Theorem (6). *The pivot columns of a matrix* $\mathbf{A}$ *form a basis for the column space* $Col(\mathbf{A})$.

Proof. The proof has two parts: show the pivot columns are linearly independent and show the pivot columns span the column space.

We use the reduced echelon matrix of $\mathbf{A}$ in the proof. We designate it by $\mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ have r pivot columns, then the pivot columns of $\mathbf{B}$ are the standard vectors $\mathbf{e}^1, \ldots, \mathbf{e}^r$ in $\mathbb{R}^m$. For example, consider the reduced echelon matrix

$$\begin{bmatrix} 1 & b_{12} & 0 & b_{14} & 0 & b_{16} \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} \\ 0 & 0 & 0 & 0 & 1 & b_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has pivot columns $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ in its first, third, and fourth columns. These are vectors in $\mathbb{R}^4$.

The solutions of the homogeneous equations $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Bx} = \mathbf{0}$ are the same. Therefore, the relations among the columns of $\mathbf{A}$ and $\mathbf{B}$ are the same. (Any non-zero solution of these homogeneous equations gives a relation among these columns.)

Lemma 1. *(a) The pivot columns of* $\mathbf{B}$ *are linearly independent.*
(b) The pivot columns of $\mathbf{A}$ *are linearly independent.*

Proof. (a) The pivot columns of $\mathbf{B}$ are the standard vectors $\mathbf{e}^1, \ldots, \mathbf{e}^r$ in $\mathbb{R}^m$ that are linearly independent. (In the example of the matrix given above, $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ are linearly independent in $\mathbb{R}^4$.)

(b) The relations among the columns of $\mathbf{A}$ and $\mathbf{B}$ are the same (as noted above), so the pivot columns of $\mathbf{A}$ are also linearly independent. $\square$

Lemma 2. *The pivot columns of* $\mathbf{A}$ *span the column space of* $\mathbf{A}$

Proof. Let $\mathbf{b}^k$ be a non-pivot column of the reduced echelon matrix $\mathbf{B}$. Assume that there are j pivot columns to the left of $\mathbf{b}^k$ in $\mathbf{B}$. These pivot columns must be $\mathbf{e}^1, \ldots, \mathbf{e}^j$. The non-pivot column $\mathbf{b}^k$ can only have nonzero entries in the first j components (or it would be a pivot column) and so is a linear combination of $\mathbf{e}^1, \ldots, \mathbf{e}^j$.

Since $\mathbf{A}$ has the same relations among its columns as $\mathbf{B}$, its non-pivot column $\mathbf{a}^k$ is a linear combination of the j pivot columns to the left of it. By Theorem 5(a), the span of the pivot columns is the same as the span of all the columns. This shows that the pivot columns of $\mathbf{A}$ span the column space of $\mathbf{A}$. $\square$

By Lemma 1(b), the pivot columns are linearly independent. By Lemma 2, the pivot columns span the column space of $\mathbf{A}$. Together, these two facts show that the pivot columns form a basis for the column space of $\mathbf{A}$. $\square$