Stable Ergodicity

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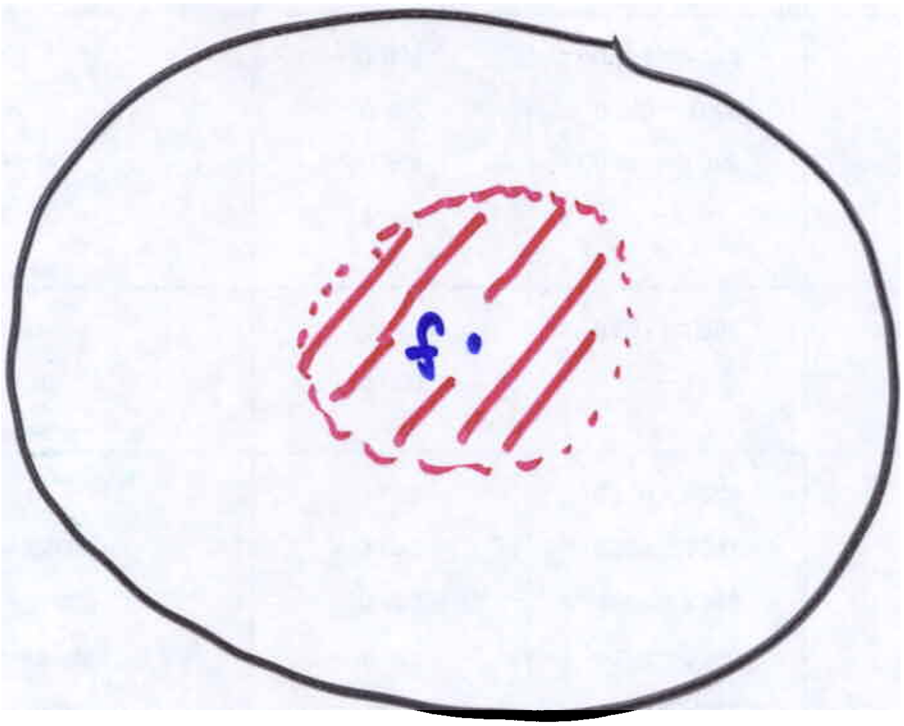
Northwestern

M compact



$\mathrm{Diff}^2_{\mathrm{Vol}}(M)$

/// = ergodic

C^1 open

Two ways to create stable ergodicity

① Robust non uniform hyperbolic (Alves Bonatti Viana)

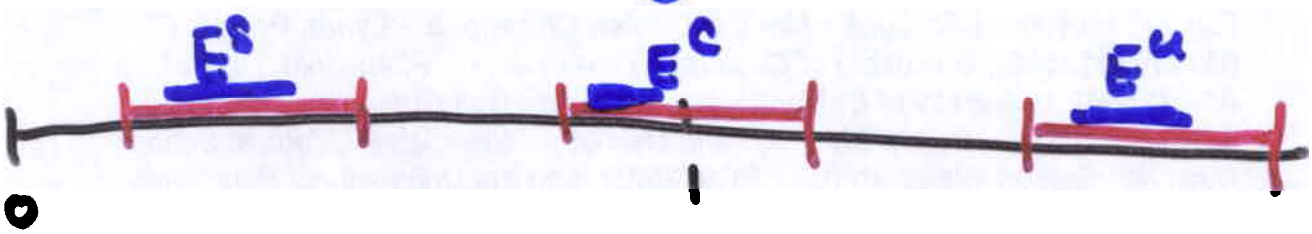
② Uniform partial hyperbolicity (Pugh- Shub)

