

1. (25 Points) Find all the Nash equilibria with pure strategies for the bimatrix game where the payoff for the player choosing the rows is given by the matrix A and the payoff for the player choosing the columns is given by the matrix B , where

$$A = \begin{pmatrix} 3 & 4 & 2 & 2 \\ 5 & 1 & 3 & 3 \\ 1 & 2 & 1 & 4 \\ 0 & 3 & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & 2 & 4 & 1 \\ 7 & 1 & 3 & 4 \\ 1 & 3 & 2 & 5 \end{pmatrix}.$$

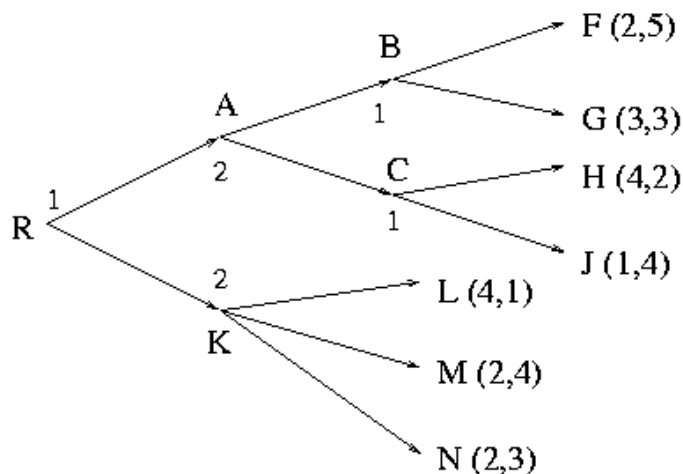
Answer:

The row chooser picks row 2 from column 1, row 1 from column 2, row 2 from column 3, and row 3 from column 4.

The column chooser picks column 2 from row 1, column 3 from row 2, column 1 from row 3, column 4 from row 4.

The two Nash equilibria with pure strategies are the common choices: (i) row 1 and column 2, (ii) row 2 and column 3.

2. (40 Points) Consider the sequential game with perfect information given in the following game tree. The vertices are labeled with a 1 or a 2 depending on whether player one or two owns the vertex.
- Find a subgame perfect equilibrium using pure strategies. Give the decision at each node.
 - Give a Nash equilibrium for the game that is not a subgame perfect equilibrium, i.e., is not a Nash equilibrium on some subgame. Explain why your answer is correct.



Answer:

- (a) At B , 1 chooses BG . At C , 1 chooses CH . At A , 2 chooses AB . At K , 2 chooses KM . At R , 1 chooses RA . The subgame perfect Nash equilibrium is player 1 chooses $\{RA, BG, CH\}$ and player 2 chooses $\{AB, KM\}$. The payoff at node G is $(3, 3)$.

- (b) One Nash equilibrium which is not subgame perfect is formed by changing KM

to KN . player 1 chooses $\{RA, BG, CH\}$ and player 2 chooses $\{AB, KN\}$.
For player 1,

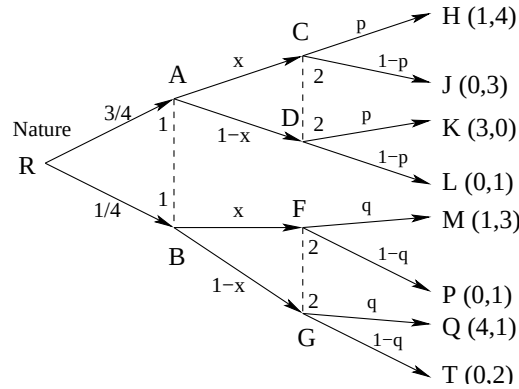
$$\pi_1(\{RA, BG, CH\}, \{AB, KN\}) = 3 \begin{cases} = 3 = \pi_1(\{RA, BG, C^*\}, \{AB, KN\}) \\ > 2 = \pi_1(\{RA, BF, C^*\}, \{AB, KN\}) \\ > 2 = \pi_1(\{RK, B^*, C^*\}, \{AB, KN\}). \end{cases}$$

For player 2,

$$\pi_2(\{RA, BG, CH\}, \{AB, KN\}) = 3 \begin{cases} = 3 = \pi_2(\{RA, BG, CH\}, \{AB, K^*\}) \\ > 2 = \pi_2(\{RA, BG, CH\}, \{AC, K^*\}). \end{cases}$$

Neither player has an incentive to change.

