Ergodicity of accessible, center bunched, partially hyperbolic diffeos

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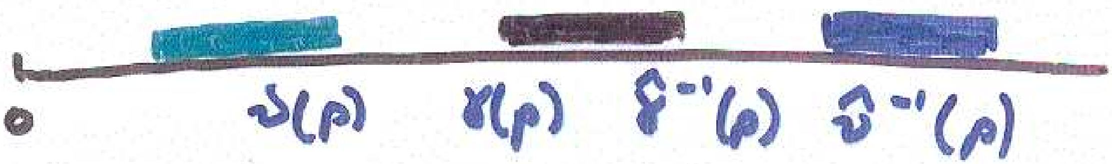
M compact

$f \in PHD^2_{vol}(M)$



$\dfrac{\|T_p f v\|}{\|v\|}$

$v \neq 0$

$\nu, \gamma, \hat\gamma^{-1}, \hat\nu$ continuous

$\nu, \hat\nu < 1$

Center bunched: $\nu, \hat\nu < \gamma\hat\gamma$

Theorem (B, Wilkinson)

$f \in PHD^2_{vol}(M)$, center bunched, essentially accessible $\Rightarrow$ ergodic

Corollary If $\dim E^c = 1$, then essential accessibility $\Rightarrow$ ergodic

(center bunching automatic)

Example

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix} \text{ on } \mathbb{T}^4$$

is center bunched & essentially accessible

Hertz:
Essential accessibility persists under C^{22} small perturbations

$\Rightarrow$ C^{22} robust ergodicity

Hopf argument

Suffices to show that Birkhoff averages of continuous functions are ae constant

$$\varphi : M \to \mathbb{R}$$

$$\hat{\varphi}^+(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(g^k x)$$

$$\hat{\varphi}^-(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(g^{-k} x)$$

Birkhoff The limits corresponding to $\hat{\varphi}^+$ and $\hat{\varphi}^-$ exist are equal ae.

Hopf $\hat{\varphi}^+$ constant on W^s leaves

$\hat{\varphi}^{-1}$ constant on W^u leaves

Natural approach

$\varphi_1, \varphi_2, \ldots$ dense set of continuous functions

$G = \{x : \hat{\varphi}_i^+(x) = \hat{\varphi}_i^-(x) \text{ for all } i\}$

G has full measure

Suppose: any two points of G can be joined by a us-path whose corners are in G (ie avoid the measure 0 set $M \setminus G$)

Then f is ergodic

$$\hat{\varphi}(x) = \frac{\hat{\varphi}^+(x) + \hat{\varphi}^-(x)}{2}$$

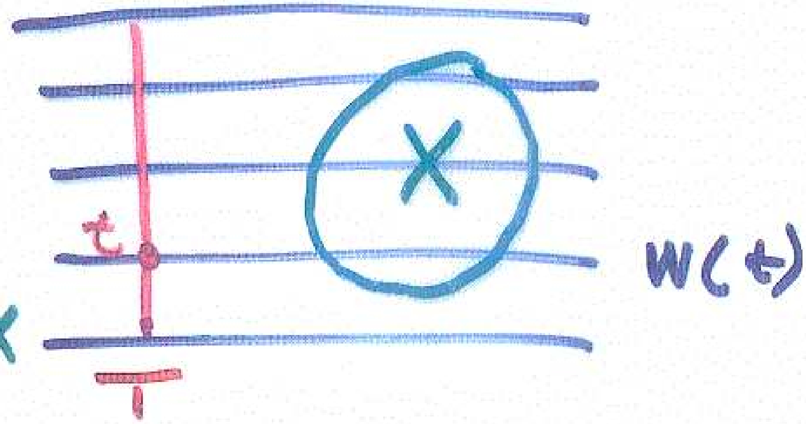
$\{x : \hat{\varphi}(x) \leq \alpha\}$ is bi-essentially-saturated for any α

