CURVE COUNTING AND MODULAR FORMS TALK

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Contents

1. Abstract

Gromov-Witten invariants ideally count genus g Riemann surfaces in a target space X by integrating over the moduli space of stable maps. When $X = \mathbb{P}^2$, these invariants generalize classical numbers in enumerative geometry considered by the likes of Euclid, Schubert, and Hilbert. In the 1990s, physicists used mirror symmetry as duality between Type IIA and Type IIB string theory to miraculously compute the Gromov-Witten invariants of the quintic threefold. When one forms the generating function of Gromov-Witten invariants over all genus g, it may have some surprising yet beautiful properties. Two of them are the Virasoro constraints, studied by Getzler et. al. and modularity, proven by a formula of Yau-Zaslow. I will explain some of the above terms, and present how when $X =$ an elliptic curve E or a K3 surface, the generating function of Gromov-Witten invariants is a modular form.

2. An enumerative question...

Schubert (1874): Given 4 lines $\ell_1, \ldots, \ell_4$ in 3-space, how many lines ℓ pass through them? The answer is 2 lines. Assume that the lines ℓ_i are in generic position by principle of conservation of number. The first line is obtained as follows: assume that

ℓ_1 and ℓ_2 intersect. They will span a plane. Lines ℓ_3 and ℓ_4 intersect the plane at some points. The line connecting the intersection points intersects all four lines. The second line is obtained as follows: assuming ℓ_1 and ℓ_2 intersect, form the plane containing the intersection point and line ℓ_3. Line ℓ_4 will intersect this plane at some point. The second line is obtained by connecting that point with the intersection point of the first two lines.

We can also use Schubert calculus. Consider the Grassmannian $Gr(2,4) = \{V \subseteq \mathbb{C}^4 | \dim_{\mathbb{C}} V = 2\}$, or the moduli of lines ℓ in $\mathbb{CP}^3$ after projectivization. As a complex manifold $\dim_{\mathbb{C}} Gr(2,4) = 4$. Define the cycles $H_i := \{\ell \in Gr(2,4) | \ell \cap \ell_i \neq \emptyset\}$. This is a codimension 1 condition on the space of lines, so we expect $H_1 \cap \ldots \cap H_4$ to be of dimension 0 and we expect to count a finite number of lines. We calculate $PD([H_1]) \cup \ldots \cup PD([H_4]) = PD([H_1 \cap \ldots \cap H_4]) \in H^8(Gr(2,4), \mathbb{Z})$ via Schubert cell decomposition of $Gr(2,4)$. Its degree is 2, in agreement with the answer from the first approach.

3. Counting maps and/or curves

We are essentially counting maps $f : \mathbb{CP}^2 \to \mathbb{CP}^3$ such that $Im(f) \cap \ell_i \neq \emptyset$ for each i. How to approach the question of counting maps? We start with the "moduli space" of all maps and impose constraints on the space so that the resulting space we are interested in counting is compact and of dimension 0, hence has finite cardinality.

As an example, suppose $f : \mathbb{CP}^1 \to \mathbb{CP}^3$ is a degree d holomorphic map, or locally f looks like $z = [z_0 : z_1] \mapsto [f_0(z) : f_1(z) : f_2(z)]$, where the f_i are homogeneous degree d polynomials. Equivalently, $f_*[\mathbb{CP}^1] = d$. What is the dimension of $\{f : \mathbb{CP}^1 \to \mathbb{CP}^2 | f_*[\mathbb{CP}^1] = d\}$? The answer is $3(d+1)-1-3 = 3d-1$. Here $d+1$ is the dimension of the space of homogeneous, degree d polynomials in 2 variables, -1 is for projectivization, -3 is for reparametrization of $\mathbb{CP}^1$ as $Aut(\mathbb{CP}^1) = PGL_2(\mathbb{C})$. We impose constraints to cut $3d-1$ to 0 by specifying incidence conditions of f.

