Every smooth cubic surface has exactly 27 lines (GSS talk)

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1 Introduction

I’ll present two proofs of the classical fact in enumerative geometry that every smooth cubic surface \( \subseteq \mathbb{P}^3 \) has exactly 27 lines. There is a “simpler” proof that uses mainly linear algebra to first show there are exactly 27 lines on the Fermat cubic \( x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0 \), and then proves every other smooth cubic must have the same number of lines. I’ll present two other proofs: 1) Realizing the smooth cubic surface as the blow up of \( \mathbb{CP}^2 \) at 6 points, and counting lines in the blow up 2) Schubert calculus on the cohomology of \( Gr(2,4) \), the Grassmannian of 2-planes in \( \mathbb{C}^4 \).

2 Counting lines on smooth cubic surface as \( Bl_{6\text{pts}}\mathbb{CP}^2 \)

Take 6 sufficiently general points in \( \mathbb{CP}^2 \), i.e. no three points are collinear, no six lie on a conic. There exists a rational map \( \varphi : \mathbb{CP}^2 \rightarrow \mathbb{CP}^3 \) as follows: Take the vector space of cubic polynomials in \( \mathbb{CP}^2 \), which is \( \binom{3+2}{2} = 10 \) dimensional. Asking that the cubic passes through 6 points imposes 6 linearly independent conditions. Hence, there exist 4 linearly independent cubics \( f_i \) vanishing at the 6 points. Define

\[
\varphi([x_0 : x_1 : x_2]) := [f_0(x_0, x_1, x_2) : f_1(x_0, x_1, x_2) : f_2(x_0, x_1, x_2) : f_3(x_0, x_1, x_2)]
\]

This map is clearly undefined at the six points. After blowing up at the 6 points, by the theorem of elimination of indeterminacy, there exists a morphism \( \varphi' : Bl_{6\text{pts}}\mathbb{CP}^2 \rightarrow \mathbb{CP}^3 \) and the commutative diagram.

\[
\begin{array}{ccc}
Bl_{6\text{pts}}\mathbb{CP}^2 & \xrightarrow{\pi} & \mathbb{CP}^2 \\
\downarrow{\varphi'} & & \downarrow{\varphi} \\
\mathbb{CP}^3 & & \end{array}
\]

, where \( \pi \) is the contraction map (Draw picture of blow up). The map \( \varphi' \) is an embedding and is the same embedding from the Kodaira embedding theorem with the anti-canonical bundle \( \mathcal{O}(3L - E_1 - \ldots - E_6) \) of \( Bl_{6\text{pts}}\mathbb{CP}^2 \). Hence, the image \( S := \varphi'(Bl_{6\text{pts}}\mathbb{CP}^2) \) is a smooth surface. We need to verify it’s cubic; it’s enough to check \( \int_S H^2 = 3 \), since by Bezout’s theorem, a cubic surface will intersect a line at 3 points (\( H = PD(\text{hyperplane}) \), so \( H^2 \) is the class of intersection of two hyperplanes or a line). The embedding map satisfies \( c_1((\varphi')^*\mathcal{O}(1)) = ((\varphi')^*H = 3H - PD(E_1) - \ldots - PD(E_6) = c_1(\mathcal{O}(3L - E_1 - \ldots - E_6)) \) So, since cup product is Poincare dual to intersection,

\[
\int_S H^2 = \int_{Bl_{6\text{pts}}\mathbb{CP}^2} ((\varphi')^*H)^2 = \int_{Bl_{6\text{pts}}\mathbb{CP}^2} (3H-PD(E_1)-\ldots-PD(E_6))^2 = (3L-E_1-\ldots-E_6)^2 = (3L-E_1-\ldots-E_6)
\]

We have that \( L \) is the class of line that doesn’t intersect \( E_i \), \( E_i^2 = -1 \), and \( E_1 \cdot E_2 = 0 \) since the exceptional divisors are disjoint from each other. Thus, the above integral is 3 and the surface is cubic.
We can characterize lines on \( S \) as rational curves with self-intersection \(-1\).

**Lemma 1** If \( \ell \) is a line on \( S \), then \( \ell^2 = -1 \) on \( S \). Conversely, if \( C \subset S \) is a smooth irreducible rational curve with \( C^2 = -1 \), then \( C \) is a line.

