Partial hyperbolicity and stable ergodicity

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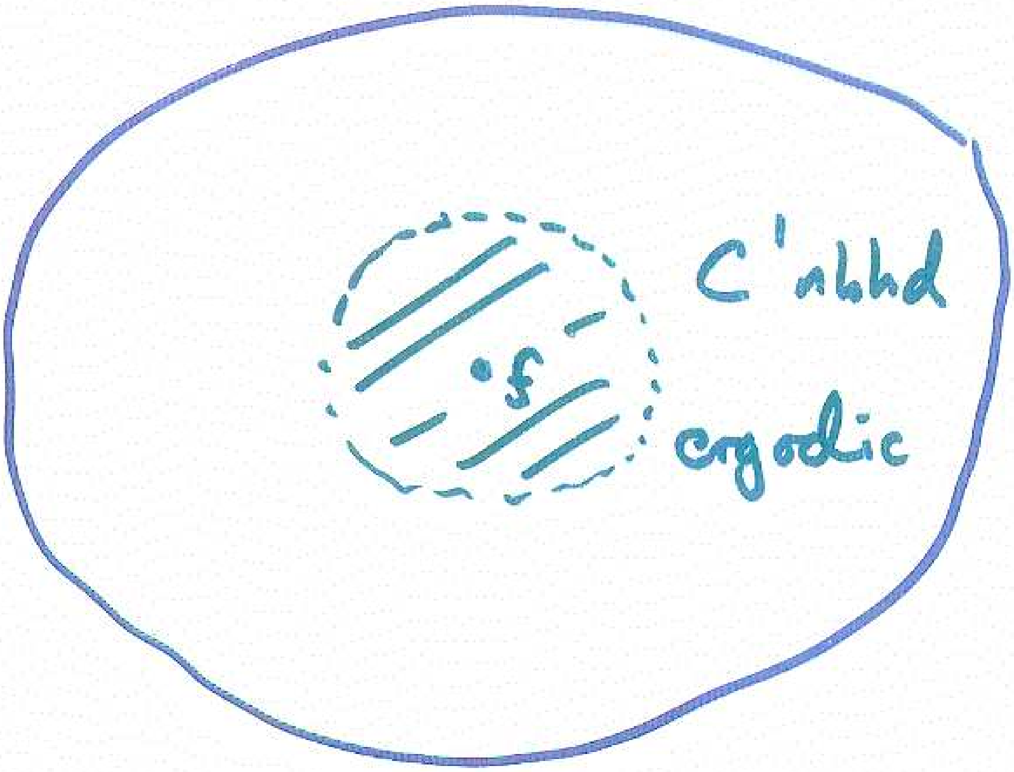
Stable ergodicity



$\mathrm{Diff}^r_{vol}(M)$

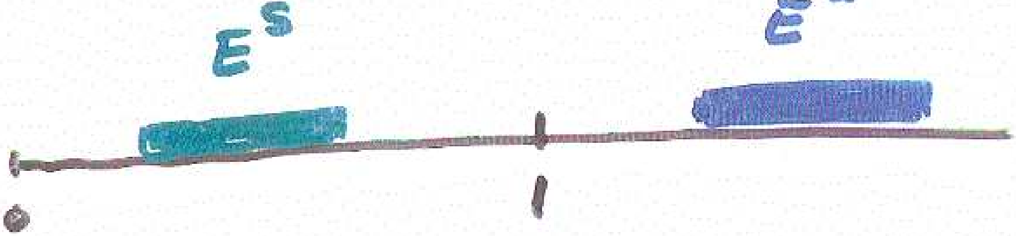
M compact

$r > 1$ (2 or $1+\alpha$)

Anosov diffeomorphisms

Tf-invariant splitting

$$TM = E^u \oplus E^s$$



$$\frac{\|Tf v\|}{\|v\|} \qquad v \neq 0$$

Foliations

W^u tangent to E^u

W^s tangent to E^s

Anosov Sinai: W^u & W^s are absolutely continuous

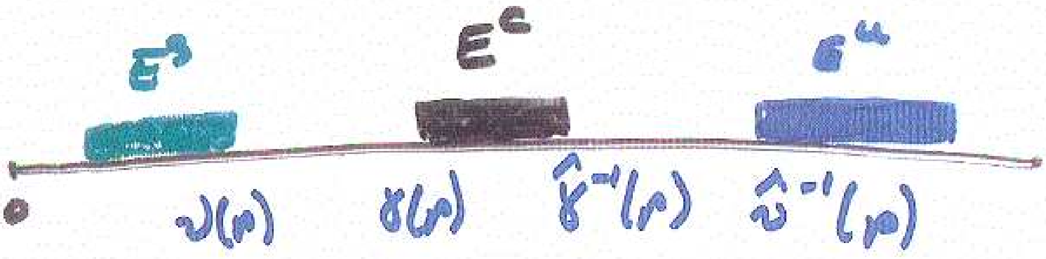
This allows Hopf argument to prove ergodicity