Essential accessibility: essentially bi-saturated sets have 0 or full measure

<u>Proposition</u>

If A is bi-essentially-saturated, then the set $\hat{A}$ of Lebesgue density points of A is bi-saturated

Absolute continuity with bounded Jacobian



$\exists\, C \geq 1$ st $\forall X$

$$\frac{1}{C} \operatorname{Vol}(X) \leq \int_T m_{W(t)} (X \cap W(t))\, dm_T(t) \leq C \operatorname{Vol}(X)$$

Brin-Pesin, Pugh-Sub après Anosov-Sinai

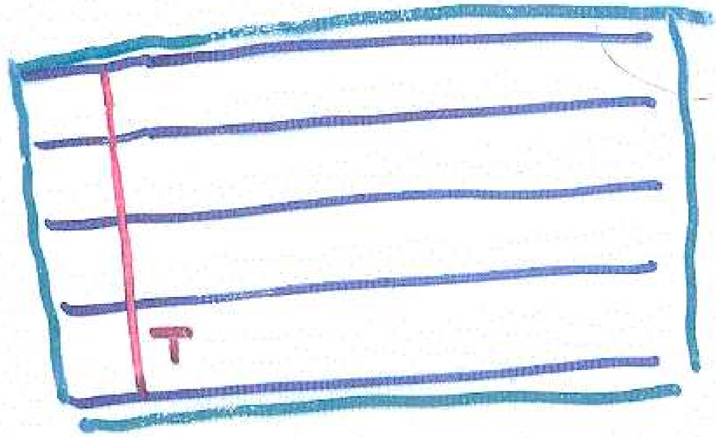
W^u and W^s are absolutely continuous

$G_1 = \{ x : \text{a.e. } y \in W^s(x) \cup W^u(x) \text{ is in } G \}$

$G_2 = \{ x : \text{ - - - - ——— } G_1 \}$

$E = \bigcap_{n \geq 1} G_n$ all have full measure

Useful observation



B = box foliated by pieces of W-leaf, all of about the same volume

If W is absolutely continuous of A is W-saturated, then density of A in B is approximately the same as the density of A in T

