HODGE THEORETIC MIRROR SYMMETRY FOR THE QUINTIC THREEFOLD AND ELLIPTIC CURVE

BENJAMIN ZHOU

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1. Introduction

Mirror Symmetry originates as a duality between Type IIA and Type IIB superstring theory [4]. For certain pairs of Calabi-Yau manifolds $X, \check{X}$, it was found that Type IIA string theory on X is equivalent to Type IIB string theory on $\check{X}$; their sigma models induce isomorphic superconformal field theories. Such manifolds are said to be a mirror pair. Type IIA string theory, or the A-model, relies on the Kähler structure ω of X, and Type IIB string theory, or the B-model, relies on the complex structure $\check{J}$ of $\check{X}$. Mirror Symmetry predicts there is a local isomorphism of the complexified Kähler moduli space of X with the complex structure moduli space of $\check{X}$. On the level of cohomology,

$$h^{p,q}(X) = h^{n-p,q}(\check{X})$$

, where n is the complex dimension of X and $\check{X}$. By equating correlation functions on the respective moduli spaces, the work of Candelas, de la Ossa, Green, and Parkes showed that the number of rational curves on a quintic threefold can be reduced to calculation of period integrals satisfying Picard-Fuchs equations on the mirror [1]. Their results partially answered deep questions in algebraic geometry, such as the Clemens conjecture. Since then, there has been great interest to mathematically formalize mirror symmetry, with work of Kontsevich conjecturing mirror symmetry to be an equivalence of derived categories [8], and Strominger, Yau, and Zaslow offering a geometric picture with special Lagrangian torus fibrations [11]. This paper seeks to provide an exposition of some early accomplishments in mirror symmetry, namely enumerative mirror symmetry for a generic quintic threefold in $\mathbb{P}^4$ and elliptic curves.

2. Mathematical Preliminaries

We first mention some mathematical preliminaries. Mirror symmetry was first formulated for Calabi-Yau 3-folds, due to their attractiveness to physicists as potential candidates for supersymmetric compactification of spacetime.

Definition 2.1. A *Calabi-Yau manifold* is a compact Kähler manifold (X, J, g, Ω) with trivial canonical bundle, i.e. $\omega_X \cong \mathcal{O}_X$. Hence there exists a nonvanishing holomorphic volume form $\Omega \in \wedge^{n,0}T^*X$.

Calabi-Yau manifolds in complex dimension 1 are elliptic curves. In complex dimension 2, they are K3 surfaces. In dimension 3 or greater, there are much more of them. Since the canonical bundle is trivial, the first Chern class vanishes $c_1(X) = 0$. By Yau's proof of the Calabi conjecture, there exists a unique Ricci flat metric whose Kähler form is in the original Kähler class. Hence, the metric on X can be chosen such that it is Ricci flat. Recall the following theorems from complex algebraic geometry.

Theorem 2.2. ***(Dolbeaut)*** *Dolbeaut cohomology is isomorphic to sheaf cohomology of the sheaf of differential forms, i.e.*

$$H^{p,q}(X) \cong H^q(X, \Omega^p_X)$$

where Ω^p_X is the sheaf of holomorphic p-forms on X.

Theorem 2.3. ***(Hodge Decomposition)*** *For a compact Kähler manifold X,*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

Let $h^{p,q} = dim\ H^{p,q}(X)$. We have $dim\ H^k(X, \mathbb{C}) = \sum_{p+q=k} h^{p,q}$.

The Hodge decomposition allows one to form the Hodge diamond, which lists the Hodge numbers $h^{p,q}$ in a diamond formation. There are various symmetries of the Hodge diamond. Complex conjugation gives $\overline{H^{p,q}(X)} = H^{q,p}(X)$, hence $h^{p,q} = h^{q,p}$. This means the Hodge diamond is symmetric under reflection across the vertical line. Serre duality says $H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$ or $h^{p,q} = h^{n-p,n-q}$. This translates to the Hodge diamond is symmetric under counterclockwise rotation by π. Finally, the Hodge *-operator gives the isomorphism $H^{p,q} \cong H^{n-q,n-p}$, which translates to the Hodge diamond is symmetric under reflection across the horizontal line.

Theorem 2.4. ***(Lefschetz Hyperplane)*** *Let X be an n-dimensional compact, complex manifold and $Y \subset X$ a smooth hypersurface with $[Y]$ positive. Then we have the map*

$$H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$$

induced by inclusion is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 2.5. ***(Bogomolov-Tian-Todorov)*** *Let X be a compact Calabi-Yau manifold with $H^0(X, T_X) = 0$, i.e. there does not exist global holomorphic vector fields on X. Then the universal deformation of X, Def(X), is a germ of a smooth manifold with tangent space $H^1(X, T_X)$.*

For quintic threefolds, this says the a tangent space in the complex moduli space has dimension $h^{1,2}$.

2.1. Variation of Hodge Structures.

Definition 2.6. A Hodge structure of weight n is a lattice $H_{\mathbb{Z}}$ of finite rank with a decomposition

$$H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

of complex subspaces with $\overline{H^{p,q}} = H^{q,p}$. A filtration

$$H = F^0 \supseteq \ldots \supseteq F^n$$

satisfying the condition $H = F^p \oplus \overline{F^{n-p+1}}$ is called a Hodge filtration.

An example is the n-th cohomology group $H^n(X, \mathbb{Z})$ of a compact Kähler manifold.

