhood of q_1.

Suppose that we could choose the map Ψ as in the definition of engulfing and with the additional property that $\Psi(Z \times \{1\}) \subset Q$. Then we could define a new map $\overline{\Psi} : Z \times [0, 1] \to N$ by setting $\overline{\Psi}(z, t) = \overline{\psi}_z(t)$ where $\overline{\psi}_z$ is the path formed by concatenating ψ_z and $\gamma_{\Psi(z,1)}$. It is easily seen that $\overline{\Psi}$ is an engulfing of q_1 from p_0.

It remains to show that Ψ can be chosen so that $\Psi(Z \times \{1\}) \subset Q$. Let $\Psi_0 : Z_0 \times [0, 1] \to \mathcal{M}$ be any engulfing of q_0 from p_0. Set $\psi_0(z) = \Psi_0(z, 1)$. Choose $\delta > 0$ so that $\overline{B(q_0, 3\delta)} \subset Q$ and $\psi_0(\partial Z_0) \cap \overline{B(q_0, \delta)} = \emptyset$. Choose a smooth map $\widetilde{\psi}_0 : Z_0 \to \mathcal{M}$ such that $\text{dist}(\widetilde{\psi}_0(z), \psi(z)) < \delta/2$ for all $z \in Z_0$. Then $\widetilde{\psi}(\partial Z_0) \subset \mathcal{M} \setminus \{q_0\}$ and ψ_0 and $\widetilde{\psi}_0$ have the same degree as maps of $(Z_0, \partial Z_0)$ into $(\mathcal{M}, \mathcal{M} \setminus \{q_0\})$.

Now choose $\rho \in (\delta, 2\delta)$ such that $\widetilde{\psi}_0$ is transverse to the geodesic sphere of radius ρ and center q_0, i.e. ρ is a regular value of $\text{dist}^2(q_0, \widetilde{\psi}_0(\cdot))$. Then $\widetilde{\psi}_0^{-1}(\overline{B(q_0, \rho)})$

is a compact smooth manifold with boundary, which must have a component Z such that $\widetilde{\psi} = \widetilde{\psi}_0|_Z$ has nonzero degree. Let $\psi = \psi_0|_Z$. Then

$$\psi : (Z, \partial Z) \to (\overline{B(q_0, 3\delta)}, \overline{B(q_0, 3\delta)} \setminus B(q_0, \delta/2)) \subset (Q, Q \setminus \{q_0\})$$

and ψ has the same degree as $\widetilde{\psi}$ as a map of $(Z, \partial Z)$ into $(Q, Q \setminus \{q_0\})$.

Finally we choose Ψ to be the restriction of Ψ_0 to $Z \times [0, 1]$. It is clear from the above Ψ has the desired properties. □

In the situation where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ for a dynamically coherent partially hyperbolic diffeomorphism $F : \mathcal{M} \to \mathcal{M}$, there is a simpler condition which implies that one point can be engulfed from another. Let c be the dimension of the leaves of $\mathcal{W}^c$.

Definition: A point q_0 can be *centrally engulfed* from a point p_0 if there is a continuous map $\Psi : Z \times [0, 1] \to \mathcal{M}$ such that:

1. Z is a compact, connected, orientable, c-dimensional manifold with boundary;

2. for each $z \in Z$, the curve $\psi_z(\cdot) = \Psi(z, \cdot)$ is a $\mathcal{W}^{u,s}(F)$-path with $\psi_z(0) = p_0$ and $\psi_z(1) \in \mathcal{W}^c(q_0)$;

3. there is a constant C such that every path ψ_z has at most C legs;

4. $\psi_z(1) \neq q_0$ for all $z \in \partial Z$;

5. the map $(Z, \partial Z) \to (\mathcal{W}^c(q_0), \mathcal{W}^c(q_0) \setminus \{q_0\})$ defined by $z \mapsto \Psi(z, 1)$ has positive degree.

Lemma 5.2 *Suppose q_0 can be centrally engulfed from p_0. Then p_0 can be engulfed from p_0.*

Proof: Let D^a denote the closed unit disc with the same dimension as the leaves of $\mathcal{W}^a$. Since $\mathcal{W}^c$ is integrable, there is a homeomorphism $\rho : D^u \times D^s \times D^c$ onto a neighborhood V of q_0 such that:

1. $\rho(0, 0, 0) = q_0$;

2. $\rho(\{0\} \times \{0\} \times D^c) \subset \mathcal{W}^c(q_0)$;

3. $\rho(D^u \times \{0\} \times \{x_c\}) \subset \mathcal{W}^u(\rho(0, 0, x_c))$ for all $x_c \in D^c$;

4. $\rho(\{x_u\} \times D^s \times \{x_c\}) \subset \mathcal{W}^s(\rho(x_u, 0, x_c))$ for all $(x_u, x_c) \in D^u \times D^s$.

By the same argument as in the proof of Proposition 5.1, we may assume that $\Psi(z,1) \in \rho(\{0\} \times \{0\} \times D^c)$ for all $z \in Z$. For $z \in Z$, let $(0,0,x_c(z)) = \rho^{-1}(\Psi(z,1))$. Now for any $(x_u,x_s,z) \in D^u \times D^s \times Z$, we can construct a $\mathcal{W}^{u,s}(F)$-path $\overline{\psi}_{(x_u,x_s,z)}$ by concatentating ψ_z with the images under ρ of the line segments from $(0,0,x_c)$ to $(x_u,0,x_c)$ and from $(x_u,0,x_c)$ to (x_u,x_s,x_c).

It is easy to see that the map $\overline{\Psi} : D^u \times D^s \times Z \times [0,1] \to N$ defined by $\overline{\Psi}(x_u,x_s,z,t) = \overline{\psi}_{(x_u,x_s,z)}(t)$ is an engulfing of q_0 from p_0. □

An immediate consequence of the above results is

Corollary 5.3 *Let $F : \mathcal{M} \to \mathcal{M}$ be a dynamically coherent, partially hyperbolic diffeomorphism. Suppose that there is a point p_0 such that any point of $\mathcal{M}$ can be reached from p_0 along a $\mathcal{W}^{u,s}(F)$-path and p_0 can be centrally engulfed from p_0. Then F is stably accessible.*

6 Right invariant equivalence relations on $M \times G$

In this section $\sim$ is an equivalence relation on $M \times G$ that is invariant under the right action of G, in the sense that

$$(x_1,g_1) \sim (x_2,g_2) \Rightarrow (x_1,g_1 g) \sim (x_2,g_2 g)$$

for all $g \in G$ and all $(x_1,g_1),(x_2,g_2) \in M \times G$. Write $g_1 \sim_x g_2$ if $(x,g_1) \sim (x,g_2)$. Observe that the following properties hold.

- $H_x = \{g \in G : g \sim_x e\}$ is a group for each $x \in M$ and the $\sim_x$ equivalence classes are closets that belong to $H_x \backslash G$.
- If $\sim$ is invariant under f_φ, in the sense that $f_\varphi(x_1,g_1) \sim f_\varphi(x_2,g_2)$ whenever $(x_1,g_1) \sim (x_2,g_2)$, then f_φ induces a map on $\bigcup_{x \in M} H_x \backslash G$.
- If $\sim$ is closed (i.e. the $\sim$ equivalence classes are closed), then each H_x is a closed Lie subgroup of G.
- If $(x_1,g_1) \sim (x_2,g_2)$, then $H_{x_2} = g_0 H_{x_1} g_0^{-1}$, where $g_0 = g_2 g_1^{-1}$. The $\sim_{x_2}$ equivalence class of g_0 is $g_0 H_{x_1} = H_{x_2} g_0$.

Suppose now that the equivalence class of (x_0,g_0) meets $\{x\} \times G$ for every $x \in M$. Then all the equivalence classes have this property and all the subgroups H_x belong to a single conjugacy class. The following properties hold.

- The map $\Phi : x \mapsto \{g \in G : (x,g) \sim (x_0,g_0)\}$ takes M to G/H_{x_0}. The map $\pi_\Phi : (x,g) \mapsto (x,\Phi(x)^{-1}g)$ takes $\sim$ equivalence classes to constant sections of $M \times H_{x_0} \backslash G$.
- If $\sim$ is closed and invariant under f_φ, then Φ induces an algebraic factor of f.

- If f_φ maps each $\sim$ equivalence class into itself, then the map induced on $M \times H_{x_0}\backslash G$ is $f \times Id$.
- If there is one $\sim$ equivalence class that meets $\{x\} \times G$ for every $x \in M$, then all the equivalence classes have this property and all the subgroups H_x belong to a single conjugacy class.

We define the *leaf-wise closure* $\overline{\sim}$ of $\sim$ by saying that $(x, g) \overline{\sim} (x', g')$ if there are sequences $g_n \to g$ and $g'_n \to g'$ such that $(x, g_n) \sim (x', g'_n)$ for each n.

The *closure* of $\sim$ is the relation $\overline{\overline{\sim}}$ defined by setting $(x, g) \overline{\overline{\sim}} (y, h)$ if there are sequences $(x_n, g_n) \to (x, g)$ and $(y_n, h_n) \to (y, h)$ such that $(x_n, g_n) \sim (y_n, h_n)$ for each n.

Lemma 6.1 *$\overline{\sim}$ is an equivalence relation. The group $\{g \in G : g \overline{\sim}_x e\}$ is the closure of H_x. Every $\overline{\sim}$ equivalence class is a union of $\sim$ equivalence classes.*

Proof: Observe that if $(x, g) \overline{\sim} (y, h)$, then there are sequences $g'_n \to g$ and $h'_n \to h$ such that $(x, g'_n) \sim (y, h)$ and $(x, g) \sim (y, h_n)$ for each n. Indeed, if g_n and h_n are as above, we can take $g'_n = g_n h_n^{-1} h$ and $h'_n = h_n g_n^{-1} g$. Hence if $(x, g) \overline{\sim} (y, h)$ and $(y, h) \overline{\sim} (z, k)$, there are sequences $g'_n \to g$ and $k'_n \to k$ such that $(x, g'_n) \sim (y, h) \sim (z, k'_n)$, which implies that $(x, g) \overline{\sim} (z, k)$. Thus $\overline{\sim}$ is transitive. Since $\overline{\sim}$ is obviously symmetric and reflexive, it is an equivalence relation.