4. Gromov-Witten Invariants

Gromov-Witten invariants are defined by the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$. An element of the moduli space is a stable map $f : (C, p_1, \ldots, p_n) \to X$ such that $f_*([C]) = \beta \in H_2(X, \mathbb{Z})$. The n marked points are distinct $p_i \neq p_j$. We have natural morphisms given by evaluation maps and a forgetful morphism to the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. The Gromov-Witten invariants are defined as follows,

Definition 4.1. Let $\gamma_i \in H^*(X)$. The genus g, n-pointed Gromov-Witten invariant of X in curve class β with incidence conditions γ_i is

$$N_{g,n}(X, \beta | \gamma_1, \ldots, \gamma_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^* \gamma_1 \cup \ldots \cup ev_n^* \gamma_n$$

We list some properties,

(1) $N_{g,n} = 0$ if $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X, \beta) \neq \sum_i deg_{\mathbb{C}} \gamma_i$
(2) If $g = 0$, $\beta = 0$, we are considering stable maps from a Riemann sphere to a point. Then $N_{g,n} = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$
(3) The expected dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is $(dimX - 3)(1 - g) + \int_\beta c_1(TX) + n$
(4) $\overline{\mathcal{M}}_{g,n}(X, \beta)$ can be compactified by allowing domain curves to degenerate to have nodes (Deligne-Mumford compactification)
(5) $\overline{\mathcal{M}}_{g,n}(X, \beta)$ may not be smooth or of pure dimension, so it may not have a fundamental class, but it may admit a virtual fundamental class that one can still integrate the define the invariants with.
(6) Often one forms the genus g generating series F_g of GW invariants

$$F_g := \sum_{\beta \in H_2(X, \mathbb{Z})} N_g(X, \beta) q^\beta$$

by summing over auxillary variables.

Remark 4.2. A node is a singular point such that locally the curve is described as $xy = 0$. The partial derivatives of the defining function vanish to order 1, i.e. $f(x, y) = xy$ with $\nabla f(0, 0) = (0, 0)$.

Example 4.3. The previous number as a GW invariant: $\int_{\overline{\mathcal{M}}_{0,4}(\mathbb{CP}^3, H)} \prod_{i=1}^4 ev_i^*(PD[\ell_i]) = 2$

5. Some results in Gromov-Witten theory

(1) GW(pt) reduces to computing intersection numbers of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. Two Fields medals of Witten and Kontsevich, also Mirzakhani. Exhibits KdV hierarchy
(2) GW(curve), Okounkov-Pandharipande, Today hierarchy
(3) GW(surfaces), have results for K3, del Pezzo, GW/SW duality.
(4) GW($\mathbb{CP}^2$). What is N_d =number of genus 0 curves of degree d passing through $3d - 1$ points in $\mathbb{CP}^2$? 1,1,12 for degrees 1,2,3. Kontsevich recursion formula,

$$N_d = \sum_{d_1 + d_2 = d} N_{d_1} N_{d_2} d_1^2 d_2 (d_2 \binom{3d-4}{3d_1 - 2} - d_1 \binom{3d-4}{3d_1 - 1})$$

Formula is equivalent to the WDVV equations in $g = 0$ quantum cohomology. Ordinary cup product counts number of point intersections. Deformed quantum cup product counts number of sphere intersections. Next few numbers are $N_4 = 620, N_5 = 87304, N_6 = 26312976$.
(5) GW(3-folds). Calabi-Yau 3-folds X, GW/DT correspondence, mirror symmetry as duality between Type IIA and Type IIB string theory. Idea is to compute $N_0(X, \beta)$ by computing $\int_{\check{X}} \Omega$, where $\check{X}$ is the "mirror" space.

Example 5.1. Quintic threefold $X = \{x^5 + y^5 + z^5 + w^5 + v^5 = 0\} \subseteq \mathbb{CP}^4$. Griffiths-Dwork mirror $\check{X} = \{t(x^5 + y^5 + z^5 + w^5 + v^5) + xyzwv = 0 | t \in \mathbb{P}^1\}$. Numbers are $N_0(X,1) = 2875, N_0(X,2) = 609250, N_0(X,3) = 317206375, N_0(X,4) = 242467530000$.

6. Gromov-Witten theory of an elliptic curve

Let $E \subseteq \mathbb{CP}^2$ be an elliptic curve. It is given by a cubic equation.

(1) $y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1$ is a smooth elliptic curve
(2) $y^2 = x^3$, cusp
(3) $y^2 = x^2(x-1)$, nodal elliptic curve.

Over $\mathbb{C}$, every elliptic curve is isomorphic to a complex torus, i.e. $E \cong \mathbb{C}/\Lambda$ where $\Lambda = \{\mathbb{Z} + \tau\mathbb{Z} | \tau \in \mathbb{H}\}$. The moduli of elliptic curves is isomorphic to $\mathbb{H}/PSL_2(\mathbb{Z})$.