For a family $f : \Psi \to S$ of compact Kähler manifolds, where S is a connected manifold, we have the inverse image sheaf $R^n(f_* \mathbb{C})$, which is the sheaf associated to the presheaf $U \to H^n(f^{-1}(U), \mathbb{C})$. This sheaf is a local system, and to prove this we appeal to the following lemma

Lemma 2.7. *Let $f : \Psi \to U$ be a family of complex manifolds with U contractible. Then there is a diffeomorphism $\Psi \cong X \times U$, where X is diffeomorphic to any fibre of f.*

Indeed, since S is a manifold, each point has a contractible neighborhood U. Hence, $H^n(f^{-1}(U), \mathbb{C}) \cong H^n(X \times U, \mathbb{C}) \cong H^n(X, \mathbb{C})$. Then by path-connectedness of S and compactness of the path, $R^n f_* \mathbb{C}$ is a local system with coefficient group $H^n(X, \mathbb{C})$.

There is a correspondence between local systems $\mathcal{E}$ with coefficients in $\mathbb{C}^r$ and pairs $(\mathcal{F}, \nabla)$, where $\mathcal{F}$ is a rank r holomorphic vector bundle and ∇ is a flat connection. Given a local system $\mathcal{E}$ on S, set $\mathcal{F} = \mathcal{E} \otimes \mathcal{O}_S$. In our context $f : \Psi \to S$, we obtain the holomorphic vector bundle $\mathcal{H}^n = (R^n f_* \mathbb{C}) \otimes \mathcal{O}_S$ and a flat holomorphic connection ∇ on $\mathcal{H}^n$, called the *Gauss-Manin connection.*

Using this connection, we may define monodromy around a singular point. Suppose $S = \Delta^*$, the punctured unit disc. The stalk of $(R^n f_* \mathbb{C})_s$ at a point can be identified with $H^n(\Psi_s, \mathbb{C})$. Using the connection to parallel transport around a loop $\gamma : [0, 1] \to \Delta^*$

around the origin gives a linear isomorphism $T : H^n(\Psi_s, \mathbb{C}) \to H^n(\Psi_s, \mathbb{C})$. This is called the *monodromy transformation.*

Example 2.8. We can explicitly compute the monodromy matrix for the family of elliptic curves. Consider the family of elliptic curves over Δ^* with coordinate t on Δ^* with modular parameter $\tau = \frac{1}{2\pi i}\log t$. Divide $\mathbb{C} \times \Delta^*$ by the action $(z,t) \to (z+1,t)$ and $(z,t) \to (z + \frac{1}{2\pi i}\log t, t)$ to obtain a complex manifold Ψ. Though $\frac{1}{2\pi i}\log t$ is multi-valued, the lattice spanned by 1 and $\frac{1}{2\pi i}\log t$ will be well-defined. This gives us a family $f : \Psi \to \Delta^*$ of elliptic curves.

Fix a point $t_0 \in \Delta^*$ and consider the monodromy transformation $T : H^1(E_{t_0}, \mathbb{C}) \to H^1(E_{t_0}, \mathbb{C})$. If $E_{t_0} \cong \mathbb{C}/\Lambda$, then by Poincare duality, $H^1(E_{t_0}, \mathbb{C}) \cong \Lambda$, which is spanned by 1 and $\tau(t_0)$. Therefore, we determine how T acts on 1 and $\tau(t_0)$. We see that T will send 1 to itself, but because of the multivaluedness of τ, T will send $\tau \to \tau + 1$. Thus, the monodromy matrix is given by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

There is a property of the Gauss-Manin connection that will be useful. We see that each fibre of the holomorphic vector bundle $\mathcal{H}^n = R^n f_* \mathbb{C} \otimes \mathcal{O}_S$ will have a Hodge filtration. This yields a filtration of $\mathcal{H}^n$ by subbundles

$$\mathcal{H}^n = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \ldots \supseteq \mathcal{F}^n$$

There is a useful relationship between the Gauss-Manin connection and the Hodge filtration of $\mathcal{H}^n$

Theorem 2.9. ***(Griffiths Transversality)*** $\nabla \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S$

Proof: It suffices to show that if $\omega_t \in \Omega^{p,q}(X, J_t)$, *then* $\frac{\partial}{\partial t}\omega_t \in \Omega^{p,q} + \Omega^{p+1,q-1} + \Omega^{p-1,q+1}$. *Locally,* $(T^{1,0}_{J_t})^*$ *is given by span*$\{dz_i(t) := dz_i - \sum_j s_{ij}(t) d\overline{z_j}\}$, *and*

$$\omega_t = \sum_{|I|=p,|J|=q} \alpha_{IJ}(t) dz_{i_1}(t) \wedge \ldots \wedge dz_{i_p}(t) \wedge d\overline{z_{j_1}}(t) \wedge \ldots \wedge d\overline{z_{j_q}}(t)$$

Taking $\frac{\partial}{\partial t}|_{t=0}$ *and* $s_{ij}(0) = 0$, *we get terms such as*

$$\alpha_{IJ}(0) dz_{i_1} \wedge \ldots \wedge (\sum_j \frac{\partial s_{ik,j}}{\partial t} d\overline{z_j}) \wedge \ldots dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}} \in \Omega^{p-1,q+1}$$

and similarly for differentiating $d\overline{z_{j_k}}$ *(the terms will be in* $\Omega^{p+1,q-1}$*).*