By the above, $g \overline{\sim}_x e$ if and only if there is a sequence $g'_n \to g$ such that $g'_n \sim_x e$ for each n. This proves the second claim.

Since $(x, g) \sim (x', g') \Rightarrow (x, g) \overline{\sim} (x', g')$, the $\overline{\sim}$ equivalence class of (x, g) contains the $\sim$ equivalence class of (x, g). It follows that every $\overline{\sim}$ equivalence class is a union of $\sim$ equivalence classes. □

There is a natural condition under which the relations $\overline{\sim}$ and $\overline{\overline{\sim}}$ coincide.

Definition: We say that $\sim$ is *continuous* if each $x \in M$ has a neighborhood $U(x)$ on which there is a continuous function $\Gamma_x : U(x) \to G$ such that $(u, \Gamma_x(u)) \sim (x, e)$ for all $u \in U(x)$.

Lemma 6.2 *Let $\sim$ be continuous. Then the relations $\overline{\sim}$ and $\overline{\overline{\sim}}$ coincide.*

Proof: It suffices to show that if $(x_n, g_n) \to (x, g)$, then there is a sequence $g'_n \to g$ such that $(x_n, g_n) \sim (x, g'_n)$ for all large enough n. But $g'_n = \Gamma_x(x_n)^{-1} g_n$ has the desired properties. □

7 $\mathcal{W}^{u,s}(f_\varphi)$-paths

In this section, we develop some properties of $\mathcal{W}^{u,s}(f_\varphi)$-paths that will be needed in the following sections. The first result is a corollary of Proposition 2.1. We will use it repeatedly. For $j \geq 1$ and $x \in M$, let

$$\varphi_j(x) = \varphi(f^{j-1}(x)) \cdots \varphi(x).$$

Note that $(f_\varphi)^j = (f^j)_{\varphi_j}$.

Proposition 7.1 *For any $g \in G$ and any $\mathcal{W}^{u,s}(f)$-path $\gamma : [0,1] \to M$, there is a unique $\mathcal{W}^{u,s}(f_\varphi)$-path $\gamma_g : [0,1] \to M \times G$ such that $\gamma_g(0) = (\gamma(0), g)$ and $\pi_M \circ \gamma_g = \gamma$. Moreover, the following properties hold:*

1. *if $\gamma' : [0,1] \to M$ is fixed-end-point homotopic to γ, then $\gamma'_g(0) = \gamma_g(0)$;*

2. *for any $g' \in G$, $\gamma_{g'} = R_{g^{-1}g'} \circ \gamma_g$.*

3. *if $\gamma(0) = x$, then $(f^j \circ \gamma)_{\varphi_j(x)} = f^j_\varphi(\gamma_e)$.*

The $\mathcal{W}^{u,s}(f_\varphi)$-paths can be viewed as being horizontal with respect to a "connection" whose horizontal distribution is $E^s_{f_\varphi} \oplus E^u_{f_\varphi}$. We do not have a connection in the usual sense, since this distribution may not be differentiable, and we are only able to lift $\mathcal{W}^{u,s}(f)$-paths rather than arbitrary piecewise smooth paths. The following proposition shows, however, that the $\mathcal{W}^{u,s}(f)$-paths will be sufficient for our purposes.

Proposition 7.2 *Let $f : M \to M$ be an Anosov diffeomorphism.*

1. *Any path can be approximated arbitrarily closely in the C^0 topology by a $\mathcal{W}^{u,s}(f)$-path with the same endpoints.*

2. *Every homotopy class of loops based at x_0 is represented by a $\mathcal{W}^{u,s}(f)$-path.*

3. *Homotopic $\mathcal{W}^{u,s}(f)$-paths are homotopic through $\mathcal{W}^{u,s}(f)$-paths.*

Proof: The first two assertions follow from the fact that $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$ are uniformly transverse.

To prove the third assertion, let γ_1 and γ_2 be $\mathcal{W}^{u,s}(f_\varphi)$-paths based at x_0, and let $A(s,t)$ be a basepoint-fixing homotopy from $\gamma_1(t)$ to $\gamma_2(t)$. We may assume that γ_2 is the constant path x_0 and that the diameter of $A([0,1]\times[0,1])$ is smaller than the injectivity radius of M. Since γ_1 and γ_2 are piecewise smooth, we may assume that the homotopy $A(s,t)$ is through piecewise smooth paths. Now chop $[0,1]$ into subintervals at points $0 = t_0 < t_1 < \cdots < t_M = 1$ so that γ_1 restricted to $[t_i, t_{i+1}]$ lies entirely in a $\mathcal{W}^{a_i}_f$-leaf , where $a_i = u$ or s. For each s replace the path $A(s,\cdot)|_{[t_i,t_i+1]}$ by a two-legged $\mathcal{W}^{u,s}(f_\varphi)$-path $\widetilde{A}(s,\cdot)|_{[t_i,t_i+1]}$, such that:

- each $\mathcal{W}_f^a$-leg of $\widetilde{A}(s,\cdot)|_{[t_i,t_i+1]}$ is a geodesic inside $\mathcal{W}_f^a$;
- the $\mathcal{W}_f^{a_i}$-leg of $\widetilde{A}(s,\cdot)|_{[t_i,t_i+1]}$ is the first leg;
- $\widetilde{A}(s,t_i) = A(s,t_i)$, and $\widetilde{A}(s,t_{i+1}) = A(s,t_{i+1})$.

Since this construction is canonical, it produces a continuous homotopy $\widetilde{A}(s,t)$ through $\mathcal{W}^{u,s}(f)$-paths from γ_1 to γ_2. □

We conclude this section with some notational conventions that will be used in the sequel. If α is a path, $\overline{\alpha}$ will be the path obtained from α by reversing direction. If β is a path whose initial point is the same as the endpoint of α, we shall denote the concatenation of α and β by $\alpha \cdot \beta$. If α and β are paths in $M \times G$ with the endpoint of α and the initial point of β in the same fiber, we define $\alpha * \beta$ to be $\alpha \cdot (R_g \circ \beta)$, where g is chosen so that R_g moves the initial point of β to the endpoint of α. We can also define $\alpha * \beta$ when α and β are paths in G.

8 Holonomy groups

The results in this section build on ideas developed in two papers of Brin ([B1] and [B2]). Let $f_\varphi : M \times G \to M \times G$ be a skew product, where $f : M \to M$ is a volume preserving C^{k-} Anosov diffeomorphism of a compact manifold, G is a Lie group with a bi-invariant metric and $f : M \to G$ is C^{k-}. We introduce several equivalence relations on $M \times G$. All of these equivalence relations will be invariant under f_φ and R_g for all $g \in G$. As we observed in the previous section, the set of $g \in G$ such that (x,g) is equivalent to (x,e) is a group for each $x \in M$. These groups will be fundamental in the later sections of this paper.

We shall say that $(x_1,g_1) \sim (x_2,g_2)$ if there is a $\mathcal{W}^{u,s}(f_\varphi)$-path from (x_1,g_1) to (x_2,g_2). We shall say that $(x,g_1) \overset{\circ}{\sim} (x,g_2)$ if there is a $\mathcal{W}^{u,s}(f_\varphi)$-path from (x,g_1) to (x,g_2) whose projection to M is a null homotopic loop. Propositions 2.1 and 7.1 imply that these relations are invariant under R_g for all $g \in G$ and each $\sim$ equivalence class meets $\{x\} \times G$ for every $g \in G$.

Definition: The *holonomy group* for f_φ at x_0 is

$$H_{x_0}(f_\varphi) = \{g \in G : (x_0,g) \sim (x_0,e)\}.$$

The *restricted holonomy group* for f_φ at x_0 is

$$H^0_{x_0}(f_\varphi) = \{g \in G : (x_0,g) \overset{\circ}{\sim} (x_0,e)\}.$$

When it is clear which diffeomorphism we are referring to, we will write H_{x_0} and $H^0_{x_0}$ instead. The conjugacy class of these subgroups is independent of x_0. Since the basepoint x_0 will never vary, we normally omit it from the notation.

The following proposition and its proof are adaptations of standard arguments in the theory of connections on principal fiber bundles. See, e.g., [N] for an analogous discussion.

Proposition 8.1 *H and H^0 have the following properties.*

1. *H^0 is an analytic subgroup (i.e. a connected Lie subgroup) of G.*
2. *H^0 is a normal subgroup of H.*
3. *There is a surjective homomorphism $\rho : \pi_1(M, x_0) \to H/H^0$.*
4. *If $f^j(x_0) = x_0$, then $\rho(f^j_*[\alpha])$ is conjugate to $\rho([\alpha])$ for all $[\alpha] \in \pi_1(M, x_0)$:*
$$\rho(f^j_*[\alpha]) = \varphi_j(x_0)\rho([\alpha])\varphi_j(x_0)^{-1}.$$
In particular, if H/H^0 is abelian, then $\rho(f^j_*[\alpha]) = \rho([\alpha])$.
5. *H is a Lie subgroup of G, and H^0 is the connected component of the identity in H.*

Proof: The proof of (2) is straightforward: if (x_0, h) is the endpoint of a $\mathcal{W}^{u,s}(f_\varphi)$-path α from (x_0, e) whose projection $\pi_M \circ \alpha$ to M is null-homotopic, and (x_0, g) is the endpoint of a $\mathcal{W}^{u,s}(f_\varphi)$-path β from (x_0, e), then $(x_0, g^{-1}hg)$ is the endpoint of the $\mathcal{W}^{u,s}(f_\varphi)$-path:

$$\gamma = \beta * \alpha * \overline{\beta} = \beta \cdot (R_g \circ \alpha) \cdot \overline{(R_{g^{-1}hg} \circ \beta)}$$

from (x_0, e). Then $g^{-1}hg \in H^0$, since

$$\pi_M \circ \gamma = (\pi_M \circ \beta) \cdot (\pi_M \circ \alpha) \cdot \overline{\pi_M \circ \beta},$$

which is null-homotopic, since $\pi_M \circ \alpha$ is null-homotopic.