3. (20 Points) Consider the sequential game with imperfect information given in the following game tree. The root R is owned by nature; there is a probability of $3/4$ of taking edge RA and a probability of $1/4$ of taking edge RB . Player one owns nodes in the information set $\{A, B\}$. Player 2 owns the nodes in the two information sets $\{C, D\}$ and $\{F, G\}$. The behavioral strategies are as given in the game tree with probabilities, $p, 1 - p, q, 1 - q, x$, and $1 - x$. What are the Bayesian consistent system of beliefs for each player on the information sets owned by them, i.e., the beliefs of player one on $\{A, B\}$, the beliefs of player two on $\{C, D\}$, and the beliefs of player two on $\{F, G\}$.

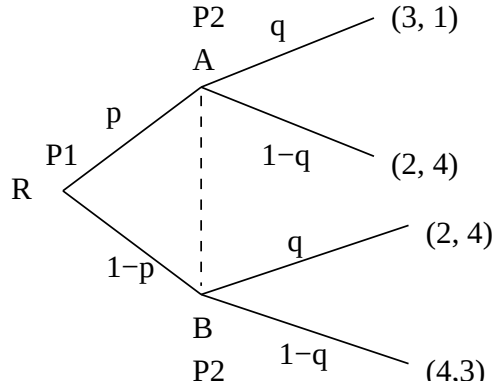


Answer:

$$\begin{aligned} \mu_1(A) &= 3/4 & \mu_1(B) &= 1/4, \\ \mu_2(C) &= x & \mu_2(D) &= 1 - x, \\ \mu_2(F) &= x & \mu_2(G) &= 1 - x. \end{aligned}$$

4. (40 Points) Consider the sequential game with imperfect information given in the following game tree. The root R is owned by player one. Player 2 owns the nodes in the information set $\{A, B\}$. The behavioral strategies are as given in the game tree with probabilities, $p, 1 - p, q$, and $1 - q$.

- (a) What is the expected payoff for player two on the information set $\{A, B\}$? Give the answer in terms of the variables for the behavioral strategies of the players. What choice of q should player two make to maximize the payoff? Give the answer in terms of the variable p for the strategy of player one.
- (b) What is the expected payoff for player one at the node R ? What choices of p should player one make to maximize the payoff? Give the answer in terms of the variable q for the strategy of player two.
- (c) What values of p and q give a sequential equilibrium?



Answer:

(a)

$$\begin{aligned} E_2(\{A, B\}) &= p[q \cdot 1 + (1 - q)4] + (1 - p)[q \cdot 4 + (1 - q)3] \\ &= p[4 - 3q] + (1 - p)[3 + q], \end{aligned}$$

$$\frac{\partial E_2(\{A, B\})}{\partial q} = -3p + 1 - p = 1 - 4p.$$

The expectation $E_2(\{A, B\})$ is maximized for

$$q = \begin{cases} 1 & \text{if } p < \frac{1}{4} \\ \text{arbitrary} & \text{if } p = \frac{1}{4} \\ 0 & \text{if } p > \frac{1}{4}. \end{cases}$$

(b)

$$\begin{aligned} E_1(R) &= p[q \cdot 3 + (1 - q)2] + (1 - p)[q \cdot 2 + (1 - q)4] \\ &= p[2 + q] + (1 - p)[4 - 2q], \end{aligned}$$

$$\frac{\partial E_1(R)}{\partial p} = 2 + q - 4 + 2q = 3q - 2.$$

The expectation $E_1(R)$ is maximized for

$$p = \begin{cases} 0 & \text{if } q < \frac{2}{3} \\ \text{arbitrary} & \text{if } q = \frac{2}{3} \\ 1 & \text{if } q > \frac{2}{3}. \end{cases}$$

(c) The only choice which satisfies both (a) and (b) is $q = 2/3$ and $p = 1/4$. This gives the sequential equilibrium.

5. (25 Points) Consider a complete information auction with 3 bidders where all the valuations are known to all the bidders. Let v_j be the valuation of the j^{th} -player and assume that $v_1 > v_2 > v_3$. The auction is sealed-bid, so the bids are not known to the other bidders. If the bids are (b_1, b_2, b_3) , let

$$\begin{aligned}
 m &= \max\{b_1, b_2, b_3\}, \\
 A &= \{i : b_i = m\}, \\
 m_s &= \begin{cases} \max\{b_i : b_i \notin A\} & \text{if } A \neq \{1, 2, 3\} \\ m & \text{if } A = \{1, 2, 3\}, \end{cases} \\
 u_i(b_1, b_2, b_3) &= \begin{cases} 0 & \text{if } b_i < m \\ (v_i - m_s) & \text{if } b_i = m \text{ and we have 1 finalists} \\ \frac{1}{2}(v_i - m_s) & \text{if } b_i = m \text{ and we have 2 finalists} \\ \frac{1}{3}(v_i - m) & \text{if } b_i = m \text{ and we have 3 finalists.} \end{cases}
 \end{aligned}$$

Thus the auction is second-price, where the price is the second highest amount bid. **Is** the set of bids $(b_1^*, b_2^*, b_3^*) = (v_1, v_2, v_3)$ a Nash equilibrium from the point of view of the second bidder? Show your work for your answer.