3. Mirror Symmetry for the Quintic Threefold

One of mirror symmetry's early successes in enumerative geometry was the prediction of the number of rational curves on a generic quintic threefold in $\mathbb{P}^4$. We consider the

Fermat quintic hypersurface $X \subseteq \mathbb{P}^4$ defined by the zero locus of the homogeneous polynomial,

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$$

Since 0 is a regular value, X is a complex hypersurface. By the adjunction formula, which states that $K_X \cong \mathcal{O}_X(\sum_i d_i - n - 1)$ for a smooth complete intersection $X = D_1 \cap D_2 \cap \ldots \cap D_n = 0 \subseteq \mathbb{P}^n, \deg D_i = d_i$, we see that X is a Calabi-Yau 3-fold.

We can compute the Hodge diamond of X. The Lefschetz Hyperplane Theorem gives an isomorphism of $H^k(X, \mathbb{C})$ with $H^k(\mathbb{P}^4, \mathbb{C})$ for $k \leq 2$. This implies $H^0(X, \mathbb{C}) \cong H^2(X, \mathbb{C}) \cong \mathbb{C}$, and $H^1(X, \mathbb{C}) = 0$. This means $h^{1,0} = h^{0,1} = 0$. Also, $h^{2,0} = h^{3,1} = h^{0,1} = 0$, where the first equality is by Hodge *-duality. Now, $h^{1,1} = 1$ by Hodge decomposition. It remains to find $h^{1,2}$. Heuristically, by Bogomolov-Tian-Todorov, $h^{1,2}$ should be the dimension of the tangent space of deformations of X, or the space of quintic hypersurfaces. The dimension should be close to $\binom{5+4}{4} = 126$, since this is the number of degree 5 homogeneous polynomials in 5 variables. One should subtract 1, since proportional polynomials give the same hypersurface, and then subtract the dimension of $PGL(5)$, the automorphism group of $\mathbb{P}^4$. This suggests $h^{1,2} = 126 - 1 - 24 = 101$, but one also needs to make sure that deformations of quintics remain quintics, and there are no exceptional isomorphism between hypersurfaces, etc. One can also calculate $h^{1,2}$ by the Gauss-Bonnet Theorem, i.e. $\chi(X) = c_3(T_X)$.

3.1. **A-model.** The A-model is the moduli space of complexified Kähler classes of X. The set of all Kähler classes form an open cone of $H^{1,1}(X, \mathbb{C})$, and is a real submanifold of dimension $h^{1,1}$. A complexified Kähler class is defined to be

$$\omega^{\mathbb{C}} := B + i\omega$$

, where ω is a Kähler class and $B \in H^{1,1}(X, \mathbb{R})$ is termed the $B-$field in physics literature. These classes are well-defined up to translation by $H^2(X, \mathbb{Z})$. The complexified Kähler moduli space is defined to be

$$\mathcal{M}_{K\ddot{a}h}(X) := (H^2(X, \mathbb{R}) + i\mathcal{K}_X)/H_2(X, \mathbb{Z})$$

For the quintic threefold, since $h^{1,1}(X) = 1$,

$$\mathcal{M}_{K\ddot{a}h}(X) \cong (\mathbb{R} + i\mathbb{R}_{>0})/\mathbb{Z} \cong \mathcal{H}/\mathbb{Z} \cong \Delta^*$$

, where Δ^* is the punctured unit disk and the isomorphism is given by $e^{2\pi i t}$.

The $(1,1)$-Yukawa coupling is a cubic form defined on the tangent space of $\mathcal{M}_{K\ddot{a}h}(X)$

$$\langle D_1, D_2, D_3 \rangle = \int_X D_1 \wedge D_2 \wedge D_3 + \sum_{\beta \neq 0} n_\beta \int_\beta D_1 \int_\beta D_2 \int_\beta D_3 \frac{e^{2\pi i \int_\beta \omega}}{1 - e^{2\pi i \int_\beta \omega}}$$

,where the sum is over $\beta \in H_2(X, \mathbb{Z})$. The definition of n_β is given in terms of the space of stable maps from a genus-zero domain curve to X. The aim of this paper is not to define the n_β rigorously or calculate them directly, but to outline a mirror calculation which predicts them.

3.2. **B-model.**

3.2.1. *Mirror family of quintics.* To construct the mirror family of Calabi-Yau 3-folds, we follow Greene and Plesser's orbifold construction. Define a family of quintic 3-folds by,

$$X_\psi = \{f_\psi := x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0x_1x_2x_3x_4 = 0\}$$

Then X_ψ can be viewed as a family $\chi \subseteq \mathbb{P}^4 \times \mathbb{A}^1 \to \mathbb{A}^1$, where ψ is a coordinate on $\mathbb{A}^1$. There exists an action of $(\mathbb{Z}/5\mathbb{Z})^5$ on $\mathbb{P}^4$ given by,

$$(x_0, x_1, x_2, x_3, x_4) \to (\zeta^{a_0}x_0, \zeta^{a_1}x_1, \zeta^{a_2}x_2, \zeta^{a_3}x_3, \zeta^{a_4}x_4)$$

, where ζ is a fifth root of unity. Notice that $\mathbb{Z}/5\mathbb{Z}$ embedded diagonally into $(\mathbb{Z}/5\mathbb{Z})^5$ acts as the identity. Therefore, we have an action of $(\mathbb{Z}/5\mathbb{Z})^5/(\mathbb{Z}/5\mathbb{Z})$. Consider the subgroup G of $(\mathbb{Z}/5\mathbb{Z})^5/\mathbb{Z}/5\mathbb{Z}$ given by,

$$G := \{(a_0, \dots, a_4) | \sum_i a_i \equiv 0 \bmod 5\}/\mathbb{Z}/5$$

G acts on each hypersurface X_ψ, and thus the X_ψ descend to a family of hypersurfaces in $\mathbb{P}^4/G$. Denote the quotient hypersurface by $Y_\psi := X_\psi/G$. A priori, Y_ψ is quite singular, and we need to perform a resolution of singularities. We first analyze which of the X_ψ are singular.