We now show (1). For any element $g \in H^0$, there is a $\mathcal{W}^{u,s}(f_\varphi)$-path $\gamma : [0, 1] \to M \times G$ such that $\gamma(0) = (x_0, e)$, $\gamma(1) = (x_0, g)$, and such that $\pi_M \circ \gamma$ is homotopic to the trivial loop through a homotopy fixing the basepoint x_0.

By Proposition 7.2, we may choose a homotopy through $\mathcal{W}^{u,s}(f)$-loops in M based at x_0 from the trivial loop to $\pi_M \circ \gamma$. By Proposition 7.1, each path in this homotopy has a lift to a $\mathcal{W}^{u,s}(f_\varphi)$-path in in $M \times G$ that starts at (x_0, e) and ends in $\pi_M^{-1}(x_0)$. This gives us a continuous map $A : [0, 1] \times [0, 1] \to M \times G$ such that, for each $s \in [0, 1]$, the path $A(s, \cdot)$ is a $\mathcal{W}^{u,s}(f_\varphi)$-path in $M \times G$ starting at (x_0, e), covering a null-homotopic loop in M and ending in the fiber over x_0. Moreover $A(0, \cdot)$ is the trivial loop at (x_0, e) and $A(1, 1) = \gamma(1) = (x_0, g)$. The path $\pi_G \circ A(\cdot, 1)$ is a path in H^0 from e to g. Thus H^0 is path-connected. By a theorem of Kuranashi-Yamabe [Y], a path-connected subgroup of a Lie group is an analytic Lie subgroup. This proves (1).

By Proposition 7.2, each element of $\pi_1(M, x_0)$ is represented by a $\mathcal{W}^{u,s}(f)$-loop based at x_0, and homotopic $\mathcal{W}^{u,s}(f)$-paths are $\mathcal{W}^{u,s}(f)$-homotopic. Let α be a $\mathcal{W}^{u,s}(f)$-loop in M that represents the class $[\alpha] \in \pi_1(M, x_0)$. Let α_e be the lift of α given by Proposition 7.1, satisfying $\alpha_e(0) = (x_0, e)$. Define $\rho([\alpha])$ by:

$$\rho([\alpha]) = p \circ \pi_G(\alpha_e(1)),$$

where $p : G \to H/H^0$ is the canonical projection. Then ρ is well-defined, for if $\alpha' \in [\alpha]$ is another representative of the same based homotopy class, then $\alpha \cdot \overline{\alpha'}$ is null-homotopic, implying that $\pi_G(\alpha_e(1))\pi_G(\alpha'_e(1))^{-1} \in H^0$. This defines a surjective homomorphism $\rho : \pi_1(M, x_0) \to H/H^0$, proving (3).

To show (4), suppose that $f^j(x_0) = x_0$, and let α be a $\mathcal{W}^{u,s}(f)$-loop based at x_0. Then $f^j \circ \alpha$ is a loop based at x_0 and, by Proposition 7.1, the $\mathcal{W}^{u,s}(f_\varphi)$-loop $f^j_\varphi \circ \alpha_e$ is a lift of $f^j \circ \alpha$, with

$$\pi_G\left(f^j_\varphi \circ \alpha_e(0)\right) = \varphi_j(x_0)$$

and

$$\pi_G\left(f^j_\varphi \circ \alpha_e(1)\right) \in \varphi_j(x_0)\rho([\alpha]).$$

Thus

$$\rho([f^j \circ \alpha]) = \varphi_j(x_0)\rho([\alpha])\varphi_j(x_0)^{-1},$$

which proves (4).

Let H' be the connected component of the identity e in H. Clearly $H^0 \subseteq H'$. Note that H' is also a connected Lie subgroup of G, being path-connected. Since ρ is surjective and $\pi_1(M)$ is countable, so is H/H^0, and therefore, so is H'/H^0. This implies that H is open in H', and so $H = H'$, proving (5). □

Observe that if γ_1 and γ_2 are two $\mathcal{W}^{u,s}(f_\varphi)$-paths that begin at (x_0, e) and end at two points (x, g_1) and (x, g_2) in the same fiber of $M \times G$, then $\gamma_1 \cdot R_{g_2^{-1}g_1}\overline{\gamma_2}$ is a $\mathcal{W}^{u,s}(f_\varphi)$-path from (x_0, e) to $(x_0, g_2^{-1}g_1)$, and hence $g_2^{-1}g_1 \in H$. It follows that the projections to G of the endpoints of all the $\mathcal{W}^{u,s}(f_\varphi)$-paths from (x_0, e) to $\{x\} \times G$ lie in a single coset belonging to G/H. Moreover the endpoints of these paths fill up the entire coset, since if γ is a $\mathcal{W}^{u,s}(f_\varphi)$-path from (x_0, e) to (x, g) and σ is a $\mathcal{W}^{u,s}(f_\varphi)$-path from (x_0, e) to (x_0, h) for some $h \in H$, then $\sigma \cdot R_h\gamma$ is a $\mathcal{W}^{u,s}(f_\varphi)$-path from (x_0, e) to (x_0, gh). Thus there is a well defined map

$$\Phi : M \to G/H$$

such that $\Phi(x)$ is the set of endpoints of the $\mathcal{W}^{u,s}(f_\varphi)$-paths from (x_0, e) to $\{x\} \times G$. More generally, for any $g_0 \in G$, the set of endpoints of the $\mathcal{W}^{u,s}(f_\varphi)$-paths from (x_0, g_0) to $\{x\} \times G$ is $\{x\} \times \Phi(x)g_0$.

We cannot always talk about smoothness of the map Φ, because G/H is not a manifold unless H is closed. However we have:

Proposition 8.2 *Let $\overline{\Phi} : M \to G/\overline{H}$ be the composition of Φ with the natural projection $G/H \to G/\overline{H}$. Then $\overline{\Phi}$ is C^{k-}. In particular Φ itself is C^{k-} if H is closed.*

Proof: By a theorem of Journé [J], it suffices to show that $\overline{\Phi}$ is uniformly C^{k-} along the leaves of the foliations $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$.

Let $B(x, \delta)$ be the geodesic ball around x in the metric that we chose on M in Section 2. There is a small enough $\delta > 0$ such that for each $x \in M$ every $y \in B(x, \delta)$ can be reached from x along a short $\mathcal{W}^{u,s}(f)$-path with two legs, the first in $\mathcal{W}^s_f(x)$ and the second in $\mathcal{W}^u_f(y)$. Let $\Gamma_x(y)$ be the endpoint of the lift of this path to a $\mathcal{W}^{u,s}(f_\varphi)$-path starting at (x, e).

Let $\mathcal{W}^s_f(x; \delta)$ and $\mathcal{W}^u_f(x; \delta)$ be the components of x in $\mathcal{W}^s_f(x) \cap B(x, \delta)$ and $\mathcal{W}^u_f(x) \cap B(x, \delta)$ respectively and define Γ^s_x and Γ^u_x to be the restrictions of Γ_x to $\mathcal{W}^s_f(x; \delta)$ and $\mathcal{W}^u_f(x; \delta)$ respectively. It follows from Propositions 2.1 and 7.1 that the functions Γ^s_x and Γ^u_x are C^{k-} and vary continuously with x in the C^{k-} topology.

Observe that if $y \in \mathcal{W}^u_f(x_0; \delta)$ and $z \in \mathcal{W}^s_f(x_0; \delta)$, then

$$\overline{\Phi}(y) = \Gamma^u_{x_0}(y)\overline{\Phi}(x) \quad \text{and} \quad \overline{\Phi}(z) = \Gamma^s_{x_0}(z)\overline{\Phi}(x).$$

The desired smoothness of $\overline{\Phi}$ follows immediately. □

Corollary 8.3 *The equivalence relation $\sim$ is continuous.*

Proof: The desired functions Γ_x were constructed in the previous proof. They are obviously continuous. □

8.1 The ergodic isotropy subgroup

We now introduce two further equivalence relations on $M \times G$. We shall say that $(x_1, g_1) \mathrel{\widehat{\sim}} (x_2, g_2)$ if $(x_1, g_1) \sim f^j_\varphi(x_2, g_2)$ for some integer j. Observe that $\mathrel{\widehat{\sim}}$ is continuous, because $\sim$ is continuous by Corollary 8.3. We define $\approx$ to be the closure of the relation $\mathrel{\widehat{\sim}}$. Let $\widehat{H} = \{g \in G : (x_0, e) \mathrel{\widehat{\sim}} (x_0, g)\}$ and $K = \{g \in G : (x_0, e) \approx (x_0, g)\}$. Then K is the closure of $\widehat{H}$ by Lemmas 6.1 and 6.2. Since it is closed, K is a Lie subgroup of G. Brin [B2] proved that the $\approx$ equivalence classes are the ergodic components for the action of f_φ on $M \times G$. For this reason, we call K the *ergodic isotropy subgroup* at x_0.

Proposition 8.4 *$\overline{H^0}$ and $\overline{H}$ are normal subgroups of K.*

Proof: $\overline{H^0}$ is normal because it is the connected component of the identity in K. To see this, recall from Proposition 8.1 that H^0 is the connected component of the identity in the Lie subgroup H, and observe that H has countable index in $\widehat{H}$. Hence H^0 is a connected Lie subgroup with countable index in $\widehat{H}$ and so $\overline{H^0}$ is a connected Lie subgroup with countable index in K.

