Math 313-1 Final December 2003 R.C. Robinson

No books, no notes. Calculators are allowed.
Show all your work in your bluebook. Start each problem on a new page.

1. (30 Points) Assume that f is a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, with $f(x_0) = x_1$, $f(x_1) = x_2$, $f(x_2) = x_0$, $f'(x_0) = 1/2$, $f'(x_1) = 2$, and $f'(x_2) = 1/3$.
a. Is the orbit $\mathcal{O}^+_f(x_0)$ attracting or repelling?
b. What is the Lyapunov exponent of x_0?

2. (30 Points) Let $f(x) = x^3 - x$ and $N(x) = x - \dfrac{x^3 - x}{3x^2 - 1} = \dfrac{2x^3}{3x^2 - 1}$ be the Newton map for f. The fixed points of N are 0 and ± 1. Also,

$$N'(x) = \frac{6x^3 - 6x}{(3x^2 - 1)^2} \begin{cases} < 0 & \text{for } \frac{1}{\sqrt{3}} < x < 1 \\ > 0 & \text{for } 1 < x. \end{cases}$$

The value $N(x)$ goes to ∞ as x goes to ∞ and also as x goes to $\frac{1}{\sqrt{3}}$ with $x > \frac{1}{\sqrt{3}}$. Using the graphical method of iteration, explain why the basin $\mathscr{B}(1; N)$ contains the interval $(1/\sqrt{3}, \infty)$. Explain with words in addition to any drawing you make.

3. (40 Points) Let

$$f(x) = \frac{2}{3}\,x^3 - \frac{1}{2}\,x.$$

a. Find the fixed points and determine their stability type as attracting or repelling.
b. Find the critical points, where $f'(x) = 0$.
c. Show the Schwarzian derivative of f is negative. Note:

$$S_f(x) = \frac{f'''(x)\,f'(x) - \frac{3}{2} f''(x)^2}{f'(x)^2}.$$

d. What is the ω-limit set of the critical points?

4. (30 Points) Let

$$f(x) = \begin{cases} 5\,x + 4 & \text{for } x \leq -0.4 \\ -5\,x & \text{for } -0.4 \leq x \leq 0.4 \\ 5\,x - 4 & \text{for } 0.4 \leq x. \end{cases}$$

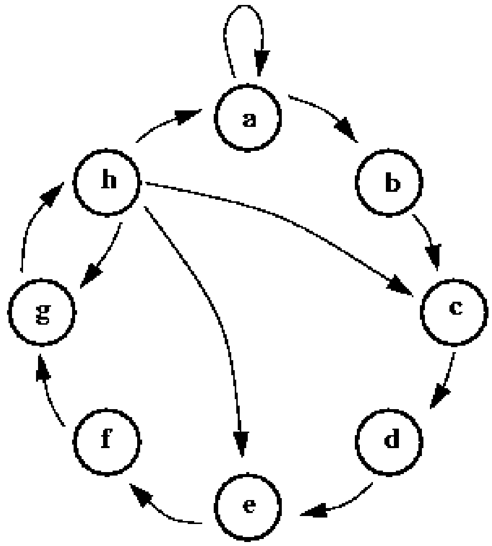
a. Sketch the graph of f. Notice that $f(-1) = -1$, $f(-0.4) = 2$, $f(0.4) = -2$, and $f(1) = 1$.
b. Consider the sets

$$\mathbf{K}_n = \{x : f^j(x) \in [-1, 1] \text{ for } 0 \leq j \leq n\} = \bigcap_{j=0}^{n} f^{-j}([-1, 1]).$$

How many intervals do $\mathbf{K}_1$ and $\mathbf{K}_2$ contain and what is the length of each of these intervals?
c. Discuss the set $\mathbf{K} = \bigcap_{n \geq 0} \mathbf{K}_n$.

(over)

5. (30 Points) Consider the transition graph given in the figure.



The map must have points of what periods? What are symbol sequences that correspond to these periodic points?

6. (40 Points) Let

$$f(x) = \begin{cases} \frac{3}{2} + \frac{5}{2}x & \text{for } x \leq 1 \\ 6 - 2\,x & \text{for } 1 \leq x \leq 3 \\ 3\,x - 9 & \text{for } 3 \leq x. \end{cases}$$

a. Show that $[0, 4]$ has a trapping region.
b. Show that f has a Markov partition on $[0, 4]$. What is its transition graph?
c. Is f topologically transitive on $[0, 4]$? If so, why?
d. Does f have a chaotic attractor? If so, why?