Anosov is C^1 robust

Partial hyperbolicity

Tf-invariant splitting

$$TM = E^u \oplus E^c \oplus E^s$$



$$\frac{\|T_p f v\|}{\|v\|}$$

$v \neq 0$

$$\nu(p) < 1 < \hat{\nu}^{-1}(p)$$

$\nu, \hat{\nu}, \gamma, \hat{\gamma}$ continuous

Foliations

W^u, W^s tangent to E^u, E^s

absolutely continuous — Brin-Pesin, Pugh-Shub

C^1 robust

Examples

1. Anosov diffeos E^c trivial

1'. Anosov diffeos, E^c corresponding to central eigenvalues

Examples with no W^c robustly

Nicholas Gourmelon

2. Time 1 maps of Anosov flows

3. Lie group extensions of Anosov diffeos $f: M \to M$ Anosov

$\varphi: M \to G$

$$f_\varphi(x, g) = (f(x), \varphi(x) g)$$

4. $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$ on $\mathbb{T}^4$

Federico Rodriguez Hertz

5 Many automorphisms of Lie groups, homogeneous spaces

6 Any extension of an Anosov diffeo in which fiber maps are dominated by the Anosov

$$\begin{tikzcd} B \arrow[r, "F"] \arrow[d] & B \arrow[d] \\ B \arrow[r, "f", "\text{Anosov}"'] & B \end{tikzcd}$$

$F|B_p$ dominated by f

Fast-slow dynamics with fast dynamics Anosov

fast dynamics non uniformly hyperbolic ?

1970's Brin-Pesin

Ergodicity for PHD's with nice E^c (W^c Lipschitz)

Not robust

1990's Grayson-Pugh-Shub

Stable ergodicity of time one maps of geodesic flow of a surface of constant neg. curv

Pugh-Shub conjecture

Stable ergodicity is C^k dense in $PHD^r_{Vol}(M)$ for any $k \leq r$ $(r \geq 2)$

