

ECONOMIC APPLICATIONS OF IMPLICIT DIFFERENTIATION

1. Substitution of Inputs

Let $Q = F(L, K)$ be the production function in terms of labor and capital. Consider the isoquant $Q_0 = F(L, K)$ of equal production. (This is the level curve of the function.) Thinking of K as a function of L along the isoquant and using the chain rule, we get

$$0 = \frac{\partial Q}{\partial L} + \frac{\partial Q}{\partial K} \frac{\partial K}{\partial L}.$$

The derivative $\frac{\partial K}{\partial L}$ measures the rate of substitution of capital for labor needed to keep the output constant.

For the Cobb-Douglas production function $Q = cL^a K^b$. If the quantity produced Q is fixed and K is considered a function of L , then differentiating

$$\begin{aligned} 0 &= \frac{\partial}{\partial L} (cbL^a K^b) \\ &= acL^{a-1} K^b + bcL^a K^{b-1} \frac{\partial K}{\partial L}, \quad \text{so} \\ \frac{\partial K}{\partial L} &= -\frac{acL^{a-1} K^b}{bcL^a K^{b-1}} \\ &= -\left(\frac{a}{b}\right) \frac{K}{L}. \end{aligned}$$

Thus, the marginal rate of technical substitution of capital for labor is a constant multiple of the average of capital to labor.

2. Comparative-Static Analysis of Balancing Supply and Demand

Assume the supply and demand functions for a single commodity are

$$\begin{aligned} Q &= S(P, T) && \text{(supply)} \quad \text{and} \\ Q &= D(P, Y) && \text{(demand)}, \end{aligned}$$

where Y is the income, T is the tax on the commodity, and P is the price. We do not assume a particular form of the supply and demand functions, but we do assume that the supply increases with price and decreases with tax while the demand decreases with price and increases with income: Assume that

$$\begin{aligned} S_P &= \frac{\partial S}{\partial P} > 0, & S_T &= \frac{\partial S}{\partial T} < 0, \\ D_P &= \frac{\partial D}{\partial P} < 0, & D_Y &= \frac{\partial D}{\partial Y} > 0. \end{aligned}$$

At an equilibrium state, supply and demand are balanced, so

$$0 = S(P, T) - D(P, Y).$$

The partial derivative of the right hand side with respect to P is $S_P - D_P > 0$, so this equation determines the price P as a function of income Y and tax T while the system stays in an equilibrium state. Taking the

partial derivatives with respect to Y and T , we get

$$\begin{aligned} 0 &= S_P \frac{\partial P}{\partial Y} - D_P \frac{\partial P}{\partial Y} - D_Y \quad \text{and} \\ 0 &= S_P \frac{\partial P}{\partial T} + S_T - D_P \frac{\partial P}{\partial T}, \end{aligned}$$

so

$$\begin{aligned} (S_P - D_P) \frac{\partial P}{\partial Y} &= D_Y \quad \text{and} \\ (S_P - D_P) \frac{\partial P}{\partial T} &= -S_T, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial P}{\partial Y} &= \frac{D_Y}{S_P - D_P} > 0 \quad \text{and} \\ \frac{\partial P}{\partial T} &= \frac{-S_T}{S_P - D_P} > 0. \end{aligned}$$

Thus, the price increases with both an increase in income and tax.

3. General Method of Implicit Differentiation

Following the book's treatment of the general implicit function theorem, assume that there are $m > 0$ equations in $n + m$ variables with $n > 0$,

$$\begin{aligned} F_1(x_1, \dots, x_n, y_1, \dots, y_m) &= C_1 \\ F_2(x_1, \dots, x_n, y_1, \dots, y_m) &= C_2 \\ &\vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_m) &= C_m \end{aligned}$$

such that at $\mathbf{p} = (\mathbf{a}, \mathbf{b})$

$$\det \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(\mathbf{p}) & \cdots & \frac{\partial F_1}{\partial y_m}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1}(\mathbf{p}) & \cdots & \frac{\partial F_m}{\partial y_m}(\mathbf{p}) \end{pmatrix} \neq 0.$$