②' Anosov

$PHD^2_{Vol}(M)$ = C^2 vol pres partially hyperbolic diffeos of M

$$TM = E^u \oplus E^c \oplus E^s$$

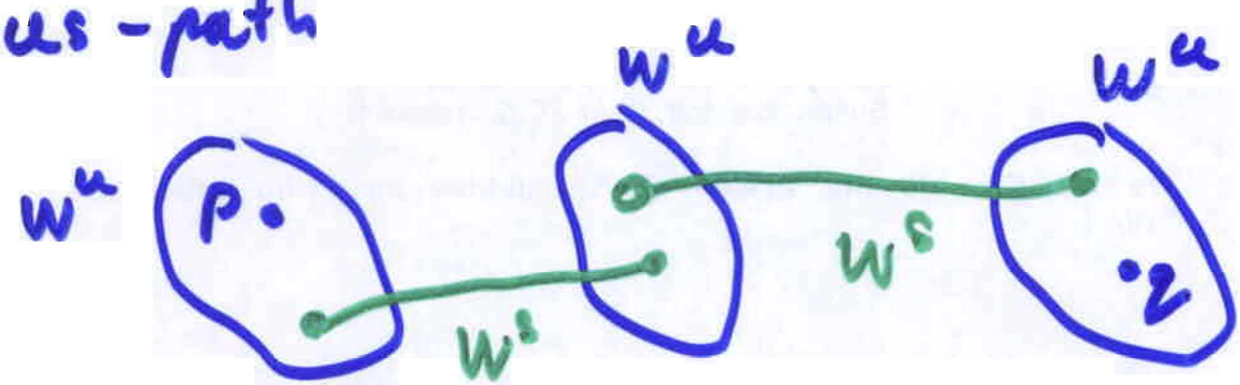
invariant splitting



$\|df v\| / \|v\| \quad v \neq 0$

W^u, W^s foliations tangent to E^u, E^s

us-path



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Examples

① Anosov diffeos

② Time 1 maps of Anosov flows

③ Lie group extensions of Anoso[v]

diffeos $f: M \to M$ Anosov

$\varphi: M \to G$

$$f_\varphi(x, g) = (f(x), \varphi(x) g)$$

④ $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$ on $\mathbb{T}^4$ other algebrai[c] examp[les]

Conjecture (Pugh & Shub)

$PHD^2_{Vol}(M)$ contains a C^2-dense C^1-open subset of ergodic diffeos

Accessibility property:

Any two points joined by a as-path

Conjecture 1

$PHD^2(M)$ & $PHD^2_{Vol}(M)$ contain C^2-dense, C^1-open subsets of diffeos with the accessibility property

Conjecture 2

If $f \in PHD^2_{Vol}(M)$ has accessibility then f is ergodic

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Results toward Conjecture 1

Dolgopyat & Wilkinson

Accessibility for C^1-open, C^1-dense subsets of diffeos

Nitica - Török

If $\dim E^c = 1$ + mild conditions

Pugh-Shub Theorem

$f \in PHD^2_{Vol}(M)$

- essentially accessible
- dynamically coherent
- center bunched

$\Rightarrow$ f is ergodic

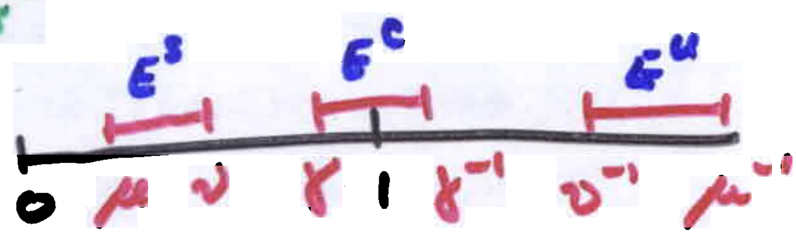
Essentially accessible: if A is W^s-saturated and W^u-satured, then A has 0 or full volume

Dynamically coherent: W^{cu} tangent to $E^c \oplus E^u$, W^{cs} tangent to $E^c \oplus E^s$

$\Rightarrow W^c$ tangent to E^c

Center bunched

$\boxed{\nu < \gamma^2}$



Hypotheses of Pugh-Shub Theorem are reasonably robust

partial hyperbolicity
center bunching } C^1 open

Proposition (based on Hirsch, Pugh, Shub)
If f is dynamically coherent and E^c is C^1, then any g sufficiently C^1 close to f is dynamically coheren

Robust essential accessibility
Katok-Kononenko — linear maps of geod flo
B, Pugh, Wilkinson — abstraction of K-K.
Rodriguez Hertz

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix} \quad \text{KAM}$$

Proof of Pugh-Shub Thm

Hopf argument

Show that if $\hat{\varphi}$ is the Birkhoff average of a continuous function $\varphi: M \to \mathbb{R}$, then $\hat{\varphi}$ is a.e. const.