The Jacobian of f_ψ is,

$$J_{X_\psi} = (5x_0^4 - 5\psi x_1x_2x_3x_4, \dots, 5x_4^4 - 5\psi x_0x_1x_2x_3)$$

If the Jacobian vanishes on all coordinates, then we have $5x_i^5 = 5\psi x_0x_1x_2x_3x_4$ for all i. This implies $\prod_i x_i^5 = \psi^5 \prod_i x_i^5$. Thus, either $x_i = 0$ or $\psi^5 = 1$. However, if $x_i = 0$ for some i, then $x_i = 0$ for all i, and $(x_0, \dots, x_4)$ does not represent a point in $\mathbb{P}^4$. Thus, X_ψ is singular at $\psi^5 = 1$. In this case, X_ψ will be singular at the points

$$(\zeta^{a_0}, \dots, \zeta^{a_4})$$

where $\sum_i a_i \equiv 0 \bmod 5$. This gives 125 distinct singular points. Notice X_ψ will also be singular when $\psi = \infty$, where $X_\psi = x_0x_1x_2x_3x_4$.

The quotient Y_ψ is singular as well. It is singular at points $x \in X_\psi$ where the action of G is not free, or where the stabilizer at x is non-trivial. A point in $\mathbb{P}^4$ has non-trivial stabilizer in G if at least two of its coordinates are 0. We see that points of the curve,

$$C_{ij} = \{x_i = x_j = 0\} \cap X_\psi$$

have stabilizer of order 5, and the points of

$$P_{ijk} = \{x_i = x_j = x_k = 0\} \cap X_\psi$$

have stabilizers of order 25. The singular locus of Y_ψ will consist of 10 curves $C_{ij}/G \cong \mathbb{P}^1$, with $C_{ij}/G, C_{jk}/G, C_{ik}/G$ meeting at the point P_{ijk}/G.

One then performs a resolution of singularities on Y_ψ for $\psi^5 \neq 1, \infty$. It can be shown there exists a resolution $\check{X}_\psi \to Y_\psi$ such that $\check{X}_\psi$ is a Calabi-Yau 3-fold whose Hodge numbers satisfy,

$$h^{1,1}(\check{X}_\psi) = 101, h^{1,2}(\check{X}_\psi) = 1$$

Indeed, the Hodge numbers are as proposed by mirror symmetry. Thus, we have a family of mirror quintics $\check{X}_\psi \to \mathbb{A}^1$. Notice that $X_\psi \cong X_{\zeta\psi}$ by the map $(x_0, \ldots, x_4) \to (\zeta x_0, \ldots, x_4)$. Hence, we instead use the coordinate $x = (5\psi)^{-5}$ on the mirror family. Notice that the singular mirror quintics occur when $x = 0, 5^{-5}$, and ∞.

3.2.2. *Canonical Coordinates.* The mirror map is a local isomorphism from the symplectic moduli space to the complex moduli space. The local parameter on $\mathcal{M}_{K\ddot{a}h}$ is given by $q = e^{2\pi i t}$. The mirror map should express q in terms of the local parameter of the complex moduli space. To do this, one finds a large complex structure limit point (LCSL) in the family of mirror quintics to compute period integrals around. Through the theory of Hodge structures, there will exist unique (up to sign) vanishing cycles β_0, β_1 around the LCSL such that under monodromy, $\beta_0 \to \beta_0$ and $\beta_1 \to \beta_1 + n\beta_0$. If $\Omega(x)$ is a holomorphic family of holomorphic volume forms on our family of mirror quintics, the mirror map expresses the Kähler parameter by,

$$q = e^{2\pi i \frac{\int_{\beta_1} \Omega(x)}{\int_{\beta_0} \Omega(x)}}$$

In order to express the integrals in terms of the complex moduli space coordinate x, we use the fact that the period integrals must satisfy Picard-Fuchs differential equations. Then, the mirror map will allow us to equate the appropriately normalized Yukawa couplings and obtain the Gromov-Witten invariants of a generic quintic $X \subset \mathbb{P}^4$.

3.2.3. *Period Integrals.* We first describe a single valued solution to the Picard-Fuchs equations. Consider the 3-cycle β_0 in X_ψ given by the set of points in $\mathbb{P}^4$ with $x_4 = 1, |x_0| = |x_1| = |x_2| = \delta$, and x_3 given by the solution to $f_\psi = 0$ that tends to 0 as $\psi \to \infty$. This means the following: defined y as $x_3 := (\psi x_0 x_1 x_2)^{\frac{1}{4}} y$. Then $f_\psi = 0$ with $x_4 = 1$ is $y = \frac{y^5}{5} + \frac{1 + x_0^5 + x_1^5 + x_2^5}{5(\psi x_0 x_1 x_2)^{\frac{5}{4}}}$. Solving for y, there are 4 solutions approaching 4th roots of 5 as $\psi \to \infty$, and 1 solution going to 0 like $\psi^{-5/4}$. Taking the latter, x_3 is well defined.