By the implicit function theorem, there is a "implicitly defined function" $\mathbf{y} = h(\mathbf{x})$ such that $\mathbf{C} = F(\mathbf{x}, h(\mathbf{x}))$ for all \mathbf{x} near \mathbf{a} . Thus, the (y_1, \dots, y_m) are the dependent variables and (x_1, \dots, x_n) are the independent variables. Let $H(\mathbf{x}) = (\mathbf{x}, h(\mathbf{x}))$, so $\mathbf{C} = F(H(\mathbf{x}))$. Differentiating this equation with respect to \mathbf{x} and using the chain rule, we get the matrix equation

$$\begin{aligned} \mathbf{0} &= DF(\mathbf{p})DH(\mathbf{a}) = [D_{\mathbf{x}}F(\mathbf{p}), D_{\mathbf{y}}F(\mathbf{p})] \begin{bmatrix} \mathbf{I} \\ Dh(\mathbf{a}) \end{bmatrix} \\ &= D_{\mathbf{x}}F(\mathbf{p}) + D_{\mathbf{y}}F(\mathbf{p})Dh(\mathbf{a}) \quad \text{or} \\ (1) \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} &= \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial F_m}{\partial x_n}(\mathbf{p}) \end{pmatrix} + \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(\mathbf{p}) & \cdots & \frac{\partial F_1}{\partial y_m}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1}(\mathbf{p}) & \cdots & \frac{\partial F_m}{\partial y_m}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial y_1}{\partial x_n}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial y_m}{\partial x_n}(\mathbf{p}) \end{pmatrix}. \end{aligned}$$

Solving for the partial derivatives of the dependent variables and taking the inverse of the square matrix involving the partial derivatives of the F_i , we get

$$(2) \quad \begin{pmatrix} \frac{\partial y_1}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial y_1}{\partial x_n}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial y_m}{\partial x_n}(\mathbf{p}) \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(\mathbf{p}) & \cdots & \frac{\partial F_1}{\partial y_m}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1}(\mathbf{p}) & \cdots & \frac{\partial F_m}{\partial y_m}(\mathbf{p}) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial F_m}{\partial x_n}(\mathbf{p}) \end{pmatrix}.$$

In a given problem, we just write down the matrix equation (1) and solve them rather than use equation (2). The reason is that it is easier to remember the correct form for the matrix equation (1): The rows are determined by the coordinate functions (or equations) of the problem; the first matrix has the derivatives with respect to the independent variables; the second matrix has the derivatives with respect to the dependent variables and is multiplied by the unknown matrix of partial derivatives of the dependent variables with respect to the independent variables. Notice in the matrix product the term $\frac{\partial F_i}{\partial y_j}$ is multiplied by $\frac{\partial y_j}{\partial x_\ell}$ as the chain rules says it must be.

4. Marginal Inputs of Prices

A firm uses two inputs, q_1 and q_2 to produce a single output Q determined by a production function. Let p_1 be the price of q_1 , and p_2 be the price of q_2 ; let 1 be the price of the output Q . (Either the price of Q is taken as the unit of money or we are looking at the ratios of the prices p_j to the price of Q .) The profit is given by $\pi = Q - p_1 q_1 - p_2 q_2$. We will see in the section on maximization later in the quarter that the inputs that maximize profits satisfy $\frac{\partial \pi}{\partial q_1} = 0$ and $\frac{\partial \pi}{\partial q_2} = 0$. Calculating these partial derivatives, we get

$$\begin{aligned} 0 &= \frac{\partial \pi}{\partial q_1} = \frac{\partial Q}{\partial q_1} - p_1 \quad \text{and} \\ 0 &= \frac{\partial \pi}{\partial q_2} = \frac{\partial Q}{\partial q_2} - p_2. \end{aligned}$$

For the Cobb-Douglas production function $Q = q_1^{\frac{1}{3}} q_2^{\frac{1}{2}}$, these equations become

$$(3) \quad \begin{aligned} 0 &= \frac{1}{3} q_1^{-\frac{2}{3}} q_2^{\frac{1}{2}} - p_1 \\ 0 &= \frac{1}{2} q_1^{\frac{1}{3}} q_2^{-\frac{1}{2}} - p_2. \end{aligned}$$

We have two equations and four variables p_1 , p_2 , q_1 , and q_2 . If we specified values for the prices, then we would have two equations for the variables q_1 and q_2 which could determine their values.

To see that these two equations in (3) determine the amounts of inputs q_1 and q_2 that optimize profit in terms of the prices p_1 and p_2 , we check the determinant of the matrix of partial derivatives with respect to q_1 and q_2 :

$$\det \begin{bmatrix} -\frac{2}{9} q_1^{-\frac{5}{3}} q_2^{\frac{1}{2}} & \frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} \\ \frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} & -\frac{1}{4} q_1^{\frac{1}{3}} q_2^{-\frac{3}{2}} \end{bmatrix} = q_1^{-\frac{4}{3}} q_2^{-1} \left[\frac{2}{36} - \frac{1}{36} \right] = \frac{1}{36} q_1^{-\frac{4}{3}} q_2^{-1} \neq 0.$$

