A FIRM MAXIMIZING PROFIT

1. **Two Products**

[1] Assume a firm makes two products with output levels of Q_1 and Q_2, prices P_1 and P_2, and revenue is $R = P_1Q_1 + P_2Q_2$. Assume the cost of production is $C = Q_1^2 + Q_1Q_2 + Q_2^2$, so the profit is

$$\pi = R - C = P_1Q_1 + P_2Q_2 - Q_1^2 - Q_1Q_2 - Q_2^2.$$

We want to maximize the profit π under two different assumptions.

First Case: In this case, we assume there is pure competition and the prices are determined externally to the firm and are considered fixed by the firm. Taking the partial derivatives with respect to Q_1 and Q_2 and setting them equal to zero, we get the two equations

$$\begin{aligned} 0 &= \frac{\partial \pi}{\partial Q_1} = P_1 - 2Q_1 - Q_2 \qquad \text{and} \\ 0 &= \frac{\partial \pi}{\partial Q_2} = P_2 - Q_1 - 2Q_2. \end{aligned}$$

Solving for Q_1 and Q_2, we get the critical values

$$Q_1^* = \frac{2P_1 - P_2}{3} \quad \text{and} \quad Q_2^* = \frac{2P_2 - P_1}{3}.$$

The second partial derivatives are

$$\begin{aligned} \frac{\partial^2 \pi}{\partial Q_1^2} &= -2 & \qquad \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} &= -1 \\ \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} &= -1 & \qquad \frac{\partial^2 \pi}{\partial Q_1^2} &= -2, \end{aligned}$$

so

$$\begin{aligned} &\frac{\partial^2 \pi}{\partial Q_1^2} = -2 < 0 \qquad \text{and} \\ &\frac{\partial^2 \pi}{\partial Q_1^2}\frac{\partial^2 \pi}{\partial Q_1^2} - \left(\frac{\partial^2 \pi}{\partial Q_2 \partial Q_1}\right)^2 = (-2)(-2) - (-1)^2 = 3 > 0, \end{aligned}$$

and the quantities (Q_1^*, Q_2^*) maximize profit.

Second Case: In this case, we assume that the firm has a monopoly and can set the prices of the two products. However, once the prices are fixed, the quantities purchased of the two products is determined by a demand function set by the consumers. We assume that the products are interchangeable, and the demand of each product depends on the prices of both products by the rules

$$\begin{aligned} Q_1 &= 40 - 2P_1 + P_2 \\ Q_2 &= 15 + P_1 - P_2. \end{aligned}$$

[1]Based on the treatment in "Fundamental Methods of Mathematical Economics" by Alpha Chiang, McGraw Hill, Inc., 1984

Substituting in the quantity demanded for the prices, we get the expression for the revenue in terms of the prices,

$$\pi = P_1(40 - 2P_1 + P_2) + P_2(15 + P_1 - P_2) - (40 - 2P_1 + P_2)^2$$
$$- (40 - 2P_1 + P_2)(15 + P_1 - P_2) - (15 + P_1 - P_2)^2.$$

We want to maximize the profit π as a function of the prices.

Taking the partial derivatives with respect to P_1 and P_2, we get

$$\begin{aligned}
0 &= \frac{\partial \pi}{\partial P_1} \\
&= (40 - 2P_1 + P_2) - 2P_1 + P_2 + 4(40 - 2P_1 + P_2) \\
&\qquad + 2(15 + P_1 - P_2) - (40 - 2P_1 + P_2) - 2(15 + P_1 - P_2) \\
&= 160 - 10P_1 + 5P_2, \qquad \text{and} \\
0 &= \frac{\partial \pi}{\partial P_2} \\
&= P_1 + (15 + P_1 - P_2) - P_2 - 2(40 - 2P_1 + P_2) \\
&\qquad - (15 + P_1 - P_2) + (40 - 2P_1 + P_2) + 2(15 + P_1 - P_2) \\
&= -10 + 5P_1 - 4P_2.
\end{aligned}$$

Thus, we have the two equations

$$\begin{aligned}
0 &= 160 - 10P_1 + 5P_2 \\
0 &= -10 + 5P_1 - 4P_2.
\end{aligned}$$

Multiplying the second equation by 2 and adding, we get $0 = 140 - 3P_2$ or $P_2^* = {}^{140}\!/_3 = 46^2\!/_3$. Then $10P_1 = 160 + 5({}^{140}\!/_3)$ and $P_1^* = {}^{118}\!/_3 = 39^1\!/_3$.