Question: Given $E = \mathbb{C}/\Lambda$, what is N_d= the number of degree d covers of E by E? The answer is $N_d = \#\{\Gamma' \subset \Gamma | |\Gamma/\Gamma'| = d\} = \sigma(d) = \sum_{n|d} n$. From physics, we have the following,

Theorem 6.1. [Djik] $\sum_{d \geq 1} \frac{N_d}{d} q^d = \frac{-1}{24} \log \Delta(q) + \frac{1}{24} \log q$, *where* $q = e^{2\pi i \tau}$.

where $\Delta(q) = q \prod_{m=1}^{\infty} (1 - q^m)^{24}$ is the modular discriminant of weight 12.

Definition 6.2. A holomorphic map $f : \mathbb{H} \to \mathbb{C}$ is a weight k modular form if $f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^k f(\tau)$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Taking $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that $f(\tau + 1) = f(\tau)$. So $f(\tau)$ admits a Fourier expansion $f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n \tau}$.

The Gromov-Witten theory of the elliptic curve was solved by Okounkov-Pandharipande. The Hurwitz numbers are $N_d = \int_{\overline{\mathcal{M}}_{1,0}(E,d)} 1$ as a Gromov-Witten invariant. They showed that Gromov-Witten generating functions have modularity properties by using the GW/Hurwitz correspondence, and previous work computing Hurwitz numbers with the character of ∞-wedge representation of $\mathfrak{gl}_\infty$.

6.1. **Brief detour on Hurwitz theory.**

The data required to define a Hurwitz number are a degree d cover $f : X \to Y$, a finite set of branch points in Y, and ramification profiles (partition of d) above them.

Degree d covers of E by E are unbranched. Example of computing $H_{0 \xrightarrow{3} 0}((3), (2,1)^2)$, in terms of monodromy representations. The formula is

$$H_{g(X) \xrightarrow{d} g(Y)}(\lambda_1, \ldots, \lambda_n) = \frac{|M|}{d!}$$

where $|M|$ is the number of monodromy representations, λ_i are ramification profiles of the branch points. Given a branch point b_j, take a loop γ_j in a small neighborhood U_j

of b_j. In U_j, the covering map looks like $z \mapsto z^{k_j}$ where k_j is a the ramification index. The loop γ_j will lift to a cycle in S_d of type given by the ramification profile λ_j.

Anyways, Okounkov-Pandharipande showed that,

Theorem 6.3. *$F_g \in \mathbb{Q}[E_2, E_4, E_6]$ for all $g \geq 0$.*

where E_k is the weight k Eisenstein series $E_k(q) = \frac{\zeta(1-k)}{2} + \sum_n (\sum_{d|n} d^{k-1}) q^n$. The latter ring is called the ring of quasimodular forms.

7. Gromov-Witten theory of a K3 surface

Let S be a K3 surface, i.e. a compact complex surface, $\pi_1(S) = 0$ and $K_S \cong \mathcal{O}_S$. K3 surfaces are the only Calabi-Yau surfaces.

Example 7.1. Take the hypersurface $\{x^4 + y^4 + z^4 + w^4 = 0\} \subseteq \mathbb{CP}^3$. By adjunction, its canonical bundle is trivial. This is the only K3 surface up to diffeomorphism.

The cohomology of S is $H^0 \cong \mathbb{Z}, H^2 \cong \mathbb{Z}^{22}, H^4 \cong \mathbb{Z}$, which implies $\chi(S) = 24$. The intersection form partly contains the E8 lattice.

7.1. **Yau-Zaslow conjecture (1995).**

Suppose that $L \to S$ is a primitive line bundle such that $c_1(L)^2 = 2g - 2$. The linear system $|L| = \mathbb{P}(H^0(S, L))$ of L is of dimension g and hence isomorphic to $\mathbb{P}^g$, and a generic curve in $|L|$ has genus g. Hence, the space of rational curves w/ g nodes is of dimension 0 in $|L|$, as a node introduces one extra linear constraint. Define $n_g = \#$ of rational curves in $|L|$ with $c_1(L) = 2g - 2$. Yau-Zaslow predicted using the duality between Type II on K3 and heterotic on T^4 that,

$$\begin{aligned}\sum_{g \geq 0} n_g q^g &= \prod_{m=1}^{\infty} (1 - q^m)^{-24} \\ &= \frac{q}{\Delta} \\ &= 1 + 24q + 324q^2 + 3200q^3 + 25659q^4 + \dots\end{aligned}$$

The numbers in the series expansion have geometric interpretations: Suppose that $S \to \mathbb{P}^1$ is an elliptic fibration. Since $\chi(S) = 24$, the fibration has 24 1-nodal fibers, by the motivic property of the Euler characteristic and $\chi(E) = 0$.