Therefore, the partial derivatives of q_1 and q_2 with respect to p_1 and p_2 satisfy the following matrix equation:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -\frac{2}{9} q_1^{-\frac{5}{3}} q_2^{\frac{1}{2}} & \frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} \\ \frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} & -\frac{1}{4} q_1^{\frac{1}{3}} q_2^{-\frac{3}{2}} \end{bmatrix} \begin{bmatrix} \frac{\partial q_1}{\partial p_1} & \frac{\partial q_1}{\partial p_2} \\ \frac{\partial q_2}{\partial p_1} & \frac{\partial q_2}{\partial p_2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial q_1}{\partial p_1} & \frac{\partial q_1}{\partial p_2} \\ \frac{\partial q_2}{\partial p_1} & \frac{\partial q_2}{\partial p_2} \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} q_1^{-\frac{5}{3}} q_2^{\frac{1}{2}} & \frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} \\ \frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} & -\frac{1}{4} q_1^{\frac{1}{3}} q_2^{-\frac{3}{2}} \end{bmatrix}^{-1}$$

$$= 36 q_1^{\frac{4}{3}} q_2 \begin{bmatrix} -\frac{1}{4} q_1^{\frac{1}{3}} q_2^{-\frac{3}{2}} & -\frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} \\ -\frac{1}{6} q_1^{-\frac{2}{3}} q_2^{-\frac{1}{2}} & -\frac{2}{9} q_1^{-\frac{5}{3}} q_2^{\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -9 q_1^{\frac{5}{3}} q_2^{-\frac{1}{2}} & -6 q_1^{\frac{2}{3}} q_2^{\frac{1}{2}} \\ -6 q_1^{\frac{2}{3}} q_2^{\frac{1}{2}} & -8 q_1^{-\frac{1}{3}} q_2^{\frac{3}{2}} \end{bmatrix},$$

and all the partial derivatives $\frac{\partial q_i}{\partial p_j}$ are negative.

Because the sum of the exponents in the production function is less than one, $\frac{1}{3} + \frac{1}{2} < 1$, the production has decreasing return to scale. Although it is not obvious from the above treatment, this property caused the partial derivatives to be negative and has the effect that the amounts of the inputs decreases with an increase in either price.

The details of this example are complicated. However, the outline of the method is not complicated.

(a) First, some equations are derived (or given) relating several variables. In our example, (3) has two equations with four variables. (b) Second, thinking of these equations as defining some variables in terms of the others, we take partial derivatives. In our case, we take the partial derivatives with respect to p_1 and p_2 .

5. Changing of Technology of Production

A firm uses two inputs to produce an output. Assume the amounts of the inputs are x and y with p the price of x and q the price of y . The amount produced Q is determined by the Cobb-Douglas production function $Q = x^a y^b$, where the technology determines the exponents a and b . The firm can vary the exponent b while keeping a fixed by changing the technology. It wants to keep the amount produced fixed, Q_0 , and the cost of the inputs fixed $px + qy = 125$. What is the rate of change of the amounts at $x = 5$, $y = 50$, $p = 5$, $q = 2$, $a = \frac{1}{3}$, and $b = \frac{2}{3}$?

Rather than use the equation $Q_0 = x^a y^b$, we take its logarithm and obtain the two equations

$$125 = px + qy \quad \text{and}$$

$$\ln(Q_0) = a \ln(x) + b \ln(y).$$

These two equations define x and y as functions of a , b , p , and q since

$$\det \begin{bmatrix} p & q \\ \frac{a}{x} & \frac{b}{y} \end{bmatrix} = \frac{pb}{y} - \frac{qa}{x} = \frac{pbx - qay}{xy} = \frac{5 \cdot 2 \cdot 5 - 2 \cdot 1 \cdot 50}{3 \cdot 5 \cdot 50} = \frac{-1}{15} \neq 0.$$

Considering x and y as functions of a , b , p , and q , and differentiating the two equations with respect to the four independent variables gives the following matrix equation:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & x & y \\ \ln(x) & \ln(y) & 0 & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ \frac{a}{x} & \frac{b}{y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{bmatrix} &= -\frac{xy}{pbx - qay} \begin{bmatrix} \frac{b}{y} & -q \\ \frac{-a}{x} & p \end{bmatrix} \begin{bmatrix} 0 & 0 & x & y \\ \ln(x) & \ln(y) & 0 & 0 \end{bmatrix} \\ &= \frac{xy}{qay - pbx} \begin{bmatrix} -q \ln(x) & -q \ln(y) & \frac{bx}{y} & b \\ p \ln(x) & p \ln(y) & -a & \frac{-ay}{x} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-xyq \ln(x)}{qay - pbx} & \frac{-xyq \ln(y)}{qay - pbx} & \frac{bx^2}{qay - pbx} & \frac{bxy}{qay - pbx} \\ \frac{xy p \ln(x)}{qay - pbx} & \frac{xy p \ln(y)}{qay - pbx} & \frac{-axy}{qay - pbx} & \frac{-ay^2 \ln(y)}{qay - pbx} \end{bmatrix}. \end{aligned}$$

At the point in question, $qay - pbx = \frac{100-50}{3} = \frac{50}{3}$ and

$$\begin{aligned} \frac{\partial y}{\partial a} &= \frac{3(25)(50) \ln(5)}{50} & \frac{\partial y}{\partial b} &= \frac{3(25)(50) \ln(50)}{50} \\ &= 75 \ln(5) & &= 75 \ln(50). \end{aligned}$$

Similarly calculations give the other partial derivatives.

