SUPPLEMENT ON EIGENVALUES AND EIGENVECTORS

We give some extra material on repeated eigenvalues and complex eigenvalues.

1. Repeated Eigenvalues and Generalized Eigenvectors

For repeated eigenvalues, it is not always the case that there are enough eigenvectors.

Let $\mathbf{A}$ be an $n \times n$ real matrix, with characteristic polynomial

$$p_{\mathbf{A}}(\lambda) = (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k}$$

with $\lambda_j \neq \lambda_\ell$ for $j \neq \ell$. Use the following notation for the eigenspace,

$$\mathbb{E}(\lambda_j) = \{\mathbf{v} : (\mathbf{A} - \lambda_j \mathbf{I})\mathbf{v} = \mathbf{0}\}.$$

We also define the *generalized eigenspace* for the eigenvalue λ_j by

$$\mathbb{E}^{\text{gen}}(\lambda_j) = \{\mathbf{w} : (\mathbf{A} - \lambda_j \mathbf{I})^{m_j}\mathbf{w} = \mathbf{0}\},$$

where m_j is the multiplicity of the eigenvalue. A vector in $\mathbb{E}(\lambda_j)$ is called a *generalized eigenvector*.

The following is a extension of theorem 7 in the book.

Theorem (7′). *Let $\mathbf{A}$ be an $n \times n$ matrix with characteristic polynomial $p_{\mathbf{A}}(\lambda) = (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k}$, where $\lambda_j \neq \lambda_\ell$ for $j \neq \ell$. Then, the following hold.*

*(**a**) $\dim(\mathbf{E}(\lambda_j)) \leq m_j$ and $\dim(\mathbb{E}^{gen}(\lambda_j)) = m_j$ for $1 \leq j \leq k$. If λ_j is complex, then these dimensions are as subspaces of $\mathbb{C}^n$.*

*(**b**) If $\mathscr{B}_j$ is a basis for $\mathbb{E}^{gen}(\lambda_j)$ for $1 \leq j \leq k$, then $\mathscr{B}_1 \cup \cdots \cup \mathscr{B}_k$ is a basis of $\mathbb{C}^n$, i.e., there is always a basis of generalized eigenvectors for all the eigenvalues. If the eigenvalues are all real all the vectors are real, then this gives a basis of $\mathbb{R}^n$.*

*(**c**) Assume $\mathbf{A}$ is a real matrix and all its eigenvalues are real. Then, the matrix $\mathbf{A}$ is diagonalizable iff $\dim(\mathbf{E}(\lambda_j)) = m_j$ for all $1 \leq j \leq k$.*

Notice that if $\mathbf{v}$ is an eigenvector for λ_j and

$$(\mathbf{A} - \lambda_j \mathbf{I})\mathbf{w} = \mathbf{v},$$

then

$$(\mathbf{A} - \lambda_j \mathbf{I})^2\mathbf{w} = (\mathbf{A} - \lambda_j \mathbf{I})\mathbf{v} = \mathbf{0}.$$

For such a generalized eigenvector, $\mathbf{A}\mathbf{w} = \lambda_j \mathbf{w} + \mathbf{v}$, so $\mathbf{A}\mathbf{w}$ is a scalar multiple of $\mathbf{w}$ plus the eigenvector $\mathbf{v}$. If there are not enough eigenvectors, then once we have solved for the eigenvector $\mathbf{v}$, then we can solve for a generalized eigenvector by solving the nonhomogeneous equation $(\mathbf{A} - \lambda_j \mathbf{I})\mathbf{w} = \mathbf{v}$.

Example 1. Let $\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix}$. The eigenvalues are -1, -1, and -2. For $\lambda = -2$, an eigenvector is $(0, 1, 1)^T$.

Now take $\lambda = -1$.

$$(\mathbf{A}+\mathbf{I}) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, there is only one independent eigenvector, which can be take to be $(1, 1, 0)^T$.

To find another generalized eigenvector for $\lambda = -1$, we solve the following nonhomogeneous equation $(\mathbf{A}+\mathbf{I})\mathbf{w} = \mathbf{v}$ by considering the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & -2 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the solution is $w_1 = w_2$ and $w_3 = 1$, or $\mathbf{w} = w_2(1, 1, 0)^T + (0, 0, 1)^T = w_2\mathbf{v} + (0, 0, 1)^T$. Notice that the solution involves an arbitrary multiple of the eigenvector $\mathbf{v}$: this is always the case. We take $w_2 = 0$ and get $\mathbf{w} = (0, 0, 1)^T$ as the generalized eigenvector.