Next, define the family of 3-forms on the open subset of X_ψ given by $x_4 = 1$,

$$\Omega(\psi) = 5\psi \frac{dx_0 \wedge dx_1 \wedge dx_2}{\frac{\partial f_\psi}{\partial x_3}} = 5\psi \frac{dx_0 \wedge dx_1 \wedge dx_2}{5x_3^4 - 5\psi x_0 x_1 x_2}$$

Apriori, this is a family of meromorphic 3-forms. However, using the implicit function theorem, it can be shown that $\Omega(\psi)$ is actually a holomorphic 3-form on non-singular X_ψ. Since these forms are G-invariant, they descend to Y_ψ and extend to the resolved hypersurfaces $\check{X}_\psi$. By definition, they are the Calabi-Yau forms that are unique up to phase.

Therefore, the period integral we want to compute is,

$$\int_{\beta_0} \frac{dx_0 \wedge dx_1 \wedge dx_2}{\frac{x_3^4}{\psi^{-1}} - x_0 x_1 x_2}$$

This integral is equivalent to the following integral,

$$\frac{1}{2\pi i} \int_{T^4} 5\psi \frac{dx_0 dx_1 dx_2 dx_3}{f_\psi}$$

where $T^4 = \{(x_0, x_1, x_2, x_3, 1) | |x_i| = \delta\}$. To see this, fixing x_0, x_1, x_2, the above integrand has poles when $f_\psi(x_0, x_1, x_2, x_3, 1) = 0$. We showed for ψ large, there was only one such value of x_3 near 0, which implies the pole is simple. Using residues,

$$\frac{1}{2\pi i} \int_{T^4} 5\psi \frac{dx_0 dx_1 dx_2 dx_3}{f_\psi} = \int_{\beta_0} 5\psi \frac{dx_0 dx_1 dx_2}{\frac{\partial f_\psi}{\partial x_3}}$$

Thus, we compute,

$$\begin{aligned}
\int_{T^4} 5\psi \frac{dx_0 dx_1 dx_2 dx_3}{f_\psi} &= \int_{T^4} \frac{dx_0 dx_1 dx_2 dx_3}{(5\psi^{-1})(1 + x_0^5 + \ldots + x_3^5) - x_0 x_1 x_2 x_3} \\
&= \int_{T^4} \frac{dx_0 dx_1 dx_2 dx_3}{x_0 x_1 x_2 x_3} \frac{1}{(5\psi)^{-1} \frac{(1 + x_0^5 + \ldots + x_3^5)}{x_0 x_1 x_2 x_3} - 1} \\
&= -\sum_{n=0}^{\infty} \int_{T^4} \frac{dx_0 dx_1 dx_2 dx_3}{x_0 x_1 x_2 x_3} \frac{(1 + x_0^5 + \ldots x_3^5)^n}{(5\psi)^n (x_0 x_1 x_2 x_3)^n}
\end{aligned}$$

The only terms that contribute to the residue calculation will be those in which one can cancel out $(x_0 x_1 x_2 x_3)^n$ in the denominator of the second integrand. Since the exponents in the numerator are divisible by 5, only the $5|n$ terms contribute,

$$-\sum_{n=0}^{\infty} \int_{T^4} \frac{dx_0 dx_1 dx_2 dx_3}{x_0 x_1 x_2 x_3} \frac{(1 + x_0^5 + \ldots x_3^5)^{5n}}{(5\psi)^{5n} (x_0 x_1 x_2 x_3)^{5n}}$$

By the binomial formula, the number of terms of the form $(x_0x_1x_2x_3)^{5n}$ in the numerator will be $(5n)!/(n!)^5$. Thus, the above integral is,

$$-\sum_{n=0}^{\infty}\int_{T^4}\frac{dx_0dx_1dx_2dx_3}{x_0x_1x_2x_3}\frac{(1+x_0^5+\dots x_3^5)^{5n}}{(5\psi)^{5n}(x_0x_1x_2x_3)^{5n}} = -(2\pi i)^3\sum_{n=0}^{\infty}\frac{(5n)!}{(5\psi)^{5n}(n!)^5}$$

Using the complex moduli coordinate $x=(5\psi)^{-5}$, we see that the period integral is proportional to,

$$\phi_0(x)=\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}x^n$$

Notice $a_n := \frac{(5n)!}{(n!)^5}$ obey the recurrence relation,

$$(n+1)^5a_{n+1}=(5n+1)(5n+2)(5n+3)(5n+4)(5n+5)a_n$$

which translates into a differential equation for $\phi_0(x)$. Letting $\Theta := x\frac{d}{dx}$, $\phi_0(x)$ satisfies the differential equation

$$(\Theta^4-5x(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4))\phi_0(z)=0$$

It turns out all period integrals satisfy the above equation, which is called the Picard-Fuchs equation. To see that they must satisfy some fourth order equation, note that $H^3(\check{X}_\psi,\mathbb{C})$ is four-dimensional, hence there is a linear dependence among,

$$\Omega(x),\frac{\partial\Omega}{\partial x},\frac{\partial^2\Omega}{\partial x^2},\frac{\partial^3\Omega}{\partial x^3},\frac{\partial^4\Omega}{\partial x^4}$$

To actually prove this, one may use residues and the Griffiths-Dwork method of pole reduction to differentiate the Calabi-Yau form and determine the correct coefficients for the differential equation.

