1.5 Solution Sets $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$

Definition. The *rank* of a matrix $\mathbf{A}$ is the number of pivots. (In Chapter 4, there is a different definition, and this is a theorem.) We write $\text{rank}(\mathbf{A}) = r$.

Definition. $\mathbf{Ax} = \mathbf{0}$ is a *homogeneous* equations and $\mathbf{Ax} = \mathbf{b} \neq \mathbf{0}$ is a *nonhomogeneous* equation.

I. Homogeneous

$\mathscr{S}_{\mathbf{A}} = \{\, \mathbf{x} : \mathbf{Ax} = \mathbf{0} \,\}$.

(i) If there are no free variables ($\text{rank}(\mathbf{A}) = n = \#\text{col}$) then only solution is $\mathbf{x} = \mathbf{0}$, $\mathscr{S}_{\mathbf{A}} = \{\, \mathbf{0} \,\}$. We say that there is only the *trivial* solution.

(ii) If there exists at least one free variable ($\text{rank}(\mathbf{A}) < n = \#\text{col}$), then there exists a nontrivial solution. Take a free variable equal to 1. $\mathscr{S}_{\mathbf{A}} \supsetneqq \{\, \mathbf{0} \,\}$.

Example.

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

These give the equations $x_1 + x_2 = 0$ and $x_3 = 0$ or $x_1 = -x_2$ and $x_3 = 0$.

$\mathscr{S}_{\mathbf{A}} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : x_2 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$\text{rank}(\mathbf{A}) = 2 < 3 = \#\text{col}$, $n - r = 3 - 2 = 1$ free variable. ■

Example.

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$. $\text{rank}(\mathbf{A}) = 3 = 3 = \#\text{col}$, $n - r = 3 - 3 = 0$ so no free variable. $\mathscr{S}_{\mathbf{A}} = \{\mathbf{0}\}$. ■

Example. *Implicit* description of a plane through $\mathbf{0}$, $\begin{bmatrix} 10 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ or $10x_1 - 3x_2 - 2x_3 = 0$. Can solve for x_1, $x_1 = 0.3x_2 + 0.2x_2$, with two free variables. General solution

$$\mathbf{x} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$

So the *parametric* form of the plane is of the form $\mathbf{x} = s\mathbf{v}^1 + t\mathbf{v}^2$. ■

II. Nonhomogeneous

Example. $\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ -1 & -1 & 1 & 5 \\ 1 & 1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = -3$, $x_3 = 2$ or $x_1 = -x_2 - 3$, $x_3 = 2$. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. $\mathbf{p} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ is one *particular* solution of (NH) and $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is general solution of (H). ■

Theorem (6). *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix and* $\mathbf{b} \in \mathbb{R}^m$. *Assume the (NH)* $\mathbf{Ax} = \mathbf{b}$ *is consistent and has a particular solution* $\mathbf{p}$. *Then the set of solution of (NH) is the set of all vectors* $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ *where* $\mathbf{v}_h$ *is a solution of (H)* $\mathbf{Ax} = \mathbf{0}$.

Proof. If $\mathbf{Ap} = \mathbf{b}$ and $\mathbf{Av}_h = \mathbf{0}$ then $\mathbf{A}(\mathbf{p} + \mathbf{v}_h) = \mathbf{Ap} + \mathbf{Av}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$. On the other hand, if $\mathbf{Aq} = \mathbf{b}$ is another solution of (NH), then $\mathbf{A}(\mathbf{p} - \mathbf{q}) = \mathbf{0}$ and $\mathbf{p} - \mathbf{q} = \mathbf{v}_h$ is a solution of (H). So $\mathbf{q} = \mathbf{p} + \mathbf{v}_h$. □

For an nonhomogeneous equations in $\mathbb{R}^3$, the *parametric* form of a general line (not necessarily through the origin) is $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. The parametric form of a plane is $\mathbf{x} = \mathbf{p} + s\mathbf{v}^1 + t\mathbf{v}^2$.

The *implicit* form of a plane (not necessarily through the origin) is $a_1x_1 + a + 2x_2 + a_3x_3 = b$ (with at least one nonzero coefficient).

The implicit form of a line is given by two equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

where $\mathbf{A}$ has rank 2 so there are $3 - 2 = 1$ free variable.