Key fact

Center bunching $\Longrightarrow$

holonomy between center leaves along stable (or unstable) leaves is Lipschitz

Fake foliations

$\exists\, r>0$ st $\forall\, p \in M$, $B(p,r)$

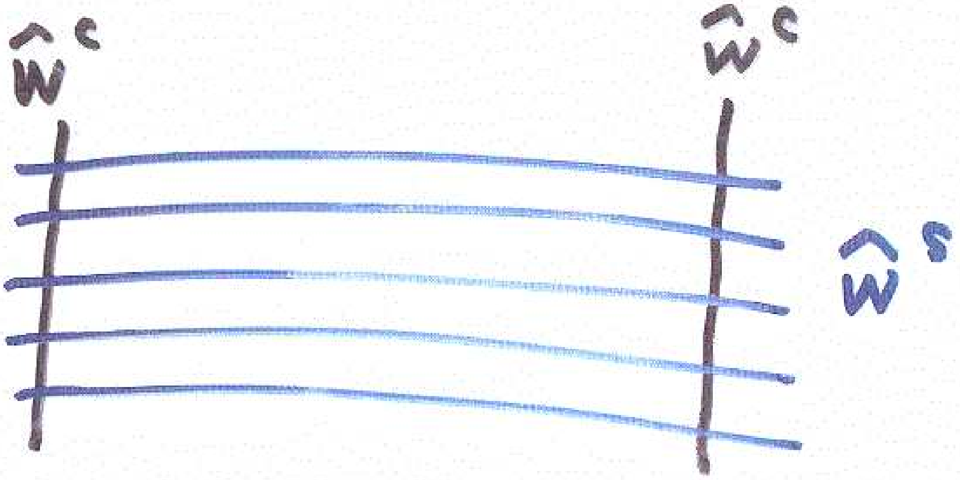
has foliations

$\hat{W}^u_p, \hat{W}^s_p, \hat{W}^c_p, \hat{W}^{cu}_p, \hat{W}^{cs}_p$

tangent to the right spaces at

p. $\hat{W}^u_p(p) = \hat{W}^u(p)$, $\hat{W}^s_p(p) = W^s(p)$

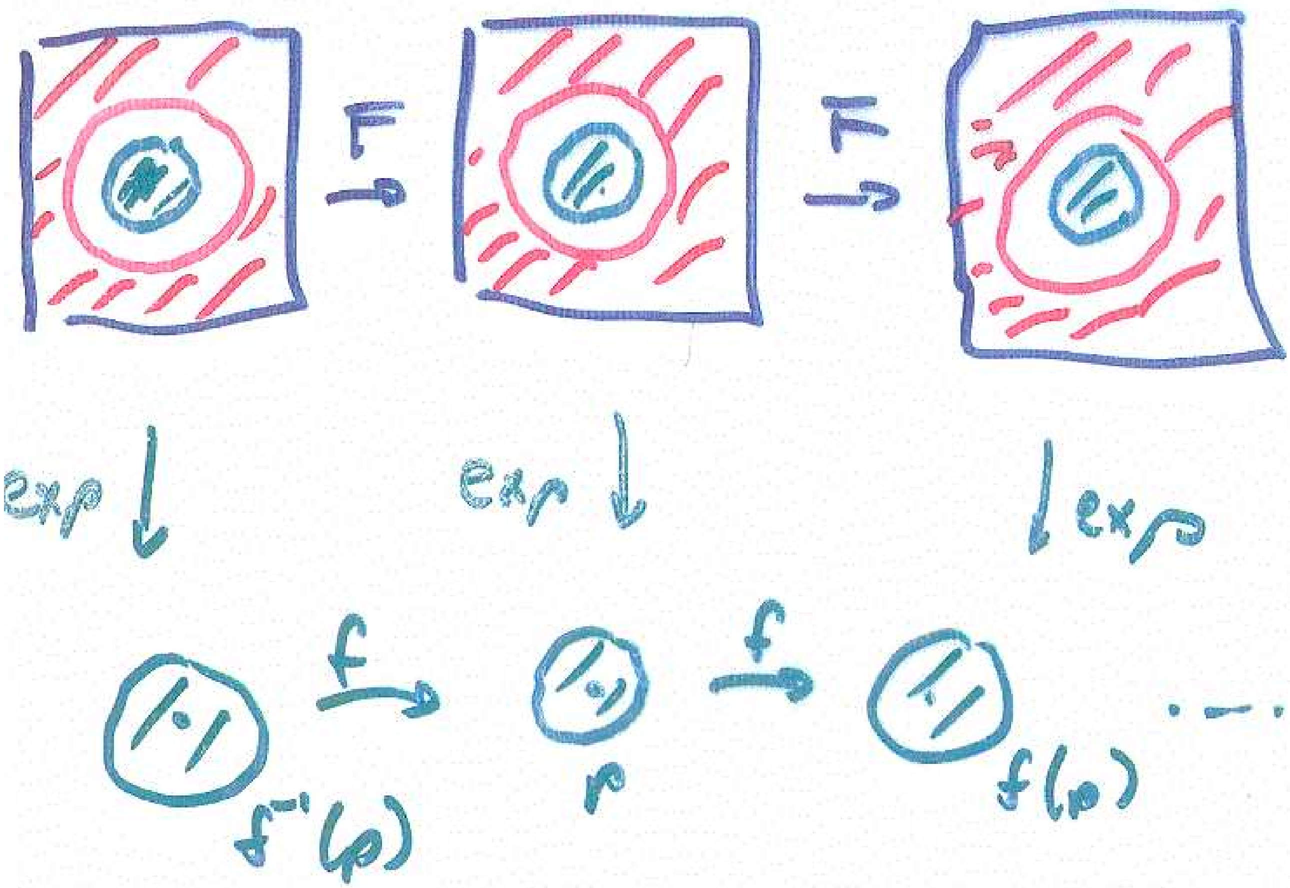
$\hat{W}^{cs}$ leaves are subfoliated by $\hat{W}^{c}$ leaves and $\hat{W}^{s}$ leaves