Key fact Sublevel sets of $\hat{\varphi}$ are essentially W^s-saturated and essentially W^u-saturated

Proof. Sublevel sets of $\hat{\varphi}^+$ are W^s-sa
sublevel sets of $\hat{\varphi}^-$ are W^u-s

Theorem (B & Wilkinson)

Under hypotheses of Pugh-Shub Thm, suppose A is essentially W^u-saturated and essentially W^s-saturated. Then the set $\hat{A}$ of Lebesgue density points of A is both W^u-saturated and W^s saturated.

Henceforth

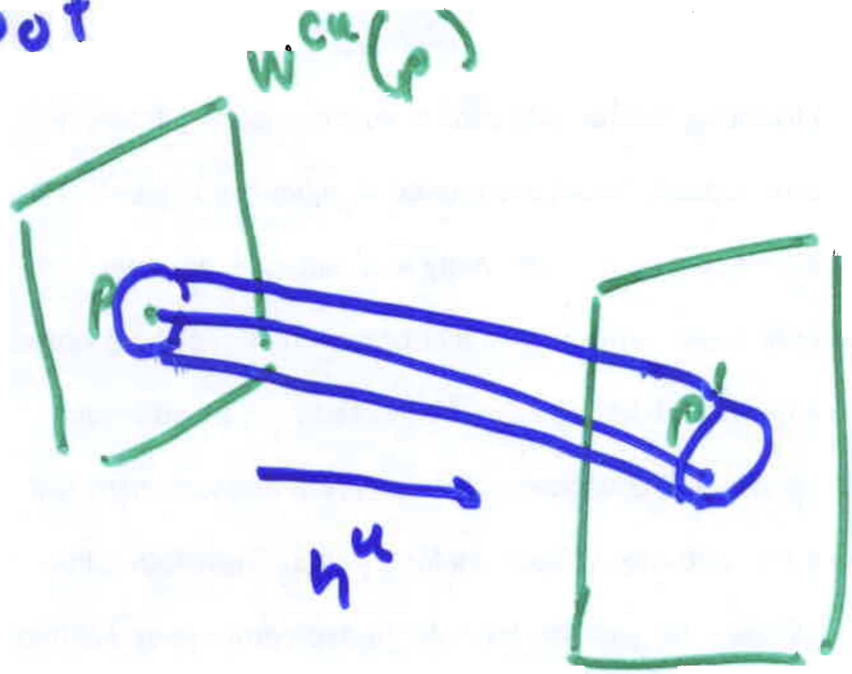
Replace A by its W^u saturate.

Assume A is W^u-saturated

essentially W^s-saturated

Prove $\hat{A}$ is W^u-saturated

"Proof"



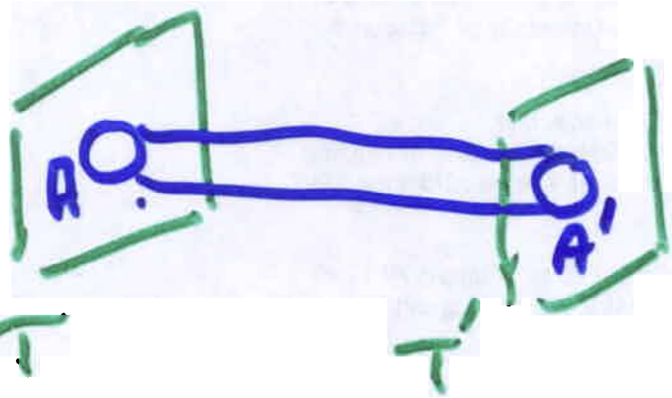
- need h^u absolutely continuous
- need h^u quasi conformal

h^u(small set)