Saturation

A set is W - saturated if it is a union of entire W leaves

bisaturated = W^u saturated & W^s saturated

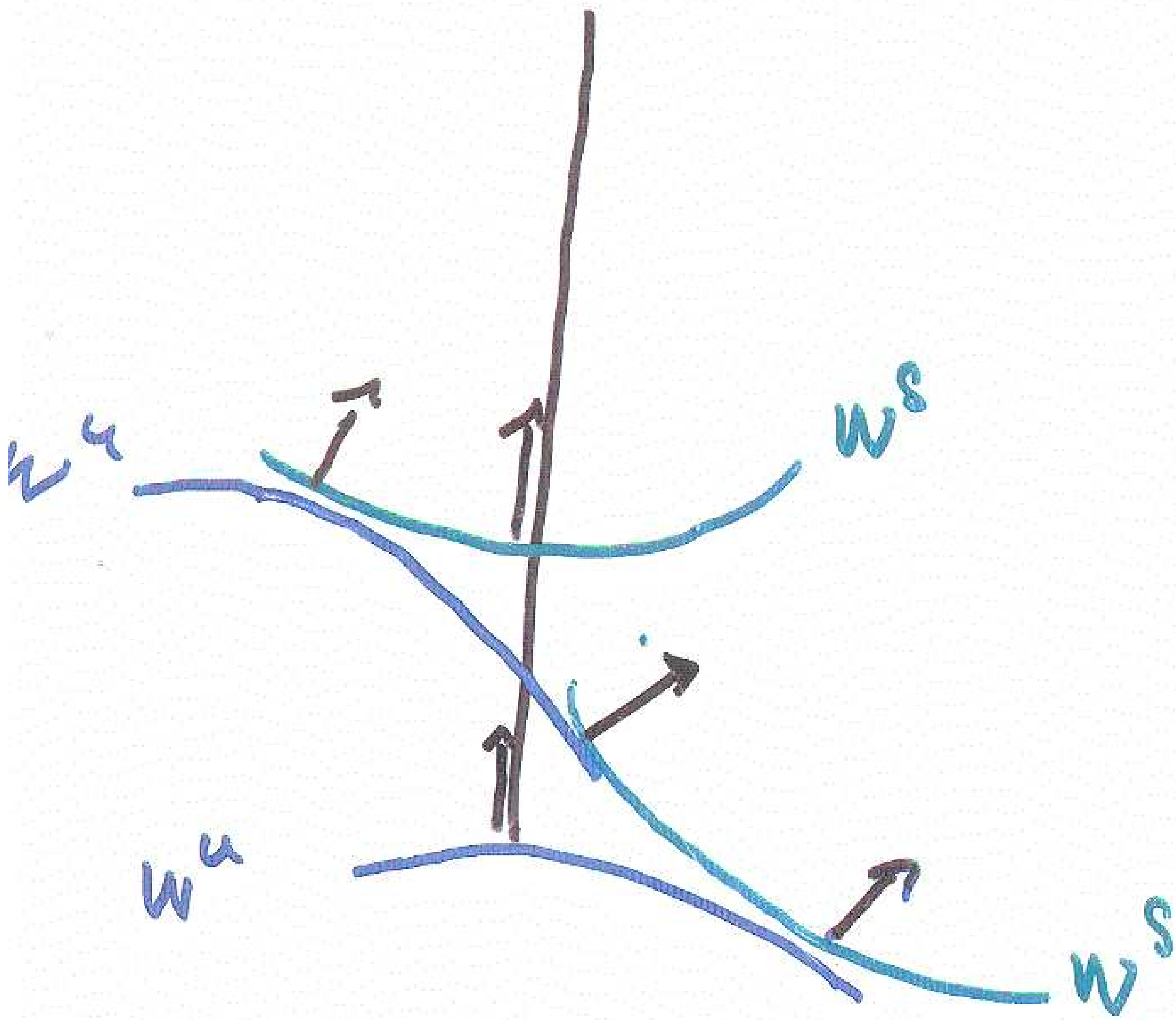
Accessibility for f

ϕ and M are the only bisaturated sets

Essential accessibility for f

Measurable bisaturated sets have zero or full measure

Time one map of geodesic flow is accessible



Pugh-Shub conjectures

Conjecture 1

Stable accessibility is C^k dense in $PHD^r(M)$ and $PHD^r_{Vol}(M)$ for any $k \le r$

Conjecture 2

If $f \in PHD^2_{Vol}(M)$ is essentially accessible, then f is ergodic

Results towards Conjecture 1

- true in many special families of PHD's
- Dolgopyat Wilkinson C^1 density
- Niţica-Török $\dim E^c = 1$ + conditions
- Hertz, Hertz, Ures true for $PHD^{r}_{Vol}(M)$ if $\dim E^c = 1$.

Another mechanism for stable ergodicity

Alves
Bonatti
Viana

Examples with dominated splitting that are not partially hyperbolic but are robustly non uniformly hyperbolic

Tahzibi Robust ergodicity of these examples

Generalization of Conjecture 2

$f \in \mathrm{Diff}^{1+\alpha}_{\mathrm{Vol}}(M)$

Definition f is non uniformly partially hyperbolic if f has a non zero Lyapunov exponent a.e.

Can define accessibility, essential accessibility using Pesin stable & unstable manifolds

Conjecture 2'

If f is non uniformly partially hyperbolic and essentially accessible, then f is ergodic

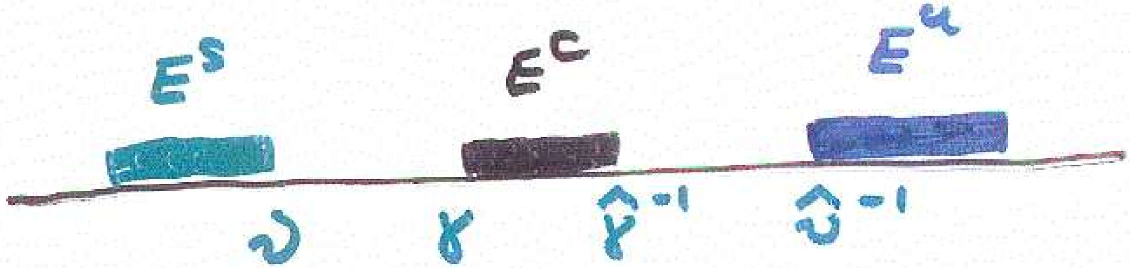
Progress toward Conjecture 2

- Grayson Pugh Shub
- Wilkinson
- Pugh Shub
- Pugh Shub

B Wilkinson

Conjecture holds if f is

<u>center bunched</u>



$$\boxed{\nu < \gamma\hat{\gamma} \text{ and } \hat{\nu} < \gamma\hat{\gamma}}$$

Definition

Set A is essentially W-saturated if there is a saturated set A' with $m(A \triangle A') = 0$