This holonomy is Lipschitz

Pugh, Shub, Wilkinson
careful verion of usual smoothness estimates applied to show $T\hat{W}^{s}$ is Lipschitz on a $\hat{W}^{cs}$ leaf

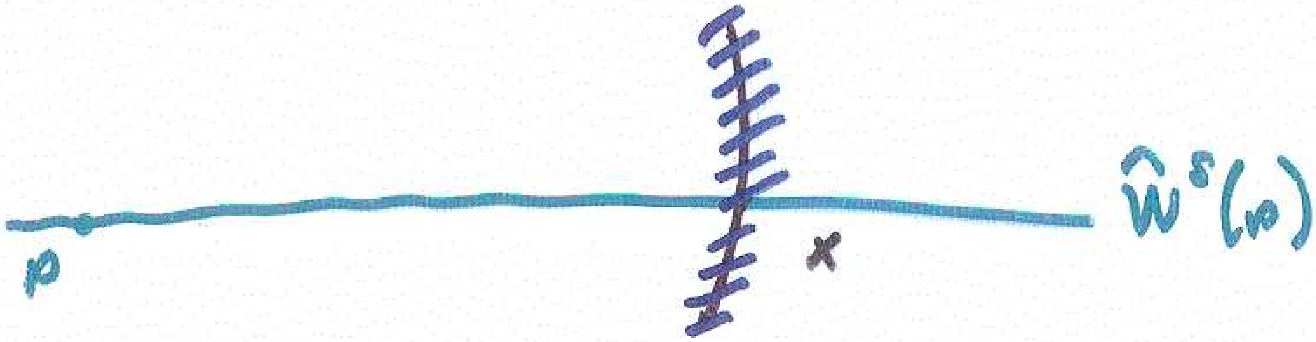
Construction of fake foliations



$$F = \exp^{-1} \circ f \circ \exp$$

$$F = df$$

Jaliennes



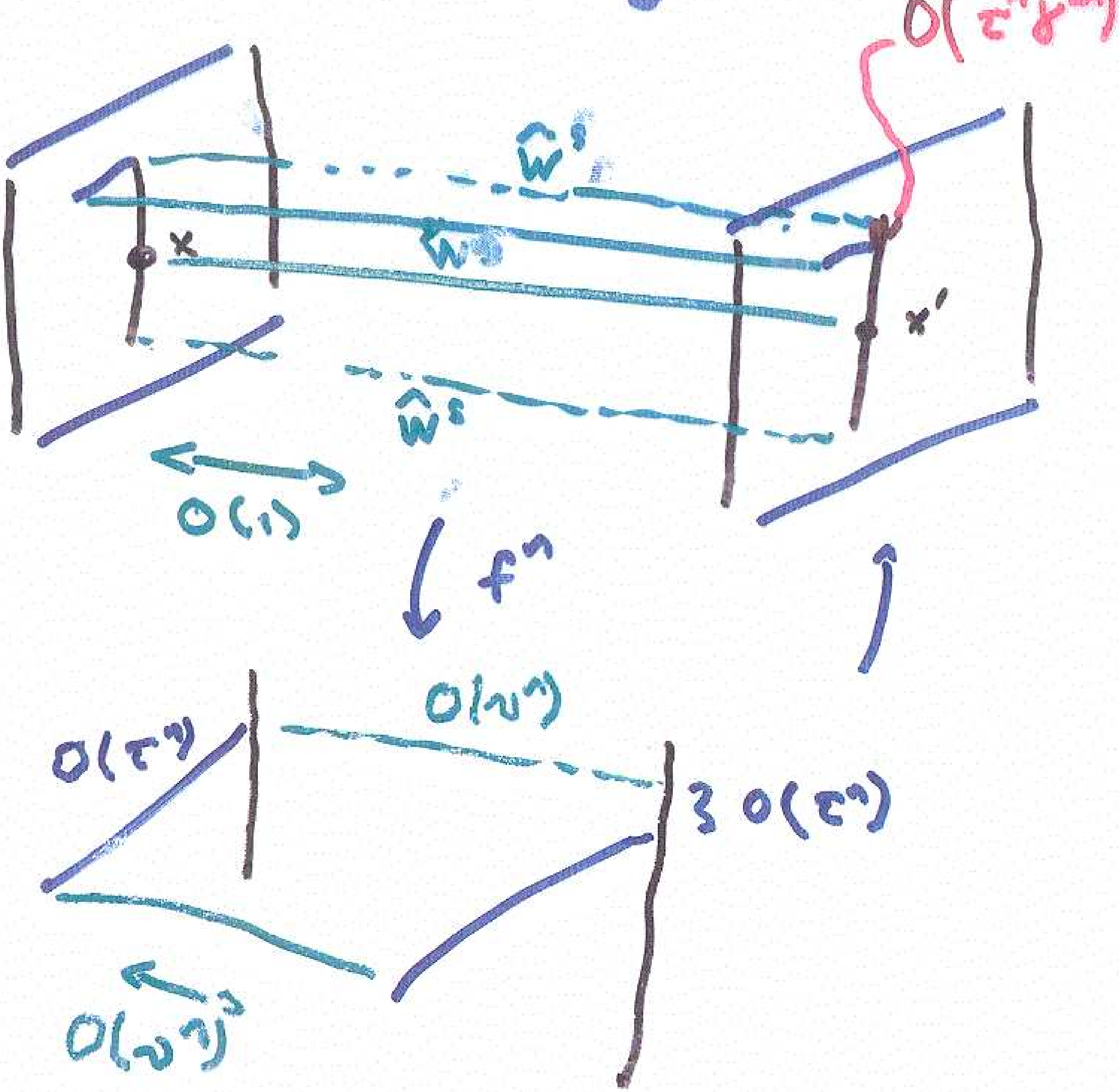
$$\hat{J}^{cu}_n(x) = \bigcup_{y \in \hat{W}^s(x, \sigma^n)} f^{-n}\hat{W}^u(f^n y, \tau^n)$$

$$\nu, \hat{\nu} < \gamma\hat{\gamma}^{-1}$$

Choose σ, τ st

$$\nu < \tau < \sigma\gamma < 1 \quad \& \quad \sigma < \min\{\hat{\gamma}, 1\}$$