In order to show that $\overline{H}$ is normal in K, it suffices to show that H is a normal subgroup of $\widehat{H}$. Let $h \in H$ and $\widehat{h} \in \widehat{H}$. Then there is a $\mathcal{W}^{u,s}(f_\varphi)$-path σ from (x_0, e) to $f_\varphi^j(x_0, \widehat{h}) = (f^j x_0, \varphi_j(x_0)\widehat{h})$, where

$$\varphi_j(x_0) = \varphi(f^{j-1}x_0)\varphi(f^{j-2}x_0)\cdots\varphi(fx_0)\varphi(x_0).$$

We claim that the $\mathcal{W}^{u,s}(f_\varphi)$-path

$$\gamma * (f_\varphi^j \circ \sigma) * \overline{\gamma} = \gamma \cdot R_{\widehat{h}}(f_\varphi^j \circ \sigma) \cdot R_{\widehat{h}^{-1}h\widehat{h}}\overline{\gamma}$$

joins (x_0, e) to $(x_0, \widehat{h}^{-1}h\widehat{h})$, which implies $\widehat{h}^{-1}h\widehat{h} \in H$. Indeed $f_\varphi^j \circ \sigma$ joins $(f^j x_0, \phi_j(x_0))$ to $(f^j x_0, \phi_j(x_0)h)$ and hence $R_{\widehat{h}}(f_\varphi^j \circ \sigma)$ joins $(f^j x_0, \phi_j(x_0)\widehat{h})$ to $(f^j x_0, \phi_j(x_0)h\widehat{h})$. Similarly, $\overline{\gamma}$ joins $(f^j x_0, \phi_j(x_0)\widehat{h})$ to (x_0, e) and hence $R_{\widehat{h}^{-1}h\widehat{h}}\overline{\gamma}$ joins $(f^j x_0, \phi_j(x_0)h\widehat{h})$ to $(x_0, \widehat{h}^{-1}h\widehat{h})$. □

Lemma 8.5 *$K/\overline{H}$ is abelian.*

Proof: The map $j \mapsto \{\widehat{h} \in \widehat{H} : (x_0, e) \sim f_\varphi^j(x_0, \widehat{h})\}$ is a surjective homomorphism from $\mathbf{Z}$ to $\widehat{H}/H$ and $\widehat{h}H \mapsto \widehat{h}\overline{H}$ is a homomorphism from $\widehat{H}/H$ to $K/\overline{H}$ whose image is dense. □

The holonomy and isotropy subgroups behave naturally under quotients:

Lemma 8.6 *Let N be a normal subgroup of a Lie group G and let $f_\psi : M \times N\backslash G \to M \times N\backslash G$ be an algebraic factor of f_φ. Let $p : G \to N\backslash G$ be the canonical projection. Then $H_x^0(f_\psi) = p(H_x^0(f_\varphi))$, $H_x(f_\psi) = p(H_x(f_\varphi))$, and $K_x(f_\psi) = p(K_x(f_\varphi))$.*

Proof: This will follow easily if we can show that

$$\mathcal{W}^a_{f_\psi}(x, p(g)) = (id \times p)(\mathcal{W}^a_{f_\varphi}(x, g)),$$

for $a = u$ or s. We consider $a = s$; the other case is similar. If $(x', g') \in W^s_{f_\varphi}(x, g)$, we have $\mathrm{dist}(f_\varphi^n(x', g'), f_\varphi^n(x, g)) \to 0$ as $n \to \infty$ and hence

$$\mathrm{dist}(f_\psi^n(x', p(g')), f_\psi^n(x, p(g))) \to 0 \quad \text{as } n \to \infty.$$

It follows that $(id \times p)(\mathcal{W}^s_{f_\varphi}(x, g)) \subset \mathcal{W}^s_{f_\psi}(x, p(g))$. On the other hand, it easy to see that $(id \times p)(\mathcal{W}^s_{f_\varphi}(x, g))$ and $\mathcal{W}^s_{f_\varphi}(x, g)$ both lie in $W^s_f(x) \times N\backslash G$ and both contain exactly one point in $\{x'\} \times N\backslash G$ for each $x' \in W^s_f(x')$. Hence $\mathcal{W}^s_{f_\psi}(x, p(g)) = (id \times p)(\mathcal{W}^s_{f_\varphi}(x, g))$.□

9 Engulfing and the proof of Theorem A

Let $f_\varphi : M \times G \to M \times G$ be as in the previous section and assume in addition that the group G is compact. Recall that we are using a bi-invariant metric on G. With this choice of metric the Lie group and Riemannian exponential maps coincide.

Choose a point $x_0 \in M$ and let $H^0 = H^0_{x_0}$ be the restricted holonomy group at x_0 as defined in the previous section. The main goal of this section is to prove

Theorem 9.1 *If $H^0 = G$, then f_φ is stably ergodic.*

Theorem A follows from this theorem and a result of Brin that we present at the end of this section. Note that if $H^0 = G$, then $H^0_x = G$ for every $x \in M$; this is evident from Proposition 8.1. The results in Section 12 imply that the converse of Theorem 9.1 holds when M is an infranilmanifold, but we do not know whether the converse of Theorem 9.1 holds without this additional hypothesis.

Proof of Theorem 9.1: Since $H^0 = G$, every $x \in M$ has the property that every point in $\{x\} \times G$ can reached from (x, e) along a $\mathcal{W}^{u,s}(f_\varphi)$-path. It follows from this and the fact that every $x \in M$ can be reached from x_0 along a $\mathcal{W}^{u,s}(f)$-path that every $(x, g) \in M \times G$ can be reached from (x_0, e) along a $\mathcal{W}^{u,s}(f_\varphi)$-path. By Corollaries 2.2 and 5.3, it suffices to prove that (x_0, e) can be centrally engulfed from (x_0, e).

For this, it will suffice to find a continuous map $\Psi : D^c \times [0, 1] \to M \times G$, where D^c is homeomorphic to the closed disc of dimension $c = \dim(G)$, with the following properties:

1. for each $z \in D^c$, the path $\psi_z(\cdot) = \Psi(z, \cdot)$ is a $\mathcal{W}^{u,s}(f_\varphi)$-path with $\psi_z(0) = (x_0, e)$ and $\psi_z(1) \in \{x_0\} \times G$;

2. there is a constant C such that each path ψ_z has at most C legs.

3. $\psi_z(1) \neq (x_0, e)$ for any $z \in \partial D^c$;

4. the map $H_c(D^c, \partial D^c) \to H_c(G, G \setminus \{e\})$ induced by $z \to \pi_G(\psi_z(1))$ is nontrivial.

Let us call a map $h : Z \to G$ *achievable* if it is the "endpoint map" of a continuous family of $\mathcal{W}^{u,s}(f_\varphi)$-paths that begin at (x_0, e), end in $\{x_0\} \times G$ and project to null-homotopic loops in M. More precisely, h is achievable if there are a continuous function $H : Z \times [0, 1] \to M \times G$ and a positive integer C such that, for each $z \in K$, the path $h_z : [0, 1] \to M \times G$ defined by $h_z(t) = H(z, t)$ is a $\mathcal{W}^{u,s}(f_\varphi)$-path from (x_0, e) to $(x_0, h(z))$ with at most C legs that is the lift of a null homotopic loop in M.

The notion of achievable map allows us to reformulate the desired properties of Ψ. What we want is an achievable map $\psi : (D^c, \partial D^c) \to (G, G \setminus \{e\})$ that induces a nontrivial map on c-dimensional homology. Moreover it is not necessary that ψ itself be achievable; it will suffice if ψ can be approximated arbitrarily closely in the C^0 topology by achievable maps.

Definition: A map $h : Z \to G$ is *approximable* if for each $\varepsilon > 0$ there is an achievable map $h^\varepsilon : Z \to G$ such that $\mathrm{dist}_{C^0}(h, h^\varepsilon) < \varepsilon$.

Lemma 9.2 *Let $h_i : Z_i \to G$, $1 \le i \le k$, be approximable. Then the product map*

$$(z_1, \ldots, z_k) \mapsto h_k(z_k)h_{k-1}(z_{k-1}) \cdots h_2(z_2)h_1(z_1)$$

is approximable.

Proof: Since the map

$$C^0(Z_1, G) \times \cdots \times C^0(Z_k, G) \to C^0(Z_1 \times \cdots \times Z_k, G)$$

that takes $(h_1, \ldots, h_k)$ to $(z_1, \ldots, z_k) \mapsto h_k(z_k)h_{k-1}(z_{k-1}) \cdots h_2(z_2)h_1(z_1)$ is continuous, it suffices to show that a product of achievable maps is achievable. This in turn reduces to showing that if $h' : Z' \to G$ and $h'' : Z'' \to G$ are achievable, then so is the map $h : Z' \times Z'' \to G$ defined by $h(z', z'') = h''(z'')h'(z')$. But this is true, since we can choose the $\mathcal{W}^{u,s}(f_\varphi)$-path $h_{(z',z'')}$ from (x_0, e) to $(x_0, h_{(z',z'')})$ whose existence is required by the definition of achievability to be $h' * h'' = h'_{z'} \cdot R_{h'(z')}h''_{z''}$. □

Suppose now that we knew that the geodesic arc, $\sigma_v : [0, 1] \to G$, $\sigma_v(s) = \exp(sv)$, was approximable for all $v \in \mathfrak{g}$. Then the map

$$\psi(z_1, \ldots, z_c) = \exp(z_c v_c) \exp(z_{c-1} v_{c-1}) \cdots \exp(z_2 v_2) \exp(z_1 v_1)$$

is approximable for any choice of $v_1, \ldots, v_c \in \mathfrak{g}$. Moreover ψ will be a diffeomorphism of $[-1, 1]^c$ onto a neighborhood of e if we choose the v_i to be short enough and to form a basis for $\mathfrak{g}$ that is orthogonal with respect to the bi-invariant metric.