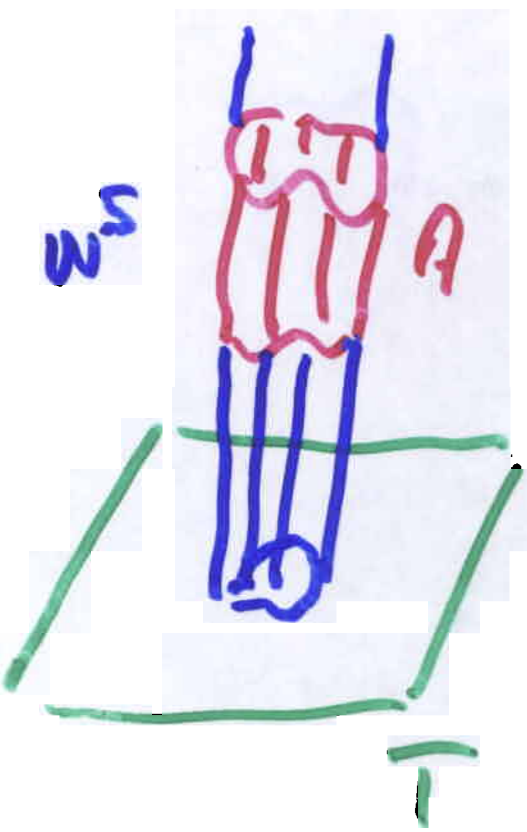
$\approx$ small set of same type

Brin-Pesin
Pugh Shub
(Anosov Sinai)

W^u & W^s are absolutely continuous with bounded jacobians

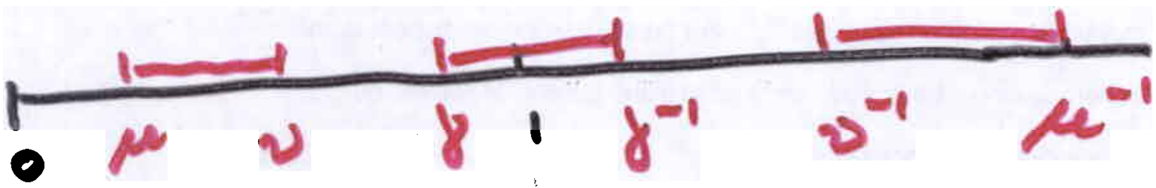


$$\frac{1}{C} \le \frac{m_T(A)}{m_{T'}(A')} \le C$$



$$\frac{1}{C} m(A) \le \int_T m_{W^s}\left(A \cap W^s_{loc}(x)\right) dm_T(x) \le C\, m(A)$$

Pugh & Shub — juliennes



$$B^c_n(p) = W^c(p, \gamma^n)$$

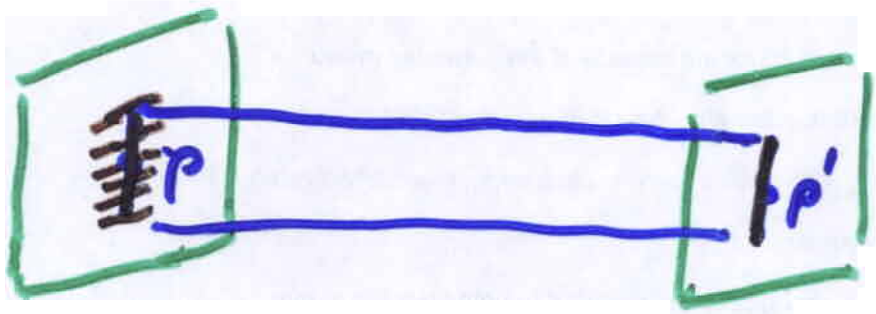
$$J^{cs}_n(p) = \bigcup_{q \in B^c_n(p)} f^n\left(W^s(f^{-n}q, \sigma^n)\right)$$