3.3. **Mirror Map.** In the definition above, $\Omega(x)$ is thought of as a section of the sheaf $R^3f_*\mathbb{C}\otimes\mathcal{O}_S$, and ∇ is the induced Gauss-Manin connection given by the Riemann-Hilbert correspondence.

In our setting, $f:\Psi\to\Delta^*$ is the family of mirror quintics. Since $x=0$ refers to the large complex structure limit point, monodromy around the point is maximally unipotent. Therefore, there exists cycles $\beta_0,\beta_1\in H^3(\check{X}_x,\mathbb{Q})$ such that $\beta_0\to\beta_0$ and $\beta_1\to\beta_1+n\beta_0$ under monodromy. Therefore, $\int_{\beta_0}\Omega(x)$ is single-valued and must be proportional to $\phi_0(x)$,

$$\int_{\beta_0}\Omega(x)=C\phi_0(x)$$

while $\int_{\beta_1}\Omega(x)$ is multi-valued and hence

$$\int_{\beta_1}\Omega(x)=D_0\phi_0(x)+D_1\phi_1(x)$$

The above $\phi_1(x)$ is a multi-valued solution of the Picard-Fuchs equation satisfying $\phi_1(e^{2\pi i}x) = \phi_1(x) + 2\pi i\phi_0(x)$. It can be written as

$$\phi_1(x) = \phi_0(x)\log x + \psi(x)$$

where $\psi(x) = 5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}(\sum_{j=n+1}^{5n}\frac{1}{j})x^n$. Now,

$$\int_{T(\beta_1)}\Omega(x) = n\int_{\beta_0}\Omega(x) + \int_{\beta_1}\Omega(x) = (nC + D_0)\phi_0 + D_1\phi_1$$

and

$$D_0\phi_0(e^{2\pi i}x) + D_1\phi_1(e^{2\pi i}x) = (D_0 + 2\pi i D_1)\phi_0(x) + D_1\phi_1(x)$$

so $nC = 2\pi i D_1$. Letting $n = 1$, the canonical coordinate is

$$\begin{aligned} w &= \frac{\int_{\beta_1}\Omega}{\int_{\beta_0}\Omega} \\ &= \frac{D_0}{C} + \frac{1}{2\pi i}\frac{\phi_1(x)}{\phi_0(x)} \\ &= \frac{1}{2\pi i}\log x + \frac{1}{2\pi i}\log c_2 + \frac{1}{2\pi i}\frac{\psi(x)}{\phi_0(x)} \end{aligned}$$

So, the mirror map is

$$q = e^{2\pi i w} = c_2 z e^{\psi/\phi_0}$$

3.4. **(1,2)-Yukawa coupling.** The (1,2)-Yukawa coupling is a cubic form defined on the complex structure moduli space. Formally, it is

Definition 3.1. Given a family of Calabi-Yau 3-folds $f : \Psi \to S$ and a holomorphically varying family of holomorphic three-forms on Ψ, $\Omega(x)$, the (1,2)-Yukawa coupling is a cubic form on the tangent bundle of S. Given a local trivialization of f, $f^{-1}(U) \cong U \times X$, let $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \in T_{S,x}$. Set

$$\langle\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\rangle = \int_X \Omega(x) \wedge \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_2}}\nabla_{\frac{\partial}{\partial x_3}}\Omega(x)$$

We can now use the mirror map to equate the Yukawa couplings from the A-model and B-model. Let

$$W_k = \int_{\check{X}_x}\Omega(x) \wedge \frac{d^k}{dx^k}\Omega(x)$$

If one rewrites the Picard-Fuchs equation as

$$(\frac{d^4}{dx^4} + \sum_{k=0}^{3} C_k(x)\frac{d^k}{dx^k})\Omega(x) = 0$$

then the couplings satisfy

$$W_4 + \sum_{k=0}^{3} C_k W_k = 0$$

By Griffiths Transversality, $W_0 = W_1 = W_2 = 0$ and $\frac{d^2}{dx^2}W_2 = 2(W_3' - W_4) + W_4)$. Hence, the coupling W_3 that we are interested in satisfies

$$W_3' + \frac{1}{2}C_3 W_3 = 0$$

Thus,

$$W_3 = \frac{c_1}{(2\pi i)^3 x^3 (5^5 x - 1)}$$

is the general solution to the above differential equation.

Now, the Yukawa couplings we want to equate need to be suitably normalized. They are computed with respect to the following family of 3-forms

$$\frac{\Omega(x)}{\int_{\beta_0}\Omega(x)}$$

Notice that if $\Omega(x) \to f(x)\Omega(x)$, then the Yukawa coupling scales by a factor of $f(x)^2$. Since we wrote the mirror map in terms of the coordinate w, by the chain rule the (1,2)-Yukawa coupling we want to compute is,

$$\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = (\frac{dx}{dw})^3 \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle$$

One then wants to express the above (1,2)-Yukawa coupling in terms of the Kähler moduli coordinate q. That will be predicted to coincide with the (1,1)-Yukawa coupling after suitable choice of constants. In this way, one obtains the Gromov-Witten invariants of the quintic threefold.