Thus we could obtain the desired central engulfing of (x_0, e) from (x_0, e) by taking $D^c = [-1, 1]^c$ and ψ as above, if we knew that the geodesic arc σ_v is approximable for every $v \in \mathfrak{g}$. Theorem 9.1 now follows from the next proposition.□

Proposition 9.3 *The geodesic arc σ_v is approximable if and only if v is tangent to H^0.*

Proof: Let us call $v \in \mathfrak{g}$ *approximable* if σ_v is approximable. The first step will be to show that if v is approximable, then so is any multiple of v. Before doing this, we note some elementary properties of approximable paths (i.e. approximable maps from $[0, 1]$ to G) that will needed. Recall that if α and β are paths in G, then $\alpha * \beta$ is the path formed by concatenating α with $R_g \circ \beta$, where g is chosen so that R_g moves the initial point of α to the endpoint of β. If the paths α and β are approximable paths, so is $\alpha * \beta$. Also any subpath, any right translate and any reparametrization (even direction reversing) of an approximable path is again approximable.

Lemma 9.4 *If $v \in \mathfrak{g}$ is approximable, then λv is approximable for all $\lambda \in \mathbf{R}$.*

Proof: The vector 0 is approximable, because σ_0 is achievable, since $\sigma_0(t) = e$ for $0 \leq t \leq 1$.

Observe that (with suitable reparametrization) $\sigma_{\lambda v}$ is a subpath of σ_v for any $\lambda \in (0,1)$ and $\sigma_{2v} = \sigma_v * \sigma_v$. Moreover σ_{-v} is (after an orientation reversing reparametrization) a right translate of σ_v, since $\sigma_{-v}(t) = \sigma_v(1-t)\exp(-v)$.

If v is approximable, then so are λv for any $\lambda \in [0,1)$, $2v$ and $-v$. The lemma follows easily. □

The lemma tells us that if a vector $v \in \mathfrak{g}$ is approximable, then every geodesic segment contained in the one parameter subgroup tangent to v is approximable. It follows that if $v_1, \ldots, v_k$ are approximable vectors, then the geodesic polygon $\sigma_{v_1} * \cdots * \sigma_{v_k}$ is approximable.

We now formulate a criterion for a vector $v \in \mathfrak{g}$ to be approximable. Let r_{inj} be the injectivity radius of G with respect to our bi-invariant metric. Then the exponential map is a diffeomorphism from the open ball B in $\mathfrak{g}$ of radius r_{inj} to the geodesic ball U of radius r_{inj} about e in G. Let $\exp^{-1} : U \to B$ be the inverse of $\exp : B \to U$.

Lemma 9.5 *A nonzero vector $v \in \mathfrak{g}$ is approximable if there is a sequence of approximable paths $\gamma_n : [0,1] \to G$ with $\gamma_n(0) = e$ such that:*

1. *there is a sequence $c_n \searrow 0$ such that $\operatorname{dist}(\gamma_n(t), e) < c_n$ for $0 \leq t \leq 1$;*
2. *if $w_n = \exp^{-1}\gamma_n(1)$ for those n such that $c_n < r_{\text{inj}}$, then $\frac{||v||}{||w_n||} w_n \to v$.*

Proof: The path $\gamma_n^k = \overbrace{\gamma_n * \cdots * \gamma_n}^{k}$ is approximable for any positive integer k. Note that σ_n is $2c_n$-approximated by γ_n because both paths lie in the geodesic ball of radius c_n around e. It follows that σ_{kw_n} is $2c_n$-approximated by γ_n^k for any k. Let k_n be the first integer larger than $||v||/||w_n||$. Then the paths $\gamma_n^{k_n}$ converge in the C^0 topology to σ_v. Since a C^0 limit of approximable paths is approximable, σ_v is approximable. □

Lemma 9.6 *The set of approximable vectors is a Lie subalgebra of $\mathfrak{g}$.*

Proof: In view of Lemma 9.4, we need to show that if v and w are approximable, then $v + w$ and $[v,w]$ are approximable. Recall (see e.g. Lemma 11.6 in Ch. II §1 of [H]) that for any vectors $v, w \in \mathfrak{g}$ we have

$$\exp((v+w)/n) = \exp(v/n)\exp(w/n) + O(1/n^2)$$

and

$$\exp([v,w]/n^2) = \exp(-v/n)\exp(-w/n)\exp(v/n)\exp(w/n) + O(1/n^3).$$

Let $\alpha_n = \sigma_{w/n} * \sigma_{v/n}$ and $\beta_n = \sigma_{w/n} * \sigma_{v/n} * \sigma_{-w/n} * \sigma_{-v/n}$. These paths are approximable if v and w are approximable, and we have $\alpha_n(1) = \exp(v/n)\exp(w/n)$ and

$$\beta_n(1) = \exp(-v/n)\exp(-w/n)\exp(v/n)\exp(w/n).$$

It follows from Lemma 9.5 that if v and w are approximable, then $v+w$ and $[v,w]$ are both approximable. □

Let $\mathfrak{g}_{\mathrm{app}}$ be the Lie algebra defined by the previous lemma and $\mathfrak{g}_{\mathrm{app}}^{\perp}$ its orthogonal complement. Let G_{app} be the subgroup tangent to $\mathfrak{g}_{\mathrm{app}}$. Proposition 9.3 follows immediately from the next proposition. □

Proposition 9.7 $G_{\mathrm{app}} = H^0$.

Proof: We begin by showing that $G_{\mathrm{app}} \subset H^0$. As above, let B be the open ball of radius r_{inj} about 0 in $\mathfrak{g}$ and U the image of B under the exponential map. We shall show that σ_v lies in H^0 for any $v \in \mathfrak{g}_{\mathrm{app}}$ with $\|v\| < r_{\mathrm{inj}}/2$. Since H^0 is a Lie subgroup, this immediately implies that $G_{\mathrm{app}} \subset H^0$. If $v \in \mathfrak{g}_{\mathrm{app}}$, the geodesic segment σ_v can be approximated arbitrarily closely in the C^0 topology by admissible paths. Since right translations carry admissible paths to admissible paths, we see that σ_v can be approximated arbitrarily closely by admissible paths that start at e. The definition of an admissible path tells us that an admissible path which starts at e must lie in H^0. If we now assume that $\|v\| < r_{\mathrm{inj}}/2$, we can conclude that σ_v can be approximated arbitrarily closely by paths in $H^0 \cap U$. Such paths must lie in the component C of $H^0 \cap U$ that contains e. Since C is a closed subset of U (even though C is not closed in G), we see that σ_v lies in C and therefore in H^0.

Now we want to show that $G_{\mathrm{app}} = H^0$. Suppose not. Then there is $g \in H^0 \setminus G_{\mathrm{app}}$. Since $g \in H^0$, there is an admissible path $\alpha : [0,1] \to G$ with $\alpha(0) = e$ and $\alpha(1) = g$. There must be a $t_0 \in [0,1)$ and a sequence $\tau_n \searrow 0$ such that $\alpha(t_0) \in G_{\mathrm{app}}$ and $\alpha(t_0 + \tau_n) \notin G_{\mathrm{app}}$ for each n. Then $\beta(t) = \alpha(t - t_0)\alpha(t_0)^{-1}$ is an admissible path with $\beta(0) = e$ and $\beta(\tau_n) \notin G_{\mathrm{app}}$ for each n.

For $r > 0$, let $B_r = \{v \in \mathfrak{g} : \|v\| < r\}$. Let $K = \exp(\mathfrak{g}_{\mathrm{app}} \cap B_{r_0})$ and $N = \exp(\mathfrak{g}_{\mathrm{app}}^{\perp} \cap B_{r_0})$ with $r_0 < r_{\mathrm{inj}}$ chosen small enough so that there is at most one point in $Kg \cap N$ for each $g \in G$. For all large enough n, $\beta(\tau_n)$ will be close enough to e so that $K\beta(\tau_n) \cap N \neq \emptyset$ and therefore contains a unique point g_n.

Observe now that g_n is joined to e by an approximable path γ_n consisting of the arc of β from e to $\beta(\tau_n)$ followed by the geodesic segment in $K\beta(\tau_n)$ from $\beta(\tau_n)$ to g_n. Both $\beta(\tau_n)$ and g_n approach e as $n \to \infty$. By passing to a subsequence, we may assume that the sequence of unit vectors $\exp^{-1}(g_n)/\|\exp^{-1}(g_n)\|$ converges to a unit vector u. Since $u_n \in \mathfrak{g}_{\mathrm{app}}^{\perp}$ for each n, we have $u \in \mathfrak{g}_{\mathrm{app}}^{\perp}$. On the other hand, it is obvious that the sequence of paths γ_n satisfies the hypotheses of Lemma 9.5. Hence u is approximable. This contradiction shows that $G_{\text{app}} = H^0$. □

Theorem A is an immediate consequence of Theorem 9.1 and the following result of Brin, which shows that the condition $H^0 = G$ is dense in $\text{Ext}^{k-}(f, G)$.

Theorem 9.8 [[B1], Proposition 2.3] *Let f_φ be a C^{k-}, compact G-extension of an Anosov diffeomorphism $f : M \to M$. Then for any $\delta > 0$, there exists a C^{k-}, G-extension f_ψ such that:*

- $d_{C^{k-}}(f_\varphi, f_\psi) < \delta$, *where j is the greatest integer less than k;*
- $H^0(f_\psi) = G$.