cu-juliennes are preserved under W^s-holonomy



$\tau < \sigma\gamma$

Suppose A is W^s-saturated & essentially W^u-saturated

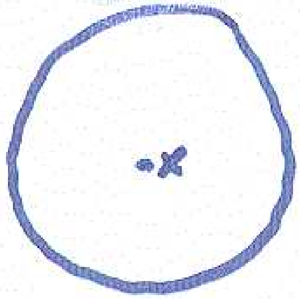
Then

x is a Lebesgue density point for A

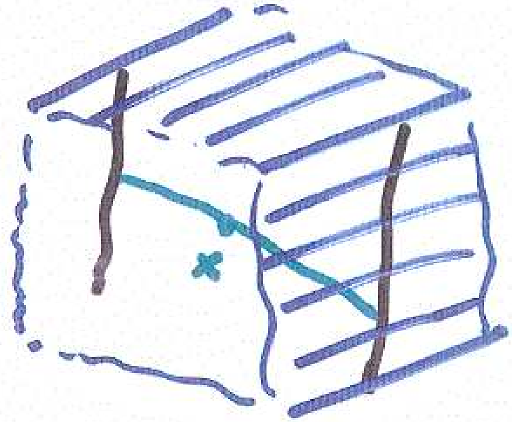
$\Longleftrightarrow$

x is a $\hat{\mathcal{J}}^{cu}$-density point of $A \cap \hat{W}^{cu}(x)$

Absolute continuity of $\hat{W}^s$ and the holonomy invariance of the $\hat{\mathcal{J}}^{cu}$'s implies that $\hat{\mathcal{J}}^{cu}$-density points are preserved under W^s holonomy



