LINEARLY CONSTRAINED QUADRATIC FORMS
AND BORDERED MATRICES

In economics, there are many situations where the variables are subject to a constraint. One example is the following. Assume the wealth (total value) is fixed as $w > 0$ and there are n goods with prices fixed as $p_i > 0$. Assume the commodity bundles $(x_1, \dots, x_n)$ are restricted to those that satisfy $p_1 x_1 + \cdots + p_n x_n = w$. Somethings, we then want to maximize another quantify, such as utility, $U(x_1, \dots, x_n)$.

When we study multidimensional calculus we will consider the method of Lagrange multipliers. In this handout, we give the linear algebra necessary to state a second derivative test for the method of Lagrange multipliers. In this context, we need to consider maximizing a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ on a linear subspace given by $\mathbf{B}\mathbf{x} = \mathbf{0}$. Here, $\mathbf{B}$ can have k rows, which corresponds to k linear constraints.

1. Quadratic Forms with Two Variables and One Linear constraint

Before considering the general case, we consider the case of two variable and one linear constraint; $Q(x, y) = a_{1,1}\, x^2 + 2a_{1,2}\, xy + a_{2,2}\, y^2$ on the points satisfying the linear constraint $b_1\, x + b_2\, y = 0$ with $(b_1, b_2) \neq \mathbf{0}$. The points on the linear constraint can be parameterized by $(x, y) = t(b_2, -b_1)$. Then,

$$\begin{aligned} Q(tb_2, \text{-}tb_1) &= a_{1,1}\,(tb_2)^2 + 2a_{1,2}(tb_2)(\text{-}tb_1) + a_{2,2}\,(\text{-}tb_1)^2 \\ &= -\left[-a_{1,1}b_2^2 + 2a_{1,2}b_2b_1 - a_{2,2}b_1^2\right] t^2 \\ &= -\det \begin{pmatrix} 0 & b_1 & b_2 \\ b_1 & a_{1,1} & a_{1,2} \\ b_2 & a_{1,2} & a_{2,2} \end{pmatrix} t^2. \end{aligned}$$

Therefore, $Q(x, y)$ is positive definite on $b_1\, x + b_2\, y = 0$, if and only if

$$-\det \begin{pmatrix} 0 & b_1 & b_2 \\ b_1 & a_{1,1} & a_{1,2} \\ b_2 & a_{1,2} & a_{2,2} \end{pmatrix} > 0.$$

2. General Theorem

In this section, we discuss the general case of the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ on the null space $\mathbf{B}(\mathbf{x}) = \mathbf{0}$. The matrix $\mathbf{A}$ is $n \times n$, so there are n variables. We assume that $\mathbf{B}$ is $k \times n$ and has rank k with $k < n$. We assume that the variables have been ordered so that $\det(\mathbf{B}_1) \neq 0$, where $\mathbf{B}$ is partitioned so that $\mathbf{B}_1$ contains the first k columns and $\mathbf{B}_2$ contains the last $n - k$ columns of $\mathbf{B}$,

$$\mathbf{B}_1 = \begin{pmatrix} b_{1,1} & \dots & b_{1,k} \\ \vdots & & \vdots \\ b_{k,1} & \dots & b_{k,k} \end{pmatrix} \quad \text{and} \quad \mathbf{B}_2 = \begin{pmatrix} b_{1,k+1} & \dots & b_{1,n} \\ \vdots & & \vdots \\ b_{k,k+1} & \dots & b_{k,n} \end{pmatrix},$$

The condition that determines whether Q is definite on the null space of $\mathbf{B}$ is the $(n+k) \times (n+k)$ symmetric matrix called the *bordered Hessian* that combines the matrix $\mathbf{A}$ with $\mathbf{B}$ and $\mathbf{B}^T$ on the borders:

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & b_{1,1} & \dots & b_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & b_{k,1} & \dots & b_{k,n} \\ b_{1,1} & \dots & b_{k,1} & a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{1,n} & \dots & b_{k,n} & a_{n,1} & \dots & a_{n,n} \end{pmatrix}.$$