Proposition

Suppose A is bi-essentially-saturated. Then there is a bi-saturated set $\hat{A}$ such that

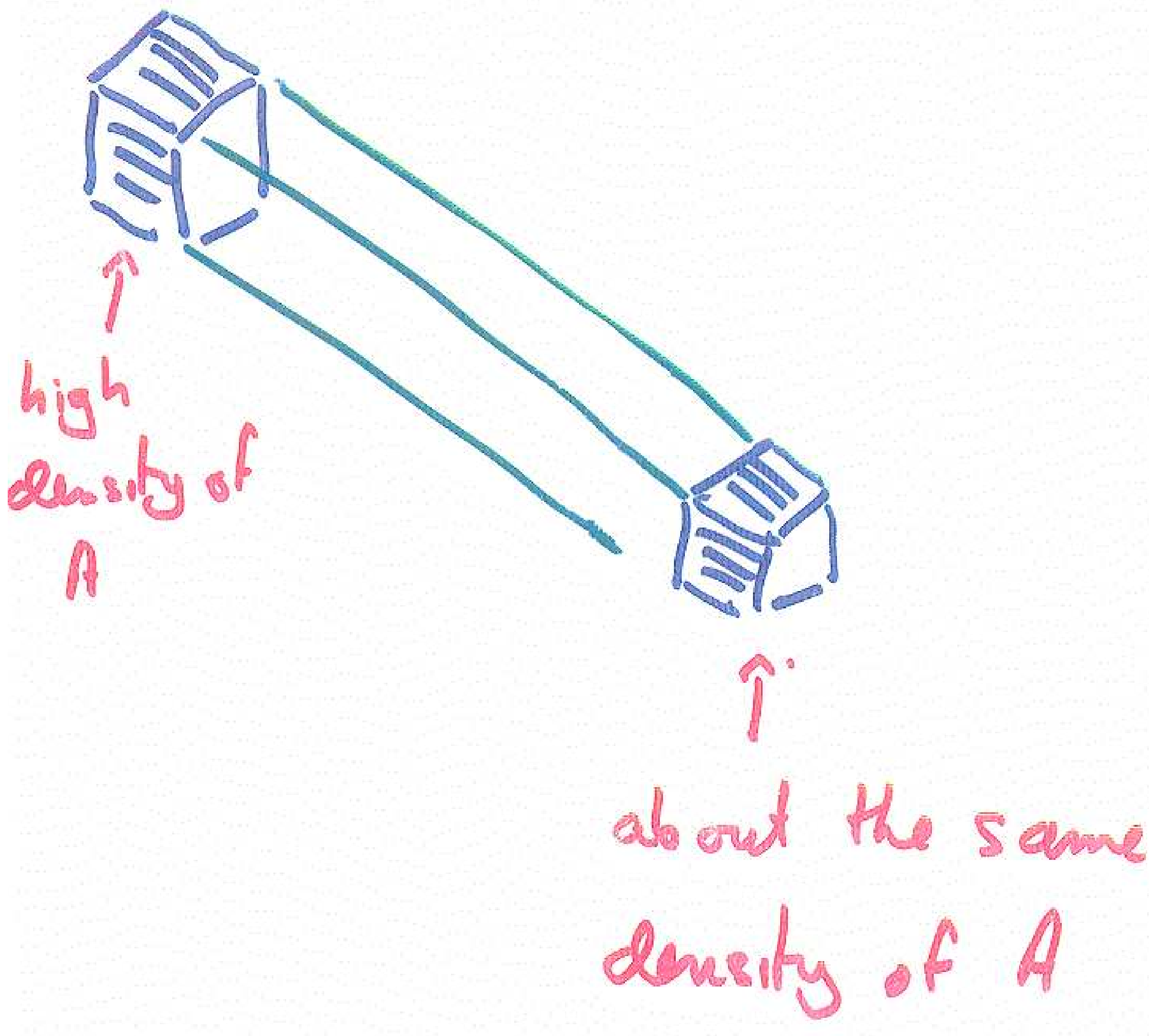
$$m(A \triangle \hat{A}) = 0$$

Proof of Theorem

Apply the proposition to sublevel sets of Birkhoff averages of continuous functions. Ess-bi-sat by Hopf

