EXTRA PROBLEMS FOR MATH 313-1 & 2

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3.3.1 Let $f(x) = 3\,x \pmod 1$ be the tripling map.
  a. Prove that if two distinct points x_0 and x_0' are within a distance $1/6$, then their iterates are at least three times as far apart.
  b. Find a pair of point whose distance is not tripled by the map.
  c. Show that f has sensitive dependence on initial conditions.

3.3.2 Let p be a fixed point for f such that $|f'(p)| > 1$. Prove that f has sensitive dependence on initial conditions at p.

3.3.5 (**Sensitive dependence**) Use a computer program to investigate the sensitive dependence for the quadratic map $G(x) = 4x(1-x)$. How many iterates does it take to get separation by 0.1 and 0.3 for the two different initial conditions x_0 and $x_0 + \delta$, for the choices $x_0 = 0.1$ and 0.48 and for $\delta = 0.01$ and 0.001? (Thus, there are four pairs of points.)

3.4.2 Consider the function $F(x) = 6x^3 - 5x$ on $[-1, 1]$. The points where $F(x) = 1$ are $\dfrac{-3 \pm \sqrt{3}}{6}$ and 1. The points where $F(x) = -1$ are $\dfrac{3 \pm \sqrt{3}}{6}$ and -1.
  a. Sketch the graph of F.
  b. Describe the set of points x such that both x and $F(x)$ are in $[-1, 1]$,

$$\{x : F^j(x) \in [-1,1] \text{ for } 0 \le j \le 1\,\} = \bigcap_{j=0}^{1} F^{-j}([-1,1]).$$

  It is made up of how many intervals? What bound can you put on the length of each of the intervals?

  c. Describe the set of points x such that x, $F(x)$, and $F^2(x)$ are all in $[-1, 1]$,

$$\{x : F^j(x) \in [-1,1] \text{ for } 0 \le j \le 2\,\} = \bigcap_{j=0}^{2} F^{-j}([-1,1]).$$

  It is made up of how many intervals? What bound can you put on the length of each of the intervals?

  d. What bound can you put on the length of one of the intervals in

$$\mathbf{K}_n = \{x : F^j(x) \in [-1,1] \text{ for } 0 \le j \le n\,\} = \bigcap_{j=0}^{n} F^{-j}([-1,1]).$$

  e. Explain why F has an invariant set that is like a Cantor set.

3.4.3 Consider the cotangent function $f(x) = \cot(x)$ on $[0, 2\pi]$. Explain why it has an invariant set that is like a Cantor set made up of points which stay in $[0, 2\pi]$ for all iterates.

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4.2.6 Let f be the map

$$f(x) = \frac{4}{\pi} \arctan(x).$$

Note that $f(0) = 0$, $f(1) = 1$, and $f(-1) = -1$). The fixed point 0 is repelling and ± 1 are attracting. Give all the attracting sets for f. Which of these sets are attractors? Why are there no chaotic attractors?

4.4.2 Explain why the rotation

$$R_\alpha(x) = x + \alpha \pmod 1$$

preserves Lebesgue measure on $[0, 1]$.

4.4.3 Show that the tripling map $f(x) = 3x \pmod 1$ preserves Lebesgue measure on $[0, 1]$.

5.1.1 Find the eigenvalues and draw the phase portrait for the linear maps which have the following matrices.

$$(a) \begin{pmatrix} 1/2 & 1/8 \\ 1/2 & 1/2 \end{pmatrix} \qquad (b) \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$(c) \begin{pmatrix} 1/4 & 1/4 \\ -1/2 & 1 \end{pmatrix} \qquad (d) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(e) \begin{pmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{pmatrix} \qquad (f) \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

5.2.1 Give the stability type of each of the linear maps of exercise 5.1.1.

5.2.2 Show that the linear map with matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is unstable.

5.2.3 Let

$$\mathbf{F} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + x^2 \\ 2\,x + 3\,y \end{pmatrix}.$$

Find the fixed points and classify them as source, saddle, sink, or none of these.

5.2.4 Let

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\,xy + y \\ 3\,y - x \end{pmatrix}.$$

Find and classify the fixed points.

5.2.5 Consider the Hénon map.

a. Show that if (x_+, x_+) and (x_-, x_-) are the two fixed points, then

$$x_+ + x_- = -1 - b.$$

b. Show that if $\{\,(x_0, y_0),\, (x_1, y_1)\,\}$ is a period-2 orbit, then

$$1 + b = x_0 + y + 0 = x_1 + y_1.$$

5.2.6 Consider the Hénon map with $b = -0.2$.

a. Show that for $a \geq -0.16 = a_0$ there are fixed points. Find the eigenvalues of the single fixed point for $a = -0.16$.

b. Show that for $a > -0.16$, $x_- < -0.4$, $\lambda_+ > 1$, and $\lambda_- = -0.2/\lambda_+$ satisfies $-1 < \lambda_- < 0$, so (x_-, x_-) is a saddle point.

c. Show that for the fixed point (x_+, x_+), the eigenvalue $\lambda_- = -1$ for $a = 0.48 = a_1$. Using the continuity of the values of the eigenvalues, conclude that the fixed point is attracting for $-0.16 < a < 0.48$.

d. Show that there is a period-2 orbit for $a > 0.48$, and the product of the two values of x on the orbit is $0.8^2 - a$, $x_0 x_1 = 0.8^2 - a$.

e. Show that the characteristic equation for this period-2 orbit is

$$\lambda^2 - (4x_0x_1 + 0.4)\lambda + 0.04 = 0.$$

Letting $-\mu = x_0 x_1$, show that one of the eigevalues is -1, when

$$0 = 3\mu + 0.4\mu - 0.84$$
$$\mu = \frac{2.8}{6}$$

Show that this occurs for $a = 0.64 + \frac{2.8}{6} = a_2$. Also, for $a_1 < a < a_2$, the period-2 orbit is attracting.

5.3.1 Consider the map

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5\,x - 4\,y^3 \\ 2\,y \end{pmatrix},$$

a. Find the inverse of **F**.

b. Find the stable and unstable manifolds of the fixed point at the origin.

5.3.2 Consider the map

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.5\,x - 4\,y^3 + 8\,z^2 \\ 2\,y \\ 4\,z \end{pmatrix},$$

a. Find the inverse of **F**.

b. Find the stable and unstable manifolds of the fixed point at the origin.