Since the dimension of the null space is $n-k$, we should need to check the signs of determinants of $n-k$ submatrices. Using the fact that $\det(\mathbf{B}_1) \neq 0$, we consider the $j \times j$ submatrices of $\mathbf{H}$, formed from $\mathbf{H}$ by deleting the last $n+k-j$ rows and columns:

$$\mathbf{H}_j = \begin{pmatrix} 0 & \dots & 0 & b_{1,1} & \dots & b_{1,j-k} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & b_{k,1} & \dots & b_{k,j-k} \\ b_{1,1} & \dots & b_{k,1} & a_{1,1} & \dots & a_{1,j-k} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{1,j-k} & \dots & b_{k,j-k} & a_{j-k,1} & \dots & a_{j-k,j-k} \end{pmatrix}.$$

Theorem 1. *Let* $\mathbf{A}$ *be* $n \times n$ *symmetric matrix and* $\mathbf{B} = [\mathbf{B}_1\, \mathbf{B}_2]$ *be* $k \times n$ *with rank* k *and* $\det(\mathbf{B}_1) \neq 0$. *Let the bordered Hessians* $\mathbf{H}_j$ *be defined as above, and* $\mathrm{Nul}(\mathbf{B}) = \{\mathbf{x} : \mathbf{B}(\mathbf{x}) = \mathbf{0}\}$ *be the null space of* $\mathbf{B}$.

- **a** *The quadratic form* $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ *is positive definite on the null space* $\mathrm{Nul}(\mathbf{B})$ *iff* $(-1)^k \det(\mathbf{H}_j) > 0$ *for* $2k+1 \leq j \leq n+k$.
- **b** *The quadratic form* $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ *is negative definite on the null space* $\mathrm{Nul}(\mathbf{B})$ *iff* $(-1)^{j-k} \det(\mathbf{H}_j) > 0$ *for* $2k+1 \leq j \leq n+k$. *(Note that the signs of* $\det(\mathbf{H}_j)$ *alternate with* j.*)*

Remark 1. (i) When we considered one linear constraint $k=1$ and $n=2$, we required that $-\det(\mathbf{H}) > 0$ which is compatible with the general result, since $2k+1 = 3 = n+k$ and we only need to consider the sign of $\det(\mathbf{H}_{n+k}) = \det(\mathbf{H})$. (ii) For $n-k > 1$, we need to consider the sign of more than one determinant.

For cases with more than one constraint, $k > 1$ and the border has more than one extra row and column.

Proof.

Since $\det(\mathbf{B}_1) \neq 0$, we can solve for the null space of $\mathbf{B}$ in terms of the last $n-k$ variables:

$$\mathbf{0} = \mathbf{B}_1 \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} + \mathbf{B}_2 \begin{pmatrix} x_{k+1} \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = -\mathbf{B}_1^{-1}\mathbf{B}_2 \begin{pmatrix} x_{k+1} \\ \vdots \\ x_{n-k} \end{pmatrix} = \mathbf{J} \begin{pmatrix} x_{k+1} \\ \vdots \\ x_{n-k} \end{pmatrix},$$

where $\mathbf{J} = -\mathbf{B}_1^{-1}\mathbf{B}_2$. Partitioning $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{A}_{22} \end{bmatrix}$ into blocks, where $\mathbf{A}_{11}$ is $k \times k$, $\mathbf{A}_{12}$ is $k \times (n-k)$, and $\mathbf{A}_{22}$ is $(n-k) \times (n-k)$, the quadratic form on the null space has the following symmetric matrix $\mathbf{E}$:

$$\begin{aligned} \mathbf{E} = \begin{bmatrix} \mathbf{J}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{I} \end{bmatrix} &= \begin{bmatrix} \mathbf{J}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}\mathbf{J} + \mathbf{A}_{12} \\ \mathbf{A}_{12}^T\mathbf{J} + \mathbf{A}_{22} \end{bmatrix} \\ &= \mathbf{J}^T\mathbf{A}_{11}\mathbf{J} + \mathbf{J}^T\mathbf{A}_{12} + \mathbf{A}_{12}^T\mathbf{J} + \mathbf{A}_{22}. \end{aligned}$$

On the other hand, we can perform a (non-orthogonal) change of basis of the $n+k$-dimensional space on which the quadratic form $\mathbf{H}$ is defined:

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^T & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{0}_k & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_1^T & \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{B}_2^T & \mathbf{A}_{12}^T & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k & \mathbf{J} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-k} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{B}_1^T & \mathbf{A}_{11} & \mathbf{C}_{12} \\ \mathbf{0} & \mathbf{C}_{12}^T & \mathbf{E} \end{bmatrix}$$

Here the matrix $\mathbf{E}$ induces the quadratic form on the null space as we showed above. Since the determinant of the change of basis matrix is one, this change of basis preserves the determinant of $\mathbf{H}$, and also the determinants of $\mathbf{H}_j$ for $2k+1 \leq j \leq n+k$.