4. Mirror Symmetry for the Elliptic Curve

We describe mirror symmetry on the level of Hodge numbers for elliptic curves. We then derive the Picard Fuchs equations that the periods of the mirror must satisfy.

Elliptic curves are the only 1-dimensional Calabi-Yau manifolds. Given a lattice $\Lambda = z_1\mathbb{Z} \oplus z_2\mathbb{Z} \cong \mathbb{Z}^2 \subseteq \mathbb{C}$, an elliptic curve can be described as $E \cong \mathbb{C}/\Lambda$. In this way, a complex elliptic curve is a torus. Complex multiplication by $\pm\frac{1}{z_1}$ gives an isomorphism between the elliptic curves $E = \mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda' = E'$, where $\Lambda' = \mathbb{Z} \oplus \tau\mathbb{Z}$, with $\tau = \frac{z_2}{z_1}$. Recall that $PSL_2(\mathbb{Z})$ is generated by the transformations $\tau \to \tau + 1$ and $\tau \to \frac{-1}{\tau}$. Hence, the moduli space of complex structures for elliptic curves is $\mathbb{H}/PSL_2(\mathbb{Z})$, where the action is given by

$$\tau \in \mathbb{H} \to \frac{a\tau + b}{c\tau + d}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{Z})$$

For the moduli space of complexified Kähler classes, recall that the Kähler classes form a cone of $H^{1,1}(E,\mathbb{R}) = H^2(E,\mathbb{R}) \cap H^{1,1}(E)$. Hence, the cone is a real manifold of dimension $h^{1,1} = 1$. Take the generator of $H^{1,1}(E, \mathbb{R})$ to be $\omega_* = \frac{\sqrt{-1}}{2\text{Im }\tau} dz \wedge d\bar{z} = \frac{1}{\text{Im }\tau} dx \wedge dy$, which is a closed, real, $(1,1)$-form. Then, a Kähler form is given by $\omega = r\omega_*$ with $r \in \mathbb{R}^{>0}$. The field $B \in H^{1,1}(E, \mathbb{R})$ is equal to $B = r'\omega_*$ for some $r' \in \mathbb{R}$. A complexified Kähler class is then

$$\omega^{\mathbb{C}} = B + i\omega = r'\omega_* + ir\omega_* = t\omega_*$$

with $t \in \mathbb{H}$. Therefore, the moduli space of complexified Kähler classes is $\mathbb{H}$. Notice that

$$\int_E \omega^{\mathbb{C}} = \int_E t\omega_* = \frac{t}{\text{Im }\tau} \int_E \frac{i}{2} dz \wedge d\bar{z} = t$$

If we denote $E_{t,\tau}$ as the elliptic curve with modular parameter τ and Kähler parameter t, then mirror symmetry for elliptic curves is

$$\check{E}_{\check{t},\check{\tau}} = E_{\tau,t}$$

4.1. **Picard-Fuchs equations.** The mirror family of $E \cong \mathbb{C}/\Lambda$ is given by the elliptic surface $\Psi \subseteq \mathbb{P}^2 \times \mathbb{P}^1$

$$F(x, y, z) = t(x^3 + y^3 + z^3) - 3xyz = 0$$

with $t \in B := \mathbb{P}^1 \setminus \{0, 1, \zeta, \zeta^2\}$, and ζ a cubic root of unity. The surface is called the *Hesse pencil*. Hence we have a smooth family $f : E \to B$. A 1-cycle $\gamma \in H_1(E_t; \mathbb{C})$ may identified with cycles of nearby fibers. Like before, we study period integrals of the family of holomorphic 1-forms $\omega(t)$ over cycles in the middle cohomology $H_1(E_t; \mathbb{C})$,

$$\pi(t) = \int_\gamma \omega(t)$$

Ratios of the periods turn out to give the modular parameter. These periods must satisfy Picard-Fuchs equations. We outline the residue approach to derive the Picard-Fuchs equation.

4.1.1. *Residue Map.* Recall from single variable complex analysis, the residue of a meromorphic 1-form returns a number or a 0-form. We want to generalize this idea to higher degree cohomology. Given a smooth hypersurface $X \subset \mathbb{P}^n$, a residue map should take a rational n-form with poles on X to a holomorphic form in $H^{n-1}(X, \mathbb{C})$. It is defined as follows: given an $(n-1)$-cycle $\gamma \in H^{n-1}(X; \mathbb{C})$, take its tube $T(\gamma)$ lying in $\mathbb{P}^n \setminus X$. The tube locally looks like $\gamma \times S^1$. Consider the map

$$\gamma \to \frac{1}{2\pi i} \int_{T(\gamma)} \omega \in \mathbb{C}$$

This defines an element $\text{Res }\omega \in H^{n-1}(X; \mathbb{C})$. Identifying singular cohomology with de Rham cohomology, this means

$$\frac{1}{2\pi i} \int_{T(\gamma)} \omega = \int_\gamma \text{Res }\omega$$

We will now use the residue map to define the family of non-vanishing holomorphic volume forms on Ψ. Hence, we compute the residue of the rational 2-form

$$\rho_t = \frac{\Omega}{F} := \frac{xdy\wedge dz - ydx\wedge dz + zdx\wedge dy}{t(x^3+y^3+z^3)-3xyz}$$