**Proof:** Let \( \ell \subset S \) be a line. By the adjunction formula \((2g - 2 = \ell \cdot (\ell + K_S))\), we have \(-2 = \ell^2 + \ell \cdot (-L)\), since the canonical divisor \( K_S = -L \). Hence \( \ell^2 = -1 \) on \( S \). Conversely, by adjunction again, \(-2 = C^2 + C \cdot K_S\), and thus \( 1 = C \cdot L \). The degree of \( C \) is 1, so \( C \) is a line.

Thus, lines on \( S \) are smooth rational curves with self intersection \(-1\). So to count the lines on \( S \), we count rational curves with self-intersection \(-1\). Using the blow up description of the cubic surface, we see that the exceptional curves \( E_1, \ldots, E_6 \) give us 6 lines. Furthermore, the proper transform \( L - E_i - E_j \) of the unique line through \( p_i \) and \( p_j \) in \( \mathbb{CP}^2 \) give us \( \binom{6}{2} = 15 \) lines, as \((L - E_i - E_j)^2 = -1\). The proper transform \( 2L - E_{i_1} - \ldots - E_{i_6} \) of the unique conic through 5 points gives us \( \binom{6}{5} = 6 \) lines (Draw pictures here). This already gives us a total of 27 lines! To see that these are the only lines, any other line in \( Bl_6 pts \mathbb{CP}^2 \) is of the form \( D = aL - b_1E_1 - \ldots - b_6E_6 \) with \( a > 0, b_i \geq 0 \) and \( D^2 = -1 \). It follows from Cauchy-Schwarz implying \( a = 1 \) or 2, and \( a^2 - \sum b_i^2 = -1 \) that the above are the only lines.

We have shown that when the cubic surface is a blow up of the projective plane at 6 sufficiently general points, then it has exactly 27 lines. But actually it’s a theorem that every smooth cubic surface is obtained in this way. Thus every smooth cubic surface has exactly 27 lines (Heuristic: we know \{\( Bl_6 pts \mathbb{CP}^2 \)\} \( \in \) \{rational cubic surfaces\}. Both sides are 19 dimensional, LHS forms a open dense subset).

### 3 Schubert calculus on \( Gr(2, 4) \)

Every 2-plane in \( \mathbb{C}^4 \) may be represented as a rank 2, \( 2 \times 4 \) matrix with complex coefficients. We are interested in \( Gr(2, 4) \), because 2-planes in \( \mathbb{C}^4 \) correspond to lines in \( \mathbb{P}^3 \), after projectivization of the row space. After Gaussian elimination, there are 6 possible row echelon forms of an element in \( Gr(2, 4) \),

\[
\Sigma_{0,0} = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \Sigma_{1,0} = \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \Sigma_{2,0} = \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \Sigma_{0,1} = \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \Sigma_{2,1} = \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \Sigma_{2,2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

, where * signifies an arbitrary complex number. We see that \( \Sigma_{0,0} \) is a complex 4-cell, etc. and they provide a cellular decomposition of \( Gr(2, 4) \). These are called the Schubert cells. We first find a geometric interpretation of the cellular decomposition. Fix a flag \( p \in L \subset H \), where we take the point \( p = [0:0:0:1] \), the line \( L = \{x_0 = x_1 = 0\} \), and the hyperplane \( \{x_0 = 0\} \). Using the matrix descriptions of the \( \Sigma_{i,j} \), it is easy to see that

\[
\Sigma_{2,2} = \{L\}, \Sigma_{2,1} = \{\ell | p \in \ell \subset H\}, \Sigma_{1,1} = \{\ell | \ell \subset H\}, \Sigma_{2,0} = \{\ell | p \in \ell\}, \Sigma_{1,0} = \{\ell | \ell \cap L \neq \emptyset\}, \Sigma_{0,0} = Gr(2, 4)
\]

and we have the cellular decomposition,

\[
\Sigma_{2,2} \subset \Sigma_{2,1} \subset \Sigma_{1,1} \cup \Sigma_{2,0} \subset \Sigma_{1,0} \subset \Sigma_{0,0}
\]

Since these are complex cells, the cohomology is even dimensional and free abelian, i.e.