By using k row interchanges

$$\det(\mathbf{H}) = \det\begin{bmatrix} \mathbf{0} & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{B}_1^T & \mathbf{A}_{11} & \mathbf{C}_{12} \\ \mathbf{0} & \mathbf{C}_{12}^T & \mathbf{E} \end{bmatrix} = (\text{-}1)^k \det\begin{bmatrix} \mathbf{B}_1^T & \mathbf{A}_{11} & \mathbf{C}_{12} \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{12}^T & \mathbf{E} \end{bmatrix}$$
$$= (\text{-}1)^k \det(\mathbf{B}_1^T) \det\begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{C}_{12}^T & \mathbf{E} \end{bmatrix}$$
$$= (\text{-}1)^k \det(\mathbf{B}_1)^2 \det(\mathbf{E}).$$

This calculation carries over to all the $\mathbf{H}_j$, $(\text{-}1)^k \det(\mathbf{H}_j) = \det(\mathbf{B}_1)^2 \det(\mathbf{E}_{j-2k})$. Therefore, we can use the signs of the determinants of the $\mathbf{H}_j$ for $2k+1 \leq j \leq n+k$ to check the signs of the determinants of the principal submatrices $\mathbf{E}_{j-2k}$ with size ranging from 1 to $n-k$.

The quadratic form $\mathbf{Q}$ for $\mathbf{A}$ is positive definite on the null space iff the quadratic form for $\mathbf{E}$ is positive definite iff

$$(\text{-}1)^k \det(\mathbf{H}_j) = \det(\mathbf{B}_1)^2 \det(\mathbf{E}_{j-2k}) > 0 \qquad \text{for } 2k+1 \leq j \leq n+k.$$

For the negative definite case, the quadratic form $\mathbf{Q}$ for $\mathbf{A}$ is negative definite on the null space iff the quadratic form for $\mathbf{E}$ is negative definite iff

$$(\text{-}1)^{j-k} \det(\mathbf{H}_j) = (\text{-}1)^{j-2k} \det(\mathbf{B}_1)^2 \det(\mathbf{E}_{j-2k}) > 0 \qquad \text{for } 2k+1 \leq j \leq n+k.$$

□

3. EXAMPLES

Example 1. Consider the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$$

and linear constraints

$$x_1 + x_2 + x_3 = 0 \quad \text{and} \quad x_1 - 9x_3 + x_4 = 0.$$

The matrices are

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \text{-}1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & \text{-}1 & 0 \\ \text{-}1 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & \text{-}9 & 1 \end{pmatrix}$$

Notice that the 2×2 matrix formed from the first two columns of $\mathbf{B}$ is invertible:

$$\det(\mathbf{B}_1) = \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \text{-}1 \neq 0.$$

We need to form the bordered matrices for $2k+1 = 4+1 = 5 \leq j \leq n+k = 6$,

$$\mathbf{H}_6 = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & \text{-}9 & 1 \\ 1 & 1 & 1 & 0 & 0 & \text{-}1 \\ 1 & 0 & 0 & 1 & 2 & 0 \\ 1 & \text{-}9 & 0 & 2 & \text{-}1 & 0 \\ 0 & 1 & \text{-}1 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{H}_5 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & \text{-}9 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 \\ 1 & \text{-}9 & 0 & 2 & \text{-}1 \end{pmatrix}.$$

Since $(\text{-}1)^k = (\text{-}1)^2 = +1$, $\det(\mathbf{H}_6) = 24 > 0$, and $\det(\mathbf{H}_5) = 140 > 0$, Q is positive definite on the linear constraint set.

Problems

1. Let Q be as in the example. Use only the one linear constraint,

$$x_1 + x_2 + x_3 = 0.$$

Is Q positive definite on the null space of this one equation?

2. Let Q be as in the example. Use the three linear constraints,

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - 9x_3 + x_4 &= 0 \\ x_2 + x_4 &= 0. \end{aligned}$$

Is Q positive definite on the null space of these three equations?

3. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \text{-}1 \\ \text{-}1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & \text{-}1 & \text{-}1 \end{pmatrix}.$$

Is the quadratic form for **A** positive definite on the null space of for **B**?

REFERENCES

[1] C. Hassell and E. Rees, “The index of a constrained critical point”, The Mathematical Monthly, October 1993, pp. 772–778.