We express ρ_t in terms of dF to make the computation explicit. Since Ψ consists of smooth elliptic curves, at any point, at least one partial derivative of F is nonzero. Assume in a neighborhood of a point that $\frac{\partial F}{\partial x}\neq 0$. We may write

$$\rho_t = \frac{ydz-zdy}{\frac{\partial F}{\partial x}}\wedge\frac{dF}{F} + \frac{3dy\wedge dx}{\frac{\partial F}{\partial x}}$$

The second term is holomorphic, so by Cauchy's Theorem, Res $\rho_t = \frac{ydz-zdy}{3(tx^2-yz)} = \frac{ydz-zdy}{\frac{\partial F}{\partial x}}$. This gives a nonvanishing holomorphic 1-form in a neighborhood where $\frac{\partial F}{\partial x}\neq 0$. Using a partition of unity, the global nonvanishing holomorphic 1-form takes a similar form. Thus, $\omega(t) = \text{Res }\rho_t$.

Define $\Omega_k(t) = (\frac{d}{dt})^k(\frac{\Omega}{F})$. It suffices to find an equation $\Omega_2(t)+B(t)\Omega_1(t)+C(t)\Omega_0(t)\equiv 0$ modulo exact forms, since we can differentiate under the integral $\frac{d^k}{dt^k}\pi(t) = \int_{T(\gamma)}\Omega_k(t)$. We will use the following two lemmas to derive the PF-equations.

Lemma 4.1. $\frac{A\frac{\partial F}{\partial x}+B\frac{\partial F}{\partial y}+C\frac{\partial F}{\partial z}}{F^k}\Omega \equiv \frac{\Omega}{(k-1)F^{k-1}}(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z})$, *modulo exact forms.*

Thus, if the numerator of a meromorphic $k-$form lies in the Jacobian ideal $J(F) := \langle\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\rangle$, then it is equivalent modulo exact forms to a meromorphic form with a pole of one degree lower.

Lemma 4.2. $H^{p,1-p}(X;\mathbb{C})\cong(\mathbb{C}[x,y,z]/J(F))_{3(1-p)}$, *where the subscript refers to the natural grading. The isomorphism is given by*

$$Res\ \frac{Q\Omega}{F^{2-p}}\leftrightarrow Q$$

This lemma will be used in the following way: if Res $\frac{Q\Omega}{F^{2-p}}=0$, *then* $Q\in J(F)$.

We begin with $\Omega_2(t) = \frac{2(x^3+y^3+z^3)^2}{F^3}\Omega$. This is a rational form that is homogeneous of degree 6. Using Lemma 3, Res $\Omega_2(t)\in H^{-1,2}(X;\mathbb{C}) = 0$. Therefore, the numerator lies in $J(F)$. Write the numerator as a linear combination of partial derivatives of F (one can explicitly calculate this with Gröbner bases) and use Lemma 2 to write, modulo exact forms,

$$\Omega_2(t)\equiv(\frac{2}{t}(x^3+y^3+z^3)+\frac{6txyz}{t^3-1}+\frac{3x^3}{t(t^3-1)})\frac{\Omega}{F^2}$$

This form has a pole of order 2 and so does $\Omega_1(t) = \frac{-(x^3+y^3+z^3)}{F^2}\Omega$. Since $H^{0,1}(X;\mathbb{C})$ is of dimension 1, $\exists B(t)$ such that Res $(\Omega_2(t)+B(t)\Omega_1(t)) = 0$. Suppose this form's expression is $\frac{Q\Omega}{F^2}$, then $Q \in J(F)$. Explicitly,

$$Q = \frac{2 - Bt}{t^2}F + \frac{x}{t^2(t^3-1)}\frac{\partial F}{\partial x} + \frac{3}{t}(\frac{4t^3-1}{t(t^3-1)} - B)xyz$$

Both F and $\frac{\partial F}{\partial x} \in J(F)$. Thus, for $Q \in J(F)$, we must have $B = \frac{4t^3-1}{t(t^3-1)}$.

Similarly, we can calculate for $C(t)$, which gives us the Picard-Fuchs equations. The associated Picard-Fuchs differential operator is thus

$$\frac{d^2}{dt^2} + \frac{4t^3-1}{t(t^3-1)}\frac{d}{dt} - \frac{2t}{t^3-1}$$

Since the equation is second order, its space of solutions is two dimensional. It is spanned by a holomorphic, single valued function $\pi_1(t)$ and another multivalued function $\pi_2(t) = \pi_1(t)\log t + \rho(t)$, with ρ holomorphic. We can find $\pi_1(t)$ using power series. Suppose $\pi_1(t) = \sum_{n=0}^{\infty} a_n t^n$. The recurrence relation that must be satisfied by the coefficients is

$$a_{n+3}(n+3)^2 = a_n(n+2)(n+1)$$

This implies

$$\pi_1(t) = \sum_{n=0}^{\infty} \frac{(3n)!}{3^{3n}(n!)^3}t^{3n}$$

Suppose $\alpha \in H_1(E_t;\mathbb{C})$ is a vanishing cycle invariant under monodromy. Then since $\int_\alpha \omega(t)$ is single valued, it is a multiple of $\pi_1(t)$.

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Benjamin Zhou, Department of Mathematics, Northwestern University, Evanston, IL, USA
Email address: byzhou01@math.northwestern.edu