W^u- holonomy quasi preserves center-stable juliennes

(Pugh-Shub)

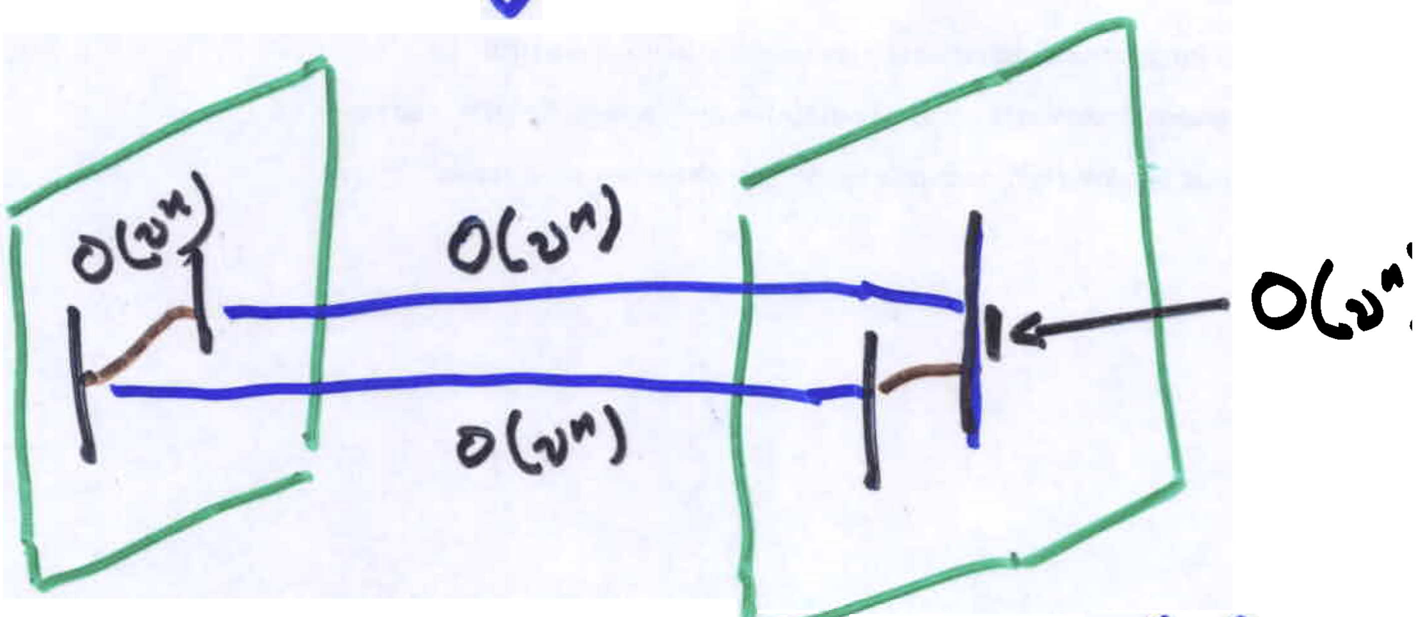
Ideas from Pugh-Shub

$\nu < \gamma^2 \Rightarrow W^u$ is C^1 inside W^{cu}-leaves



$h^u(B^c_n(p)) \approx B^c_n(p')$

$\Big\downarrow f^{-n}$



$\nu < \gamma^2 \Rightarrow \gamma^{-n}\nu^n < \gamma^n$