We sketch the proof. First we prove the slightly easier result that φ can be perturbed so as to make $H = G$. We may choose the basepoint x_0 of our holonomy groups to be a periodic point, since periodic points for f are dense in M by the Anosov closing lemma. There are infinitely many distinct homoclinic orbits for x_0; indeed both $\mathcal{W}^u(x_0)$ and $\mathcal{W}^s(x_0)$ are dense in M, which is a basic set for f. Let $z_1, \ldots, z_c$, where $c = \dim G$ be homoclinic points for x_0 that belong to distinct orbits of f. For each z_i choose a neighborhood U_i with the properties that $x_0 \notin U_i$ and $f^k z_j \in U_i$ if and only if $j = i$ and $k = 0$. Now choose for each i a two-legged $\mathcal{W}^{u,s}(f)$-loop formed by a path from x_0 to z_i in $\mathcal{W}^u(x_0)$ and a path from z_i to x_0 in $\mathcal{W}^s(x_0)$. Let γ_i, $1 \leq i \leq c$, be the lifts of these loops to $\mathcal{W}^{u,s}(f_\varphi)$-paths that begin at (x_0, e) and end in $\{x_0\} \times G$. If $\widetilde{\varphi}$ is a perturbation of φ, let $\widetilde{\gamma}_i$ be the $\mathcal{W}^{u,s}(f_{\widetilde{\varphi}})$-path that starts at (x_0, e) and has the same projection to M as γ_i. Observe that if φ and $\widetilde{\varphi}$ differ only in U_i, then γ_j and $\widetilde{\gamma}_j$ both have the same endpoint for each $j \neq i$. It is not difficult to perturb φ within U_i so as to move the endpoint of $\widetilde{\gamma}_i$ in any desired direction within $\{x_0\} \times G$. Consequently we can arrange that the projections to G of the endpoints of the paths $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_c$ do not lie in any subgroup of G with dimension less than c. Since these projections belong to $H(f_{\widetilde{\varphi}})$, we obtain $H(f_{\widetilde{\varphi}}) = G$.

In order to adapt the preceding argument so that it applies to H^0, we need to change the paths γ_i so that their projections to M are null homotopic. To this end, we choose periodic points x_i, $1 \leq i \leq c$, close to x_0, and for each i a pair of heteroclinic points z_i^0 and z_i^1 such that there are short paths from x_0 to z_i^0 in $\mathcal{W}^u(x_0)$, from z_i^0 to x_i in $\mathcal{W}^s(x_i)$, from x_i to z_i^1 in $\mathcal{W}^u(x_i)$, and from z_i^1 to x_0 in $\mathcal{W}^s(x_0)$. This loop is null homotopic and its lift to a $\mathcal{W}^{u,s}(f_\varphi)$-path starting at (x_0, e) is the new γ_i. Of course we ensure that all of the points were chosen from different orbits of f. The sets U_i are now neighborhoods of the points z_i^0 with the properties that none of the periodic points $x_0, x_1, \ldots, x_c$ is in any U_i and $f^k z_j^a \in U_i$ if and only if $j = i$, $a = 0$ and $k = 0$. After these changes, the argument proceeds as before.

10 Reduction and algebraic conjugacy

In the following sections we shall frequently form algebraic factors in two special ways.

The first is a process that we call *reduction*. Suppose N is a closed normal subgroup of G and $p : G \to N\backslash G = G/N$ is the projection. Then the skew product $f_{p\circ\varphi} : M \times N\backslash G \to N\backslash G$ is the algebraic factor of f_φ induced by the constant map $\Phi(x) = N$ from M to G/N. By Lemma 8.6, we have $K(f_{p\circ\varphi}) = p(K(f_\varphi))$, $H(f_{p\circ\varphi}) = p(H(f_\varphi))$ and $H^0(f_{p\circ\varphi}) = p(H^0(f_\varphi))$.

The second process is *algebraic conjugacy*. In this case we quotient by the trivial group. Suppose $\varphi : M \to G$ and $\psi : M \to G$ are cohomologous with respect to f, i.e. there is a function $\Phi : M \to G$ such that

$$\psi(x) = \Phi(fx)^{-1}\phi(x)\Phi(x).$$

Let $\pi_\Phi(x, g) = (x, \Phi(x)^{-1}g)$. Then the following diagram commutes.

$$\begin{array}{ccc} M \times G & \xrightarrow{\quad f_\varphi \quad} & M \times G \\ \pi_\Phi \downarrow & & \pi_\Phi \downarrow \\ M \times G & \xrightarrow{\quad f_\psi \quad} & M \times G \end{array}$$

This says that f_ψ is the algebraic factor of f_φ induced by $\Phi : M \to G/\{e\} = G$. Similarly f_φ is the algebraic factor of f_ψ induced by the map $x \mapsto \Phi(x)^{-1}$.

When the group G is abelian, we shall use additive notation. The condition that φ and ψ are cohomologous becomes

$$\psi(x) = \varphi(x) - \Phi(f(x)) + \Phi(x)$$

and we have $\pi_\Phi(x, g) = g - \Phi(x)$.

We shall make several successive reductions and algebraic conjugacies. This is permissible, since it is an easy exercise to show that an algebraic factor of an algebraic factor is again an algebraic factor.

11 The Parry-Pollicott reduction

This section is preparatory to the proof of Theorem B.

Not every function $\varphi : M \to G$ can be lifted to a function from M to the universal cover of G; the group $\varphi_*(\pi_1(M))$ is an obstruction. Functions that can be lifted are much easier to work with, especially when one is interested in perturbing them in a specified manner. Parry and Pollicott [PP] have shown that, when G is abelian, and under certain assumptions on f, every skew product f_ϕ has an algebraic factor f_ψ such that ψ can be lifted. We recall their argument here.

Let M be an infranilmanifold. Then, as was shown in [Ma1], f^* is a hyperbolic linear automorphism of the torsion free part of $H^1(M, \mathbf{Z})$. It follows that

$$(f^* - Id)(H^1(M, \mathbf{Z})) \text{ is a finite index subgroup of } H^1(M, \mathbf{Z}) \quad (*).$$

The next proposition is from [PP].

Proposition 11.1 [PP] *Let $f_\varphi : M \times G \to M \times G$ be a C^{k-} skew product such that f satisfies property $(*)$ and the group G is a torus. Then f_φ has an algebraic factor $f_\psi : M \times G_0 \to M \times G_0$ such that G_0 is the quotient of G by a finite subgroup and $\psi : M \to G_0$ is homotopic to a constant map.*

Since ψ is homotopic to the identity, it can be lifted. After choosing an identification of G_0 with the standard torus $\mathbf{T}^d$, where $d = \dim G_0$, we can express ψ in the form $\psi(x) = (\exp(ir_1(x)), \ldots, \exp(ir_d(x)))$, where $r_i : M \to \mathbf{R}$, $i = 1, \ldots, d$.

Proof of Proposition 11.1: We shall consider the special case $G = \mathbf{T}$. The general result then follows by expressing G as a product of circles and applying the special case to each component of φ.

We use additive notation for $\mathbf{T}$.

The set $C^0(M, \mathbf{T})$ of continuous functions from M to $\mathbf{T}$ is an abelian group under pointwise addition. The *Bruschlinsky group* $\pi^1(M)$ is $C^0(M, \mathbf{T})$ modulo homotopy equivalence. We could just as well have defined $\pi^1(M)$ to be C^{k-} functions modulo C^{k-} homotopy; the proof is a standard exercise.

There is an isomorphism from $\pi^1(M)$ to $H^1(M, \mathbf{Z})$, defined as follows. Fix a choice of generator ζ for $H^1(\mathbf{T}, \mathbf{Z})$ (which amounts to choosing an orientation of $\mathbf{T}$). Observe that if φ_1 and φ_2 represent the same class in $\pi^1(M)$, then $\varphi_1^* \zeta = \varphi_2^* \zeta$. Thus there is a well defined homomorphism

$$i_\zeta : \pi^1(M) \to H^1(M, \mathbf{Z})$$

that takes φ to $\varphi^* \zeta$. In fact i_ζ is an isomorphism. For a proof, see e.g. Chapter II.7 of [Hu].

Let $\omega = i_\zeta[\varphi]$ and let m be the index of $(f^* - Id)H^1(M, \mathbf{Z})$ in $H^1(M, \mathbf{Z})$. Since $m\omega$ is in $(f^* - Id)H^1(M, \mathbf{Z})$, there is an $\omega' \in H^1(M, \mathbf{Z})$ such that $(f^* - Id)\omega' = m\omega$. Pick any C^{k-} representative $\Phi \in i_\zeta^{-1}\omega'$ and note that

$$i_\zeta[\Phi \circ f - \Phi] = (f^* - Id)\omega' = m\omega = i_\zeta[m\varphi].$$

The functions $\Phi \circ f - \Phi$ and $m\varphi$ are therefore homotopic, and so their difference $\overline{\psi} = m\varphi - \Phi \circ f + \Phi$ is homotopic to a constant.

Observe that $m\varphi$ is algebraically conjugate to $\overline{\psi}$.

Now let Γ_m be the finite subgroup of $\mathbf{T}$ with m elements. The group G_0 in the statement of the proposition will be $\mathbf{T}/\Gamma_m$. For $t \in \mathbf{T}$, let $p_m(t) = t + \Gamma_m$

and $q_m(t) = t/m + \Gamma_m$. The map $p_m : \mathbf{T} \to \mathbf{T}/\Gamma_m$ is the natural projection and $q_m : \mathbf{T} \to \mathbf{T}/\Gamma_m$ is a group isomorphism. Set $\psi = q_m \circ \overline{\psi}$. Then ψ is homotopic to a constant and is algebraically conjugate to $q_m \circ m\varphi$; the conjugacy is induced by $q_m \circ \Phi$. But $q_m \circ m\varphi = p_m \circ \varphi$, and $f_{p_m \circ \varphi}$ is the factor of φ obtained by reducing by Γ_m.

Thus, if we reduce f_φ by the finite subgroup Γ_m, we obtain a skew product that is algebraically conjugate to the skew product f_ψ, and $\psi : M \to \mathbf{T}/\Gamma_m = G_0$ is homotopic to a constant. It follows immediately that f_φ has an algebraic factor with the desired property. □

Remark: Instead of $m\varphi$, it would have sufficed to consider $m'\varphi$, where m' is the smallest positive integer such that $m'\omega \in (f^* - Id)H^1(M, \mathbf{Z})$. By taking $\varphi = m\chi$, we see that, for every $\chi : M \to \mathbf{T}^d$, $m\chi$ is cohomologous to a function which is homotopic to a constant map.