There is not a basis of just eigenvectors, but we have a basis for $\mathbb{R}^3$ of eigenvectors and generalized eigenvectors: $(0, 1, 1)^T$ for $\lambda = -2$, and $(1, 1, 0)^T$ and $(0, 0, 1)^T$ for $\lambda = -1$. If we let the matrix $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then

$$\mathbf{AP} = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -2 & -1 & 1 \\ -2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \mathbf{PB}.$$

Thus, conjugation by $\mathbf{P}$ changes $\mathbf{A}$ to $\mathbf{B} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, which is an upper triangular matrix in a very simple form, but not a diagonal matrix.

We consider an arbitrary real generalized eigenvector. Assume that λ_j is a real eigenvalue of multiplicity $m_j > 1$. Assume that $\mathbf{v}^{(r)}$ is a generalized eigenvector with

$$(\mathbf{A} - \lambda_j \mathbf{I})^r \mathbf{v}^{(r)} = \mathbf{0} \qquad \text{but}$$
$$(\mathbf{A} - \lambda_j \mathbf{I})^{r-1} \mathbf{v}^{(r)} \neq \mathbf{0},$$

for some $1 < r \leq m_j$. Setting

$$\mathbf{v}^{(r-\ell)} = (\mathbf{A} - \lambda_j \mathbf{I})^{\ell} \mathbf{v}^{(r)} \qquad \text{for } \ \ell = 1, \ldots, r-1,$$

we get

$$\begin{aligned}(\mathbf{A}-\lambda_j\mathbf{I})\mathbf{v}^{(r)} &= \mathbf{v}^{(r-1)},\\ (\mathbf{A}-\lambda_j\mathbf{I})\mathbf{v}^{(r-1)} &= \mathbf{v}^{(r-2)},\\ \vdots \qquad & \qquad \vdots\\ (\mathbf{A}-\lambda_j\mathbf{I})\mathbf{v}^{(2)} &= \mathbf{v}^{(1)}, \qquad \text{and}\\ (\mathbf{A}-\lambda_j\mathbf{I})\mathbf{v}^{(1)} &= \mathbf{0},\end{aligned}$$

or

$$\begin{aligned}\mathbf{A}\mathbf{v}^{(r)} &= \lambda_j\mathbf{v}^{(r)}+\mathbf{v}^{(r-1)},\\ \mathbf{A}\mathbf{v}^{(r-1)} &= \lambda_j\mathbf{v}^{(r-1)}+\mathbf{v}^{(r-2)},\\ \vdots \quad & \quad \vdots\\ \mathbf{A}\mathbf{v}^{(2)} &= \lambda_j\mathbf{v}^{(2)}+\mathbf{v}^{(1)}, \qquad \text{and}\\ \mathbf{A}\mathbf{v}^{(1)} &= \lambda_j\mathbf{v}^{(1)}.\end{aligned}$$

The matrix $\mathbf{A} = \begin{bmatrix} a & 1 & 0\\ 0 & a & 1\\ 0 & 0 & a\end{bmatrix}$ has a as an eigenvalue of multiplicity 3, but only one eigenvector. The vector $\mathbf{e}^3 = (0,0,1)^T$, has $(\mathbf{A}-a\mathbf{I})^3 = \mathbf{0}$ but $(\mathbf{A}-a\mathbf{I})^2 \neq \mathbf{0}$: $\mathbf{A}\mathbf{e}^3 = a\mathbf{e}^3+\mathbf{e}^2$, $\mathbf{A}\mathbf{e}^2 = a\mathbf{e}^2+\mathbf{e}^1$, and $\mathbf{A}\mathbf{e}^1 = a\mathbf{e}^1$.

2. Complex Eigenvalues

We give an example of finding a complex eigenvector for a 3×3 matrix by row reduction.

Example 2. Let $\mathbf{A} = \begin{bmatrix} -3 & 0 & 2\\ 1 & -1 & 0\\ -2 & -1 & 0\end{bmatrix}$. The characteristic equation is $0 = -\lambda^3 - 4\lambda^2 - 7\lambda + 6$, which has one real eigenvalue of $\lambda = -2$. By performing synthetic division, we get that $0 = -(\lambda+2)(\lambda^2+2\lambda+3)$. Using the quadratic formula, we get the other eigenvalues are $\lambda = -1 \pm \sqrt{2}i$.