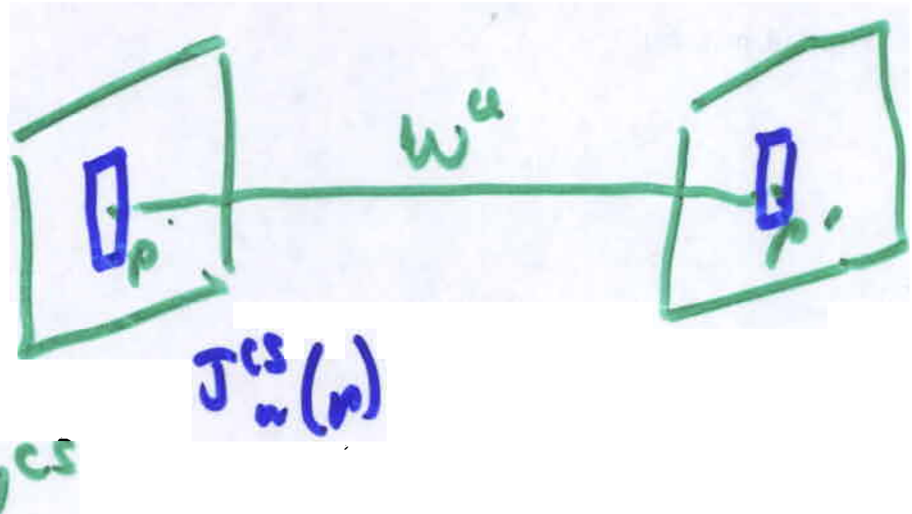
want to show

A W^u-saturated & ess. W^s-sat

$\Rightarrow$ Lebesgue density points of A are W^u-saturated

julienne quasiconformality and absolute continuity of W^u

$\Rightarrow$ J^{cs}-density points of A are W^u-saturated

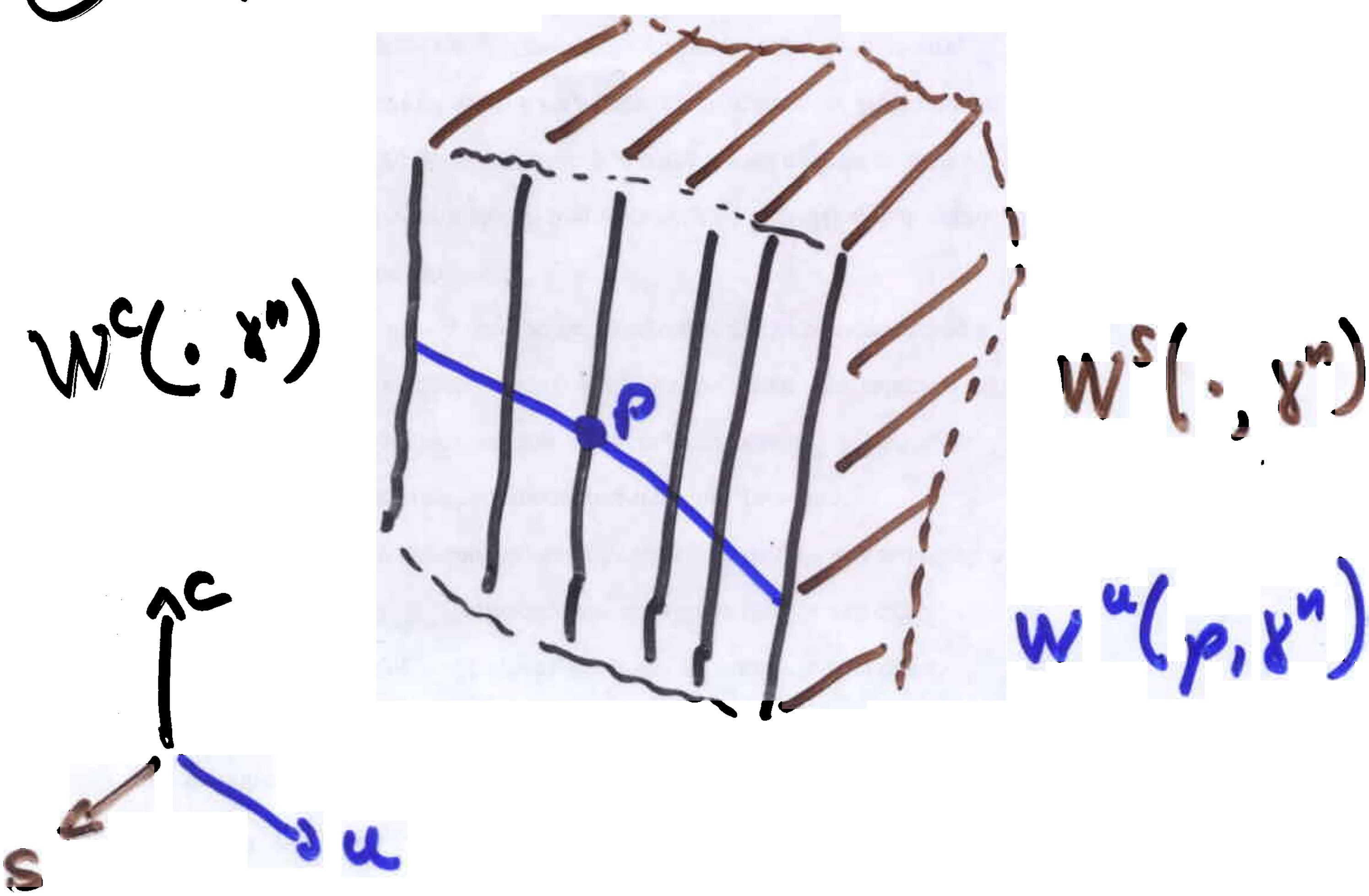