12 Proof of Theorem B

In this section, $f_\varphi : M \times G \to M \times G$ is a skew product, where $f : M \to M$ is a volume preserving C^{k-} Anosov diffeomorphism of a compact infranilmanifold, G is a compact Lie group and $f : M \to G$ is C^{k-}. We choose a point $x_0 \in M$ and consider the groups $H^0 = H^0_{x_0}(f_\varphi)$, $H = H_{x_0}(f_\varphi)$, and $K = K_{x_0}(f_\varphi)$ that were defined in Section 8.

From Theorem 9.1, we know that $H^0 = G$ implies that f_φ is stably ergodic. Theorem B follows from

Theorem 12.1 *If $H^0 \neq G$ then f_φ contains an algebraic factor of type* (1), (2) *or* (3)*, and so is not stably ergodic.*

As we remarked after the statement of Theorem B, it is easy to see that factors of types (1), (2) or (3) are not stably ergodic. Thus it follows from Theorem 12.1 that f_φ is stably ergodic only if $H^0 \neq G$.

The proof of Theorem 12.1 breaks naturally into four parts.

1. If $K \neq G$ then f_φ has an algebraic factor of type (1).
2. If $K = G$ and $\overline{H} \neq G$ then f_φ has an algebraic factor of type (2).
3. If $\overline{H} = G$, then $\overline{H^0} = G$.
4. If $\overline{H^0} = G$ and $H^0 \neq G$ then f_φ has an algebraic factor of type (3).

We remark here that the proofs of first two parts do not use the hypothesis that M is an infranilmanifold. Only the last two parts (Propositions 12.4 and 12.7) require the extra hypothesis.

Proposition 12.2 *If $K \neq G$ then f_φ has an algebraic factor of type* (1).

Proof: Let $\Phi : M \to G$ be the function given by Proposition 8.2 and let $\Phi_0 = p \circ \Phi : M \to G/K$, where $p : G \to G/K$ is the canonical projection. Then Φ_0 is C^{k-} and maps $\approx$ equivalence classes in $M \times G$ to constant sections of $M \times K\backslash G$, where $\approx$ is the equivalence relation defined in Section 8.1. The diagram

$$\begin{array}{ccc} M \times G & \xrightarrow{\quad f_\varphi \quad} & M \times G \\ {\scriptstyle (x,g) \mapsto (x, \Phi_0(x)^{-1} g)} \downarrow & & \downarrow {\scriptstyle (x,g) \mapsto (x, \Phi_0(x)^{-1} g)} \\ M \times K\backslash G & \longrightarrow & M \times K\backslash G \end{array}$$

induces a map from $M \times K\backslash G \to M \times K\backslash G$ that is an algebraic factor of f_φ. Since f_φ maps each $\approx$ equivalence class into itself, this induced map must be $f \times Id$. Thus f_φ has an algebraic factor of type 1. □

Proposition 12.3 *If $K = G$ and $\overline{H} \neq G$ then f_φ has an algebraic factor of type* (2).

Proof: By Proposition 8.4, we know that $\overline{H}$ is a normal subgroup of K. Since we are assuming that $K = G$, this implies that $\overline{H}$ is normal in G. Thus we can reduce by $\overline{H}$ to obtain a skew product $f_{\varphi_1} : M \times K/\overline{H} \to M \times K/\overline{H}$ that has $H(f_{\varphi_1}) = \overline{H}/\overline{H} = \{e\}$. The group $K/\overline{H}$ is abelian by Lemma 8.5. Hence we can reduce again, this time by a closed codimension 1 subgroup of $K/\overline{H}$, to obtain a skew product $f_{\varphi_2} : M \times \mathbf{T} \to M \times \mathbf{T}$ that has $H(f_{\varphi_2}) = \{e\}$.

We now construct an algebraic conjugacy from f_{φ_2} to $f \times R_\alpha$ for a suitable $\alpha \in \mathbf{T}$. Note that $f \times R_\alpha$ is the skew product induced by the constant map that takes all points in M to α.

Recall the equivalence relation $\sim$ introduced in Section 8 and its properties. Since $H(f_{\varphi_2}) = \{e\}$, each of the $\sim$ equivalence classes for f_{φ_2} intersects each fiber of $M \times \mathbf{T}$ in a single point, and is therefore a graph. In particular, there is a function $\Phi : M \to \mathbf{T}$ such that the equivalence class of (x_0, e_0) is the graph of Φ. It follows from Proposition 8.2 that Φ is C^{k-}.

Both f_φ and translations of $M \times \mathbf{T}$ carry $\sim$ equivalence classes into $\sim$ equivalence classes. Thus if E is the $\sim$ equivalence class of (x_0, e), there is $\alpha \in \mathbf{T}$ such that $f_{\varphi_2} E = R_\alpha E$. Now for any $x \in M$, we can work out in two different ways where $f_{\varphi_2} E$ intersects $\{f(x)\} \times \mathbf{T}$. We obtain

$$(f(x), \Phi(f(x)) + \alpha) = (f(x), \Phi(x) + \varphi_2(x)).$$

Hence $\varphi_2(x) = \Phi(f(x)) - \Phi(x) + \alpha$, and the following diagram commutes.

$$\begin{array}{ccc} M \times \mathbf{T} & \xrightarrow{\quad f_{\varphi_2} \quad} & M \times \mathbf{T} \\ {\scriptstyle (x,g) \mapsto (x, g - \Phi(x))} \Big\downarrow & & \Big\downarrow {\scriptstyle (x,g) \mapsto (x, g - \Phi(x))} \\ M \times \mathbf{T} & \xrightarrow{\quad f \times R_\alpha \quad} & M \times \mathbf{T} \end{array}$$

Thus $f \times R_\alpha$ is an algebraic factor of f_{φ_2} and hence of f_φ. It is a factor of type 2. □

In the remaining propositions we use the assumption that M is an infranilmanifold. Since all Anosov diffeomorphisms of infranilmanifolds have fixed points, we may now assume that the point x_0, where our holonomy groups are based, is a fixed point of f. This is convenient in our proof of Lemma 12.6.

Proposition 12.4 *If* $\overline{H} = G$*, then* $\overline{H^0} = G$*.*

Proof: We suppose that $\overline{H^0} \neq G$ and derive a contradiction. It follows from Proposition 8.1 (part 2) that $\overline{H^0}$ is a normal subgroup of $\overline{H} = G$. Thus we can reduce by $\overline{H^0}$ to obtain a skew product $f_{\varphi_1} : M \times G_1 \to M \times G_1$, where $G_1 = \overline{H^0} \backslash G$. Let $H_1^0 = H^0(f_{\varphi_1})$ and $H_1 = H(f_{\varphi_1})$. By Lemma 8.6,

$$H_1^0 = H^0(f_{\varphi_1}) = p(H^0) = \{e\},$$

while

$$\overline{H_1} = \overline{H(f_{\varphi_1})} = p(\overline{H}) = p(G) = G_1.$$

By Proposition 8.1, there is a surjective homomorphism

$$\rho : \pi_1(M) \to H_1/H_1^0 = H_1.$$

Since M is an infranilmanifold, $\pi_1(M)$ has a normal nilpotent subgroup of finite index, and so H_1 has a normal nilpotent subgroup N of finite index. Since $\overline{H_1} = G_1$, the closure of N is a normal nilpotent subgroup of finite index in G_1. Since $\overline{N}$ is compact, it contains a torus of finite index, and so G_1 contains a torus of finite index. But G_1 is connected, so G_1 is a torus, which has positive dimension by our assumption that $\overline{H^0} \neq G$. After reducing by a closed codimension 1 subgroup of G_1, we may assume that G_1 is a circle.

We will need:

Lemma 12.5 *Let* $f : M \to M$ *be an Anosov diffeomorphism of a compact infranilmanifold and let* $f_{\varphi_1} : M \times \mathbf{T} \to M \times \mathbf{T}$ *be a skew product such that* $H^0(f_{\varphi_1}) = \{0\}$*. Then* $H(f_{\varphi_1})$ *is finite and is trivial if* φ *is homotopic to a constant.*

Proof: By Proposition 11.1, f_{φ_1} has an algebraic factor $f_\psi : M \times \mathbf{T} \to M \times \mathbf{T}$, where $\psi(x) = \exp(ir(x))$, for some function $r : M \to \mathbf{R}$. Furthermore, $H(f_\psi) = H(f_{\varphi_1})/F$, where F is finite. If φ is homotopic to a constant, we can take $\psi = \varphi$. Thus it suffices to show that $H(f_\psi)$ is trivial.

Let $p_1 : \mathbf{R} \to \mathbf{T}$ be the projection $p_1(x) = \exp(ix)$. Then $\psi = p_1 \circ r$, and the skew product $f_r : M \times \mathbf{R} \to M \times \mathbf{R}$ has f_ψ as a factor. According to Lemma 8.6, $H(f_\psi) = p_1(H(f_r))$ and $p_1(H^0(f_r)) = H^0(f_\psi)$. Since $H^0(f_\psi)$ is trivial, $H^0(f_r)$ is a proper analytic subgroup of $\mathbf{R}$, and therefore $H^0(f_r) = \{0\}$. Lemma 12.5 now follows from the next lemma.□

Lemma 12.6 *Let $f : M \to M$ be an Anosov diffeomorphism of a compact infranilmanifold and let $f_r : M \times \mathbf{R} \to M \times \mathbf{R}$ be a skew product such that $H^0(f_r) = \{0\}$. Then $H(f_r) = \{0\}$.*

Proof: Since $H^0(f_r)$ is trivial, part 3 of Proposition 8.1 gives us a surjective homomorphism $\rho : \pi_1(M) \to H(f_r)$. Since $H(f_r) \subset \mathbf{R}$, $H(f_r)$ is an abelian group and ρ must factor through $H_1(M, \mathbf{Z})$. Since $\mathbf{R}$ is also torsion-free, ρ must actually factor through the torsion-free part of $H_1(M, \mathbf{Z})$. Thus ρ induces a linear map L_ρ on $H_1(M, \mathbf{R})$, which in turn corresponds to a class $\omega_\rho \in H^1(M, \mathbf{R})$.