Taking $\lambda = -1+\sqrt{2}i$, we need to row reduce the following matrix:

$$\mathbf{A} - (-1+\sqrt{2}\,i) = \begin{bmatrix} -2-\sqrt{2}\,i & 0 & 2\\ 1 & -\sqrt{2}\,i & 0\\ -2 & -1 & 1-\sqrt{2}\,i\end{bmatrix}$$

multiplying row 1 by $-2+\sqrt{2}\,i$

$$\sim \begin{bmatrix} 6 & 0 & -4+2\sqrt{2}\,i\\ 1 & -\sqrt{2}\,i & 0\\ -2 & -1 & 1-\sqrt{2}\,i\end{bmatrix}$$

interchanging rows 1 & 2 and dividing the new row 2 by 2

$$\sim \begin{bmatrix} 1 & -\sqrt{2}\,i & 0\\ 3 & 0 & -2+\sqrt{2}\,i\\ -2 & -1 & 1-\sqrt{2}\,i\end{bmatrix}$$

$$\text{clearing column 1}$$

$$\mathbf{A} - (\text{-}1 + \sqrt{2}\,i) \sim \begin{bmatrix} 1 & \text{-}\sqrt{2}\,i & 0 \\ 0 & 3\sqrt{2}\,i & -2+\sqrt{2}\,i \\ 0 & \text{-}1 - 2\sqrt{2}\,i & 1 - \sqrt{2}\,i \end{bmatrix}$$

$$\text{multiplying row 2 by } -\sqrt{2}\,i \text{ and row 3 by -1} + 2\sqrt{2}\,i$$

$$\sim \begin{bmatrix} 1 & \text{-}\sqrt{2}\,i & 0 \\ 0 & 6 & 2+2\sqrt{2}\,i \\ 0 & 9 & 3+3\sqrt{2}\,i \end{bmatrix}$$

$$\text{dividing row 2 by 2 and eliminating row 3}$$

$$\sim \begin{bmatrix} 1 & \text{-}\sqrt{2}\,i & 0 \\ 0 & 3 & 1+\sqrt{2}\,i \\ 0 & 0 & 0 \end{bmatrix}.$$

These give us the equations $v_1 = \sqrt{2}\,i\,v_2$ and $(1+\sqrt{2}\,i)v_3 = \text{-}3v_2$, so we can get the solutions $v_3 = 3$, $v_2 = \text{-}1 - \sqrt{2}\,i$, and $v_1 = \sqrt{2}\,i(\text{-}1 - \sqrt{2}\,i) = 2 - \sqrt{2}\,i$:

$$\mathbf{v} = \begin{bmatrix} 2 \\ \text{-}1 \\ 3 \end{bmatrix} - i \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}.$$

The eigenvector for $\lambda = \text{-}1 - \sqrt{2}i$ is the complex conjugate

$$\begin{bmatrix} 2 \\ \text{-}1 \\ 3 \end{bmatrix} + i \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}.$$

In general, if $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ is an eigenvector for the complex eigenvalue $\lambda = a + ib$ for a real matrix $\mathbf{A}$, then

$$\begin{aligned} \mathbf{A}(\mathbf{u} + i\mathbf{w}) &= \mathbf{Au} + i\mathbf{Aw} \qquad \text{(by linearity of matrix multiplication)} \\ &= (a+ib)(\mathbf{u} + i\mathbf{w}) \qquad \text{(because it is an eigenvector)} \\ &= (a\mathbf{u} - b\mathbf{w}) + i(b\mathbf{u} + a\mathbf{w}). \end{aligned}$$

Equating the real and imaginary parts,

$$\mathbf{Au} = a\mathbf{u} - b\mathbf{w}$$
$$\mathbf{Aw} = b\mathbf{u} + a\mathbf{w}.$$

In two dimensions, we have the following theorem.

Theorem (9). *Let* $\mathbf{A}$ *be a* 2×2 *real matrix with complex eigenvalue* $\lambda = a + ib$, $b \neq 0$, *with corresponding eigenvector* $\mathbf{v} = \mathbf{u} + i\mathbf{w}$. *Let* $\mathbf{P}$ *be the matrix with columns* $\mathbf{u}$ *and* $\mathbf{w}$. *Then* $\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} a & b \\ \text{-}b & a \end{bmatrix}$.

If $r = \sqrt{a^2 + b^2}$, $a = r\cos(\phi)$, $\text{-}b = r\sin(\phi)$, then $\begin{bmatrix} a & b \\ \text{-}b & a \end{bmatrix}$ is a rotation by ϕ and an expansion (or contraction) by r.