The second partial derivatives are

$$\frac{\partial^2 \pi}{\partial P_1^2} = -10 \qquad \frac{\partial^2 \pi}{\partial P_2 \partial P_1} = 5$$
$$\frac{\partial^2 \pi}{\partial P_1 \partial P_2} = 5 \qquad \frac{\partial^2 \pi}{\partial P_1^2} = -4,$$

so

$$\frac{\partial^2 \pi}{\partial P_1^2} = -10 < 0 \qquad \text{and}$$
$$\frac{\partial^2 \pi}{\partial P_1^2}\frac{\partial^2 \pi}{\partial P_1^2} - \left(\frac{\partial^2 \pi}{\partial P_2 \partial P_1}\right)^2 = (-10)(-4) - (5)^2 = 15 > 0,$$

and the prices (P_1^*, P_2^*) maximize profit.

2. **Multiple inputs and one product**

Assume a firm makes one product from n inputs. Let p be the price of the output, x_j be the amount of the j^{th} input used, and p_j be the price of the j^{th} input. Let $G(x_1, \dots, x_n)$ be the production function, which gives the amount of output in terms of the inputs. The profits is

$$\pi = p\, G(x_1, \dots, x_n) - p_1 x_1 - \cdots - p_n x_n.$$

Assuming the prices are fixed, the inputs which maximize profit satisfy

$$0 = \frac{\partial \pi}{\partial x_i} = p\,\frac{\partial G}{\partial x_i} - p_i \qquad \text{or}$$
$$\frac{\partial G}{\partial x_i} = \frac{p_i}{p}.$$

Thus, at the critical point, the marginal product of each input equals the price of the input relative to the price of the output.

The matrix of second partial derivatives is

$$\left(\frac{\partial^2 \pi}{\partial x_i \partial x_j}\right) = \left(p\,\frac{\partial^2 G}{\partial x_i \partial x_j}\right).$$

Thus, for the critical point to be a maximum it is necessary that not only $p\,\frac{\partial^2 G}{\partial x_i^2} < 0$, but also that the principal determinants have the correct signs,

$$(-1)^k \det\left(p\,\frac{\partial^2 G}{\partial x_i \partial x_j}\right)_{1 \le i,j \le k} > 0$$

for each $1 \le k \le n$. The requirement that $p\frac{\partial^2 G}{\partial x_i^2} < 0$ could be viewed as saying that a small change in the input x_i makes more difference for small values of x_i than for large input: diminishing returns.

For two inputs, we need

$$p\,\frac{\partial^2 G}{\partial x_1^2} < 0 \qquad \text{and}$$
$$p^2\,\frac{\partial^2 G}{\partial x_1^2}\frac{\partial^2 G}{\partial x_2^2} - p^2\left(\frac{\partial^2 G}{\partial x_1 \partial x_2}\right) > 0.$$

If we have a Cobb-Douglas production function for two inputs, $Q = kx_1^{\alpha}x_2^{\beta}$, then

$$\frac{\partial^2 Q}{\partial x_1^2} = \alpha(\alpha-1)kx_1^{\alpha-2}x_2^{\beta} \qquad \frac{\partial^2 Q}{\partial x_2^2} = \beta(\beta-1)kx_1^{\alpha}x_2^{\beta-2}$$
$$\frac{\partial^2 Q}{\partial x_1 \partial x_2} = \alpha\beta kx_1^{\alpha-1}x_2^{\beta-1}$$

$$\begin{aligned} p^2\,\frac{\partial^2 G}{\partial x_1^2}\frac{\partial^2 G}{\partial x_2^2} - p^2\left(\frac{\partial^2 G}{\partial x_1 \partial x_2}\right) &= p^2k^2x_1^{2\alpha-2}x_2^{2\beta-2}\left(\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2\right) \\ &= p^2k^2x_1^{2\alpha-2}x_2^{2\beta-2}\alpha\beta\left(1-\alpha-\beta\right). \end{aligned}$$

To make the critical point a maximum, we need $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta < 1$, i.e., we need a production function with diminishing return to scale.