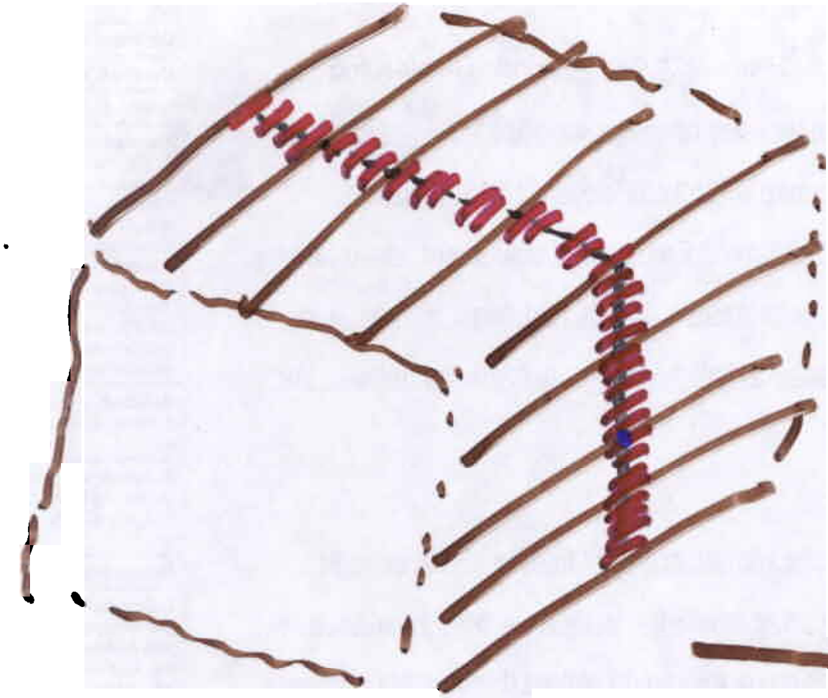
B, Wilkinson

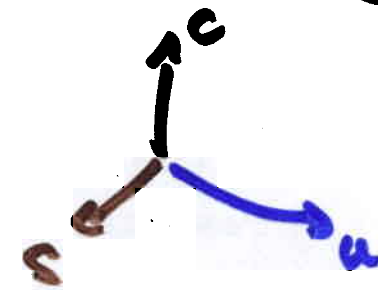
p is a Lebesgue density point of $A \iff p$ is a $\mathcal{J}^{cs}$-density poin

A is W^u-sat
ess W^s-sat.

① Replace $B(p, \gamma^n)$ by a "cube







— $W^s(\cdot, \gamma^n)$

— $f^n(W^s(f^{-n}(\cdot), v^n$

$J^{cs}_n(p)$