\[
H^0(Gr(2, 4); \mathbb{Z}) = \mathbb{Z}\sigma_{0,0}, H^2(Gr(2, 4); \mathbb{Z}) = \mathbb{Z}\sigma_{1,0}
\]

\[
H^4(Gr(2, 4); \mathbb{Z}) = \mathbb{Z}\sigma_{2,0} \oplus \mathbb{Z}\sigma_{1,1}, H^6(Gr(2, 4); \mathbb{Z}) = \mathbb{Z}\sigma_{2,1}, H^8(Gr(2, 4); \mathbb{Z}) = \mathbb{Z}\sigma_{2,2}
\]

, where \( \sigma_{i,j} \) are the generators of cohomology corresponding to the \( \Sigma_{i,j} \).
The calculation of 27 lines on the cubic surface will be a characteristic class calculation involving the Schubert classes. So, we calculate part of the cohomology ring of Gr(2,4), namely the classes $\sigma_{11}\sigma_{20}, \sigma_{10}^2$, and $\sigma_{11}^2$. First, fix another flag $p' \in L' \subset H'$. In order the make intersections transverse. We see that $\sigma_{11}\sigma_{20}$ represent the lines $\ell$ which are contained in $H$ and meet $p'$. Generically $p' \notin H$, so there are no such lines, i.e. $\sigma_{11}\sigma_{20} = 0$. We see that $\sigma_{11}^2$ represent the lines $\ell$ contained in the hyperplanes $H$ and $H'$. The intersection $H \cap H'$ will be only one line, so $\sigma_{11}^2 = \sigma_{22}$. To calculate $\sigma_{10}^2$, we know that $\sigma_{10}^2 = a\sigma_{20} + b\sigma_{11}$ for some integers $a, b$. From this, we have $\sigma_{10}\sigma_{20} = a\sigma_{22}$ and $\sigma_{10}\sigma_{11} = b\sigma_{11}$. Taking a third flag $p'' \in L'' \subset H''$, the class $\sigma_{10}\sigma_{20}$ represents the lines $\ell$ that meet $L, L'$ and contain $p''$. Lines meeting $L'$ and containing $p''$ span a plane. The line $L$ will intersect the plane at only one point, so there is only one line satisfying all three conditions. Therefore, $\sigma_{10}\sigma_{20} = \sigma_{22}$. Similarly, $\sigma_{10}^2\sigma_{11}$ will represent lines $\ell$ intersecting $L, L'$ and contained in the hyperplane $H''$. Each line $L, L'$ will intersect $H''$ at one point respectively, and there exists a unique line connecting the two points. Thus $a = b = 1$, i.e. $\sigma_{10}^2 = \sigma_{20} + \sigma_{11}.$

Recall that we have the tautological bundle $E$ over $Gr(2,4)$. We will calculate the classes of $Sym^3E^*$, the fiber of which are degree 3 homogeneous forms on the line $\ell$. Now take a smooth cubic surface $S = \{F = 0\} \subseteq \mathbb{P}^3$, which is the zero-set of a degree 3 homogeneous polynomial $F$. Given $\ell \subseteq \mathbb{P}^3$, we can restrict $F|\ell$, and thus get a section $s$ of $Sym^3E^*$. The top Chern class of $Sym^3E^*$ represents the 0-set of a generic section. The zeros of the section are the lines contained in the cubic surface $S$! Thus we calculate

$$\int_{Gr(2,4)} c_4(Sym^3E^*)$$

with $rk_{Gr}Sym^3E^* = \binom{2+3}{1} = 4$. The splitting principle can be used to calculate $c_4(Sym^3E^*)$: any formula of Chern classes of a vector bundle derived from assuming $E$ is a direct sum of line bundles actually holds. Using this, we have

$$c_4(Sym^3E^*) = 9c_2(E^*)(2c_1(E^*)^2 + c_2(E^*)),$$

with $c_1(E^*) = \sigma_{10}, c_2(E^*) = \sigma_{11}$. Thus, $c_4 = 9\sigma_{11}(2\sigma_{10}^2 + \sigma_{11}) = 27\sigma_{22}$ \Rightarrow $\int_{Gr(2,4)} c_4 = 27$. This number is up to multiplicity, but if we assume the section $s$ has transverse zeros, then there exactly 27 lines. The section is transverse precisely when $X$ is a smooth cubic surface.

References

[1] Vakil’s course on complex algebraic surfaces
[4] Griffiths Harris chapter on surfaces