Consider a double cover of $\mathbb{P}^2 \to \mathbb{P}^2$ branched along a smooth sextic curve. This surface is K3 by the Rieman-Hurwitz formula $K_X = f^* K_Y + R$. It has 324 2-nodal rational curves covering the 324 bitangents of the sextic curve. Similarly the number 3200 is the number of tri-tangent planes a quartic surface has in 3-space.

7.1.1. *Proof 1 without using GW invariants.* Consider the compactified Jacobian $J \to |L|$. The fiber above a curve C is $\{(C, L) | \deg L = 0\}$, the moduli space of degree 0 line bundles on C. In fact, J is smooth and hyperkahler as a moduli space of sheaves, and J is birational to $S^{[g]}$, the Hilbert scheme of g points on S. Gottsche used the Weil conjectures to show that

$$\sum_{g\geq 0} e(S^{[g]})q^g = \frac{q}{\Delta}$$

Batyrev tells us that birational CYs have the same Betti numbers, hence $\sum_{g\geq 0} e(J)q^g = \frac{q}{\Delta}$. One can show that $e(J) = \sum_{g(C)=0} e(J(C)) = \sum_{g(C)=0} +1$, if $[C]$ is primitive in $H_2(S)$. This is n_g the count of $g = 0$ curves. This shows that $\sum_{g\geq 0} n_g q^g = \frac{q}{\Delta}$.

7.1.2. *Proof 2 without using GW invariants.* The second proof by Bryan-Leung uses family Gromov-Witten invariants. Deformation of complex structure of the target does not change the GW invariant, and in fact one can always deform to a complex structure of the K3 such that the GW invariant is 0 (one can compute $\overline{\mathcal{M}}_{g=0}(S, \beta) = -1$). However S is hyperkahler, so it has an S^2-family of Kahler structures. Use this 1 parameter family to define family Gromov-Witten invariants (the expected dimension goes from -1 to 0).

We also have the following theorem,

Theorem 7.2. *(Maulik-Pandharipande-Thomas) Descendant theory of K3 is in $\frac{1}{\Delta(q)}QMod_{\leq 2g+2r}$, where r is the number of insertions. Proof uses vanishing of tautological cohomology of moduli of curves, and uses descendant theory of the elliptic curve.*

8. Conjectures for $S \times E$

Turn to Calabi-Yau 3-folds. There is not a single case of CY3 with exact solutions for every genus. Generalization of $SL_2(\mathbb{Z})$-modular forms via Siegel modular forms. Genus g curve classes are of the form $[C] = (\beta, d) \in H_2(S) \times H_2(E)$ such that $\beta^2 = 2h - 2$. Define a generating series summing over g, h, d.

$$\sum_{g,h,d} N_{g,h,d} = \frac{1}{\chi_{10}(\Omega)}$$

Related to heterotic duality, black hole counts, Katz-Klemm-Vafa. The RHS is the Igusa cusp form - it's a weight 10 Siegel modular form.

9. Local $\mathbb{P}^2$

Coates proves quasi-modularity of generating functions, Jie Zhou.

10. Quintic threefold

$X_5 \subset \mathbb{CP}^4$. Example of a Calabi-Yau 3-fold. Compute $N_{g,d}(X_5)$. Schubert found $N_{0,1}(X_5) = 2875 =$ lines on X_5 using Schubert calculus.

CdOGP '91 computed $F_0(q) = \sum_d N_{0,d}(X_5)q^d$ in physics as an explicit formula. Later, Givental, Lian-Liu-Yau proved the mirror conjecture, and proved this formula.

BCOV in '93 computed $F_1(q)$ in physics. Zinger, Li, Vakil, Zagier proved this in '07. BCOV in '93 computed $F_2(q)$ in physics. Janda-Ruan-Chen-Guo prove genus 2 conjecture. What about the structure of $F_g(q)$? Modularity, Holomorphic-Anomaly equations. Together with Castelnuovo bound, can compute $F_g(q)$ for all $g \leq 51$.

Computation of GW invariants of X_5 is based on the Quantum Lefschetz Principle, which determines the former via GW invariants of $\mathbb{P}^4$. The latter can be computed via Atiyah-Bott localization with its torus action. Can apply Lefschetz principle unless there exists f such that $H^1(f^*\mathcal{O}(5)) \neq 0$. Problem is if there exists $\eta \neq 0$ holomorphic differential on C vanishing with order 5 along each intersection of C w/ a fixed hyperplane. If C is smooth, then this only happnes if $5d \leq 2g - 2$. Define effective invariants.

References

[Djik] Mirror Symmetry and Elliptic Curves

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