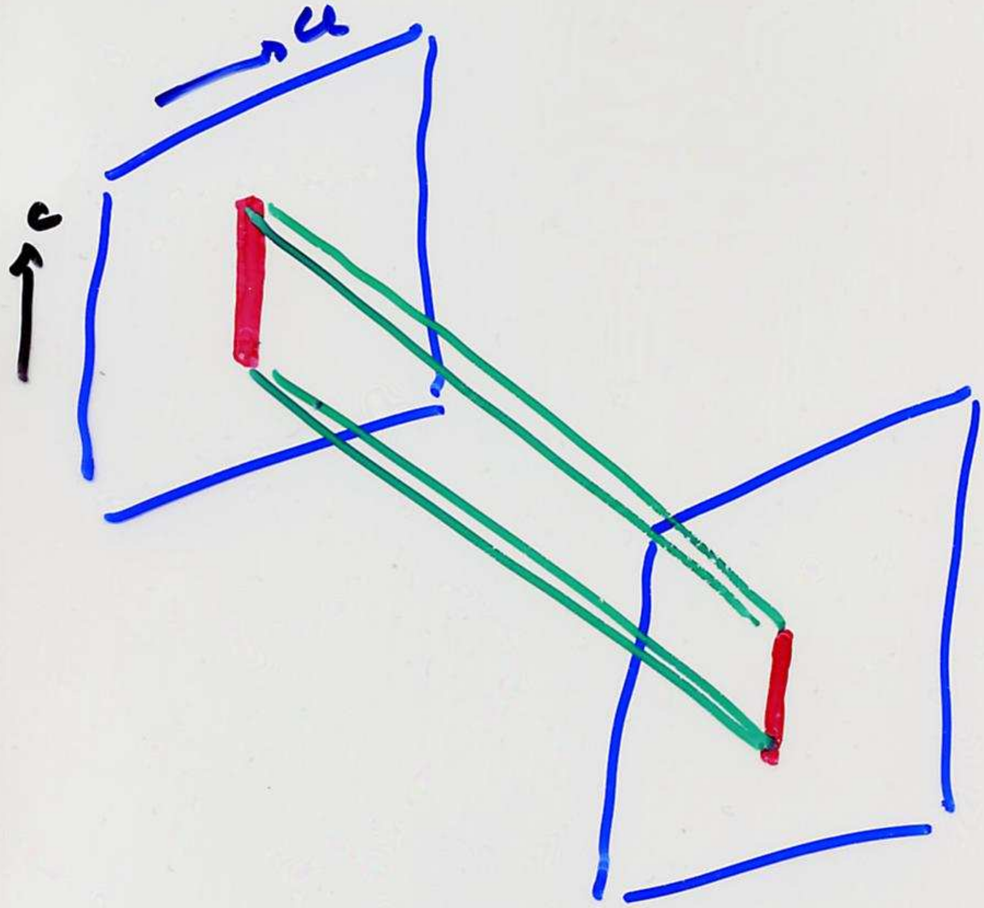
$\hat{A}$ = Lebesgue density points of A

$$\lim_{r \to 0} \frac{m(A \cap B(x,r))}{m(B(x,r))} = 1$$

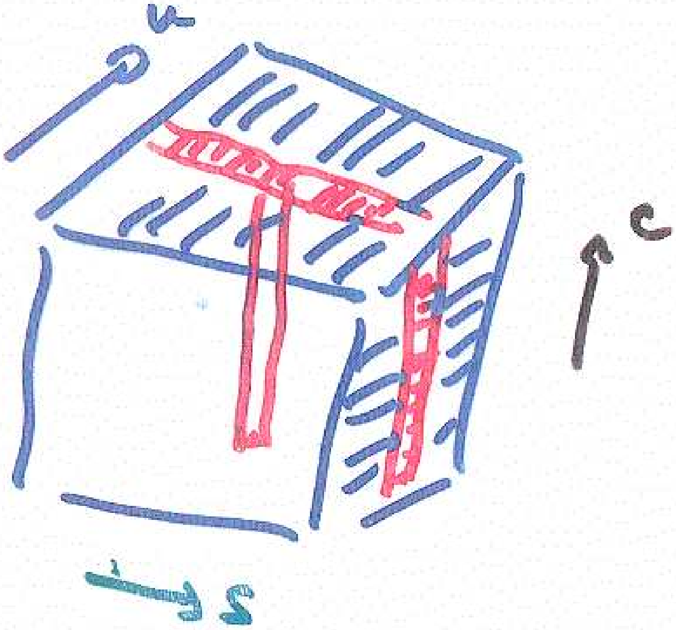


Grayson
Pugh Shub

juliennes



Unlike boxes, juliennes
survive under holonomy



density of Λ doesn't change much when we collapse the box