FUNCTIONS IN ECONOMICS

Cost Function

Assume that a firm must pay P_E for each unit of equipment and P_L for each unit of labor. If E units of equipment and L units of labor are used, then the cost is $C = P_E E + P_L L$. The cost can be thought of as a (linear) function of E and L.

In general, if the prices are p_i for $i = 1, \dots, n$ with price vector $\mathbf{p} = \langle p_1, \dots, p_n \rangle$, and $\mathbf{q} = \langle q_1, \dots, q_n \rangle$ is the commodity bundle of the quantities of the goods, then the cost of the commodity bundle is $C = \mathbf{p} \cdot \mathbf{q} = p_1 q_1 + \cdots + p_n q_n$.

Revenue

The revenue is the total amount of money earned by selling a commodity or commodities. If a firm produces one commodity in a quantity q and can sell it for a price p, then the revenue is $r = pq$. If a firm produces two commodities in quantities q_1 and q_2 respectively and can sell them for prices p_1 and p_2 respectively, then the revenue is $r = p_1 q_1 + p_2 q_2$.

Profit

A firm's profit π is the amount earned as revenue minus the cost of producing the goods, $\pi = r - C$. Assume the there are two goods which are sold in quantities q_1 and q_2 for prices p_1 and p_2 respectively. For the costs, assume there is a fixed cost of C_0 and the items cost p'_1 and p'_2 per unit. The cost of producing them is then $C = C_0 + p'_1 q_1 + p'_2 q_2$. Thus, the profit is $\pi = p_1 q_1 + p_2 q_2 - (C_0 + p'_1 q_1 + p'_2 q_2) = (p_1 - p'_1) q_1 + (p_2 - p'_2) q_2 - C_0$.

Utility function

A fundamental problem in consumer theory deals with a single agent who consumes n commodities. Assume $q_i \geq 0$ is the amount of the i^{th} commodity. The utility function is then a real valued function $U(q_1, \dots, q_n)$ which depends on the amounts of the commodities held. It indicates the preferences for the agent. Thus, if $U(q_1, \dots, q_n) > U(q'_1, \dots, q'_n)$ then the agent prefers to have the amounts $(q_1, \dots, q_n)$ of the commodities to the amounts $(q'_1, \dots, q'_n)$. If $U(q_1, \dots, q_n) = U(q'_1, \dots, q'_n)$, then the agent likes the two holdings equally well, i.e., the agent is indifferent to the two holdings. The set of all the points where $U(q_1, \dots, q_n) = C$ corresponds to all holdings that the person values equally. This latter set is the *indifference set* that we call *level sets* (level curves or level surfaces).

For two commodities, a typical function is $U(q_1, q_2) = q_1 q_2$. A person with this utility function values holding some of each good more than a lot of the first and little of the second good. The indifference set is just $q_1 q_2 = C$ or $q_2 = {}^{C}\!/_{q_1}$, which is a "hyperbola". (Draw the level curves for $C = 0.5$, 1, and 2.)

A more general form of the utility function is $U(q_1, q_2) = q_1^{a_1} q_2^{a_2}$, where $a_1, a_2 > 0$ are fixed constants. For more commodities, this generalized form would be $U(q_1, \dots, q_m) = q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}$.

Production Function

Another situation where a real valued function of several variable arises in economics is the production function from the theory of a firm. Assume that $\langle x_1, \dots, x_n \rangle$ is the bundle of goods used to produce a single product. Let q be the amount of the output product and $q = f(x_1, \dots, x_n)$ the function that gives the amount produced in terms of the amount of the input goods. This function is called the *production function*.

There are various forms of the production function that have been used.

Linear: $q = a_1 x_1 + a_2 x_2$.

Input-output:

$$q = \min \left\{ {}^{x_1}/_{c_1}, {}^{x_2}/_{c_2} \right\}.$$

This means the producer needs $x_i \geq c_i q$ for each i to produce q amount of the output.

Cobb-Douglas:

$$q = f(x_1, x_2) = A\, x_1^a\, x_2^b,$$

where $A > 0$, $a > 0$, and $b > 0$ are constants We discuss below how this function scales with increasing amounts of inputs.

Constant elasticity of substitution:

$$q = k \left[c_1 x_1^{-a} + c_2 x_2^{-a} \right]^{-b/a}.$$

Cobb-Douglas Production Function

If the two inputs are labor and capital then the quantities are often designed by L for labor and K for capital. In this case, the production function is

$$q = f(L, K) = A\, L^a\, K^b$$

where A is a constant. If both labor and capital are scaled by a factor of s, then

$$\begin{aligned} f(sL, sK) &= A\,(sL)^a\,(sK)^b \\ &= s^{a+b}\, A\, L^a\, K^b \\ &= s^{a+b}\, f(L, K) \end{aligned}$$

and the production increases by a factor of s^{a+b}.

Case 1: $a + b = 1$. Often, $0 < a < 1$ and $0 < b = 1 - a < 1$; for example,

$$q = f(L, K) = A\, L^{1/3}\, K^{2/3}.$$

In this case, if both labor and capital are doubled, then $f(2L, 2K) = 2\, f(L, K)$ and the production is doubled. In fact, if both labor and capital are scaled by a factor of s, then $f(sL, sK) = s\, f(L, K)$, and the production is scaled by the same factor of s. Thus, for this case, the function is said to have *constant returns to scale.*

Case 2: $a + b > 1$. For an increase in inputs by $s > 1$, $f(sL, sK) = s^{a+b}\, f(L, K) > s\, f(L, K)$, and the production is increased by a greater factor than the inputs; the function is said to have *increasing returns to scale.*

Case 3: $a + b < 1$. For an increase in inputs by $s > 1$, $f(sL, sK) = s^{a+b}\, f(L, K) < s\, f(L, K)$, and the production is increased by a smaller factor than the inputs; function is said to have *decreasing returns to scale.*

Level sets (equal production): If the amount of production q_0 is fixed, then the level set $q_0 = f(L, K)$ is the set of all combinations of labor and capital which produce this amount, and is called the *isoquant.* For the general Cobb-Douglas production function with $a > 0$ and $b > 0$, these points satisfy

$$\begin{aligned} q_0 &= A\, L^a\, K^b \\ K^b &= \frac{q_0}{A\, L^a} \\ K &= \left(\frac{q_0}{A} \right)^{1/b} \frac{1}{L^{a/b}}. \end{aligned}$$

Because $a/b > 0$, K goes to zero as L goes to infinity, and K goes to infinity as L goes to zero. Thus, the general shape of the level set is like $xy = 1$.