6. Nonlinear Keynesian Model

This is a nonlinear Keynesian IS-LM model for national income. The details of solving for partial derivatives in terms of other is more complicated than the other ones we considered since it involve more variables and two equations that the system must satisfy. However, it does uses the general ideas of implicit differentiation as well as the chain rule.

The variables are as follows:

Y	Gross domestic product (GDP)
C	Consumption
I	Investment expenditure
G	Government spending
T	Taxes
r	Interest rate
M	Money supply

The GDP is assume to be the sum of the consumption, investment, and government spending, $Y = C + I + G$. The consumption is a function of after taxes income or the difference $Y - T$, $C = C(Y - T)$; the investment is a function of the interest rate, $I = I(r)$; and the money supply is a function (called the liquidity function) of GDP and the interest rate, $M = L(Y, r)$. The assumptions on the partial derivatives are that

$$0 < C'(x) < 1, \quad I'(r) < 0, \quad \frac{\partial L}{\partial Y} > 0, \quad \text{and} \quad \frac{\partial L}{\partial r} < 0.$$

Note that the partial derivative of C with respect to Y and T are $\frac{\partial C}{\partial Y} = C'$ and $\frac{\partial C}{\partial T} = -C'$.

We have obtained the following two functional relationships between the five variables:

$$(4) \quad \begin{aligned} 0 &= -Y + C(Y - T) + I(r) + G & \text{and} \\ 0 &= L(Y, r) - M \end{aligned}$$

We consider G , M , and T as the independent variables which can be controlled; C and I are intermediate variables (or functions); Y and r are dependent variables determined by G , M , and T through the variables C and I .

Taking the partial derivatives of the two equations with respect to Y and r is an invertible matrix,

$$\Delta = \det \begin{bmatrix} -1 + C' & I' \\ L_Y & L_r \end{bmatrix} = (C' - 1)L_r - I'L_Y > 0 :$$

The sign of Δ is positive because $(C' - 1) < 0$, $L_r < 0$, $(C' - 1)L_r > 0$, $I' < 0$, $L_Y > 0$, and $-I'L_Y > 0$. Therefore, these two equations define Y and r as functions of G , T , and M .

Taking the partial derivatives of the two equations with respect to G , T , and M (in that order), and considering Y and r as functions of these variables, we get the following matrix equation:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & -C' & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 + C' & I' \\ L_Y & L_r \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial G} & \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial M} \\ \frac{\partial r}{\partial G} & \frac{\partial r}{\partial T} & \frac{\partial r}{\partial M} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial Y}{\partial G} & \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial M} \\ \frac{\partial r}{\partial G} & \frac{\partial r}{\partial T} & \frac{\partial r}{\partial M} \end{bmatrix} &= \frac{1}{\Delta} \begin{bmatrix} L_r & -I' \\ -L_Y & C' - 1 \end{bmatrix} \begin{bmatrix} -1 & C' & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} -L_r & L_r C' & -I' \\ L_Y & -L_Y C' & C' - 1 \end{bmatrix}. \end{aligned}$$

The partial derivatives of Y and r with respect to G are positive:

$$\frac{\partial Y}{\partial G} = \frac{-L_r}{\Delta} > 0 \quad \text{and} \quad \frac{\partial r}{\partial G} = \frac{L_Y}{\Delta} > 0.$$

Thus, both the GDP, Y , and the interest rate, r , increase if government spending is increased while holding the money supply and taxes fixed. The partial derivatives of Y and r with respect to T are negative, and both Y and r decrease with increased taxes:

$$\frac{\partial Y}{\partial T} = \frac{L_r C'}{\Delta} < 0 \quad \text{and} \quad \frac{\partial r}{\partial T} = \frac{-L_Y C'}{\Delta} < 0,$$

Finally,

$$\frac{\partial Y}{\partial M} = \frac{-I'}{\Delta} > 0 \quad \text{and} \quad \frac{\partial r}{\partial M} = \frac{C' - 1}{\Delta} < 0,$$

and Y increases and r decreases with an increased money supply.

This is the type of qualitative information that this type of analysis can give without knowing the particular form of the functional dependence but only the signs of the rates of change.