Recall that the base x_0, at which we are computing the holonomy groups, is a fixed point of f. Since $H(f_r)$ is abelian, part 4 of Proposition 8.1 tells us that $\rho \circ f_* = \rho$.

It follows immediately that $f^*\omega_\rho = \omega_\rho$. Since M is an infranilmanifold, the map $(f^* - I) : H^1(M, \mathbf{R}) \to H^1(M, \mathbf{R})$ is invertible. This implies that $\omega_\rho = 0$. Hence ρ is the trivial homomorphism and $H(f_r) = \{0\}$. □

We now complete the proof of Proposition 12.4. Applying Lemma 12.5 to f_{φ_1}, we obtain that H_1 is finite. This contradicts the fact that $\overline{H_1} = G_1$. Going back to f_φ, we must have $\overline{H^0} = G$. □

Proposition 12.7 *If $\overline{H^0} = G$ and $H^0 \neq G$ then f_φ has an algebraic factor of type* (3).

Proof: Our first step is to reduce to the case where G is a torus. A similar reduction appears in Field-Parry [FP]. Let G' be the derived subgroup of G, i.e. the analytic subgroup whose Lie algebra is $[\mathfrak{g}, \mathfrak{g}]$.

Lemma 12.8 *G' is a closed normal subgroup. If D is a dense analytic subgroup of G, then $G' \subset D$.*

Note that H^0 is a dense analytic analytic subgroup by Proposition 8.1 and our hypothesis that $\overline{H^0} = G$.

Proof of Lemma 12.8: Let $\mathfrak{z}$ be center of $\mathfrak{g}$ and let $\mathfrak{g}'$ be the orthogonal complement of $\mathfrak{z}$ with respect to our bi-invariant inner product. Bi-invariance means that the inner product is Ad-invariant. Differentiating shows that for any $X, Y, Z \in \mathfrak{g}$ we have

$$\langle [Y, X], Z \rangle + \langle X, [Y, Z] \rangle = 0.$$

If $Z \in \mathfrak{z}$, we obtain

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle = 0$$

for any $X, Y \in \mathfrak{g}$. Hence $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}'$ and $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$ (as is $\mathfrak{z}$). The center of $\mathfrak{g}'$ is trivial because an element of the center of $\mathfrak{g}'$ would also commute with all the elements of $\mathfrak{z}$ and would have to belong to $\mathfrak{z}$. On the other hand, the Killing form on $\mathfrak{g}'$ is negative semidefinite, because $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$, which is the Lie algebra of a compact Lie group. It follows that $\mathfrak{g}'$ is semisimple. Hence $\mathfrak{g}' = [\mathfrak{g}', \mathfrak{g}']$; see e.g. section 5.2 of [Hum].

Thus $\mathfrak{g}'$ is a semisimple ideal and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. It follows immediately that G' is a compact semisimple Lie group. In particular G' is closed and normal as a subgroup of G.

Now suppose that D is a dense analytic subgroup of G. Let A be the analytic subgroup of G tangent to $\mathfrak{z}$. Then A is normal and G/A is compact and semisimple, because its Lie algebra is isomorphic to $\mathfrak{g}'$. The projection D_A of D to G/A is a dense analytic subgroup of G/A. Since D_A is dense, any element of the center of D_A must commute with all elements of G/A. The center of G/A is finite, so D_A must also have finite center and the Lie algebra of D_A must have trivial center. On the other hand, D_A supports a bi-invariant metric because we can take a bi-invariant metric on G/A and restrict it to D_A. These two properties imply that D_A is compact (see e.g. Corollary 21.4 in [Mi]). Thus D_A must be the whole of G/A since it is both closed and dense. Consequently the Lie algebra $\mathfrak{d}$ of D contains a complement to $\mathfrak{z}$. But this implies that $[\mathfrak{d}, \mathfrak{d}] = [\mathfrak{g}, \mathfrak{g}]$. Hence $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{d}$ and $G' \subset D$. □

It follows from the previous lemma that we can reduce by G'. This yields a skew product $f_{\varphi_1} : M \times G_1 \to M \times G_1$, where $G_1 = G/G'$, which is a torus since it is a connected abelian Lie group. Lemma 8.6 tells that $\overline{H^0}(f_{\varphi_1})$ and $H^0(f_{\varphi_1})$ are the projections to G_1 of $\overline{H^0}$ and H^0 respectively. Hence $\overline{H^0}(f_{\varphi_1}) = G_1$. We also have $H^0(f_{\varphi_1}) \neq G_1$. For we have $G' \subset H^0$ by Lemma 12.8, and if H^0 projected to the whole of G_1, that would force $H^0 = G$.

Proposition 12.7 now follows by applying the next lemma to f_{φ_1}. □

Lemma 12.9 *Let $f : M \to M$ be an Anosov diffeomorphism of a compact infranilmanifold M, and let $f_\varphi : M \times \mathbf{T}^d \to M \times \mathbf{T}^d$ be a skew product such that $H^0 \neq G$, but $\overline{H^0} = G$. Then f_φ has an algebraic factor of type (3).*

Proof: By Proposition 11.1, f_φ has an algebraic factor $f_\psi : M \times \mathbf{T}^d \to M \times \mathbf{T}^d$, where $\psi(x) = (\exp(ir_1(x)), \ldots, \exp(ir_d(x)))$. We shall show below that there are real numbers $\lambda_1, \dots, \lambda_d, \beta$, and a C^{k-} function $\Phi : M \to \mathbf{R}$ such that for all $x \in M$,

$$\lambda_1 r_1(x) + \cdots + \lambda_d r_d(x) = \Phi(f(x)) - \Phi(x) + \beta.$$

From this it follows that ψ is cohomologous to a map $\chi : M \to \mathbf{T}^d$ such that $\chi(x) = (\exp(is_1(x)), \dots, \exp(is_d(x)))$ and

$$\lambda_1 s_1(x) + \cdots + \lambda_d s_d(x) = \beta.$$

Indeed, if $\lambda_j \neq 0$, we can take $s_j(x) = r_j(x) - \lambda_j^{-1}[\Phi(f(x)) - \Phi(x)]$ and $s_i(x) = r_i(x)$ for $i \neq j$. Then f_ψ is algebraically conjugate to f_χ, which is the desired algebraic factor of type (3).

Let $p_1 : \mathbf{R}^d \to \mathbf{T}^d$ be the projection

$$p_1(\theta_1, \dots, \theta_m) = (\exp(i\theta_1), \dots, \exp(i\theta_d)),$$

and let $r : M \to \mathbf{R}^d$ be the function

$$r(x) = (r_1(x), \dots, r_d(x)).$$

Then $\psi = p_1 \circ r$, and the skew product $f_r : M \times \mathbf{R}^d \to M \times \mathbf{R}^d$ has f_ψ as a factor. Let $H_1 = H(f_r)$, and let $H_1^0 = H^0(f_r)$. According to Lemma 8.6, $p(H_1^0) = H^0 \neq G$, and so H_1^0 is a proper subgroup of $\mathbf{R}^d$, i.e., a proper subspace of $\mathbf{R}^d$. Let $\Lambda : \mathbf{R}^d \to \mathbf{R}$ be a nontrivial linear map such that $H_1^0 \subset \ker \Lambda$.

Let $s = \Lambda \circ r$. For the skew-product $f_s : M \times \mathbf{R} \to M \times \mathbf{R}$, we have, by Lemma 8.6, that

$$H^0(f_s) = \Lambda(H_1^0) = \{0\}.$$

By Lemma 12.6, it follows that $H(f_s) = \{0\}$. Let $\Phi : M \to \mathbf{R}$ be the C^{k-} function defined by applying Proposition 8.2 to f_s. As in the proof of Proposition 12.3, we see that there is a real number β such that

$$s(x) = \Phi(f(x)) - \Phi(x) + \beta.$$

In other words,

$$\Lambda \circ r(x) = \Phi(f(x)) - \Phi(x) + \beta,$$

and there are real numbers $\lambda_1, \dots, \lambda_d$ such that

$$\lambda_1 r_1(x) + \cdots + \lambda_d r_d(x) = \Phi(f(x)) - \Phi(x) + \beta.$$

□

Proof of Corollary B3: Suppose that $m\varphi = \Phi \circ f - \Phi + c$. Then the proof of Proposition 12.3 shows that $f_{m\varphi}$ is algebraically conjugate to $f \times R_c$, Since $f_{m\varphi}$ is an algebraic factor of f_φ by Proposition 11.1, it follows from Lemma 3.1 that f_φ is not stably ergodic.

Conversely, if f_φ is not stably ergodic, then $H^0(f_\varphi) \neq \mathbf{T}$ and hence $H^0(f_\varphi) = \{e\}$. Since $f_{m\varphi}$ is an algebraic factor of f_φ, we obtain $H^0(f_{m\varphi}) = \{e\}$, by Lemma 8.6. By the remark after Proposition 11.1, $m\varphi$ is cohomologous to a function $\psi : M \to \mathbf{T}$ that is homotopic to a constant. Since $f_{m\varphi}$ and f_ψ are algebraically conjugate, we have $H^0(f_\psi) = \{e\}$. Hence $H(f_\psi) = \{e\}$ by Lemma 12.5. It now follows from the proof of Proposition 12.3 that ψ is cohomologous to a constant. Hence $m\varphi$ is cohomologous to the same constant.□

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