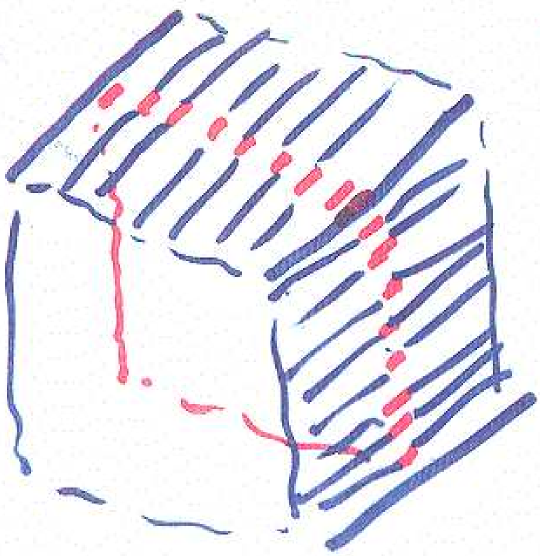
$B(x, \sigma^n)$



$= W^s(x, \sigma^n)$

$= \hat{W}^c(\cdot, \sigma^n)$

$= W^u(\cdot, \sigma^n)$

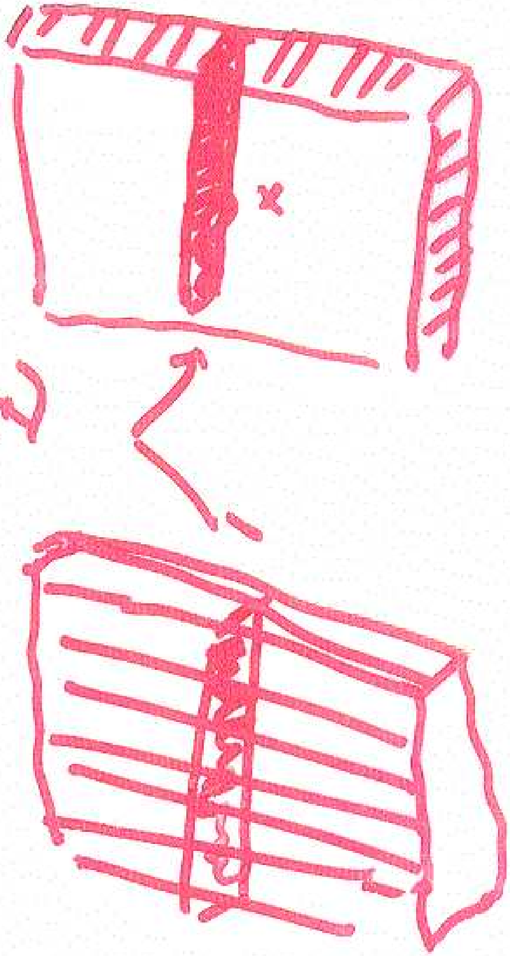


$= J_n(\cdot)$

$= f^{-n} W^u(f^n(\cdot), \tau^n)$

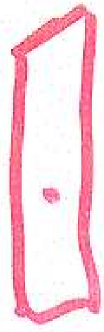
Replace $J^u_n(\ )$ by $\hat{J}^u_n(\cdot) = f^{-n}\hat{W}^u(f^n(\cdot), \varepsilon^n)$

$\hat{J}^{cu}_n(x)$



Switch to union of W^s holonomy images of $\hat{J}^{cu}_n(x)$

Now collapse along W^s leaves!



$\hat{J}^{cu}_n(x)$