Answer:

$$u_2(b_1^*, b_2^*, b_3^*) = u_2(v_1, v_2, v_3) = 0$$

since $v_2 < v_1$ is not the highest bid. On the other hand

$$u_2(v_1, b_2, v_3) = \begin{cases} v_2 - v_1 < 0 & \text{if } b_2 > v_1 \\ \frac{1}{2}(v_2 - v_3) > 0 & \text{if } b_2 = v_1 \\ 0 & \text{if } b_2 < v_1. \end{cases}$$

Clearly, $b_2 = v_1$ gives the highest payoff. Since this payoff is larger than choosing $b_2 = v_2 < v_1$, the choice $(b_1^*, b_2^*, b_3^*) = (v_1, v_2, v_3)$ is not a Nash equilibrium.

6. (25 Points) Consider a private value auction with 2 bidders. Assume that each bidder knows that the other bidder's valuation is a random variable that is uniformly distributed on the interval $[a, A]$. (Each bidder knows their own exact valuation.) **Show** that the bidding rules $b_j^*(v_j) = \frac{1}{2}v_j + \frac{1}{2}a$ satisfy the conditions for a Nash equilibrium for the first bidder. Hint: You may use the fact that from the perspective of the first bidder with bid b_1 the probability

$$P(b_1 > b_2^*) = \begin{cases} 0 & \text{if } b_1 \leq a \\ \frac{2(b_1 - a)}{A - a} & \text{if } a \leq b_1 \leq \frac{1}{2}(a + A) \\ 1 & \text{if } \frac{1}{2}(a + A) \leq b_1. \end{cases}$$

Answer:

In the interval of b_1 where the maximum occurs,

$$(Eu_1)(b_1) = P(b_1 > b_2^*)(v_1 - b_1) = \frac{2}{A - a}(b_1 - a)(v_1 - b_1),$$

$$\frac{\partial(Eu_1)}{\partial b_1} = \frac{2}{A - a}[(v_1 - b_1) - (b_1 - a)] = 0,$$

$$0 = a + v_1 - 2b_1$$

$$2b_1 = a + v_1$$

$$b_1 = \frac{1}{2}(a + v_1) = b_1^*(v_1).$$

The second derivative

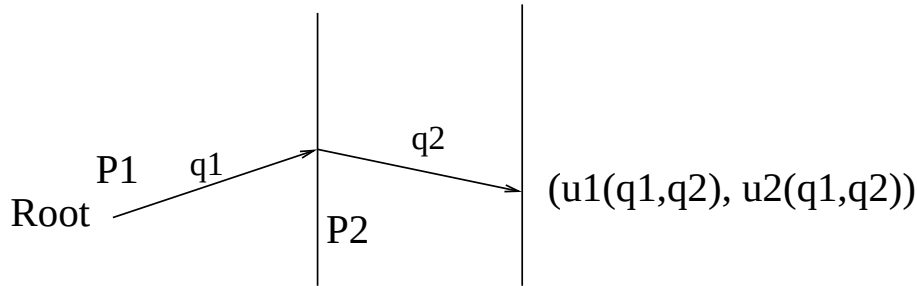
$$\frac{\partial^2(Eu_1)}{\partial b_1^2} = \frac{2}{A - a}[-2]$$

is negative, so the value of b_1 maximizes $(Eu_1)(b_1)$.

7. (25 Points) Two firms produce identical products. Each firm picks a quantity $q_i \geq 0$ to produce. Firm one chooses first and then firm two. See the game tree below. The price is given by $p = A - q_1 - q_2$, and the profit for each firm is given by

$$\pi_i(q_1, q_2) = q_i(A - q_1 - q_2 - c_i),$$

where c_i is the fixed cost per unit produced. What is the Nash equilibrium for this sequential game with a continuous choice variables q_1 and q_2 ?



Answer:

$$\pi_2(q_1, q_2) = q_2(A - q_1 - q_2 - c_2)$$

$$\frac{\partial \pi_2}{\partial q_2} = A - q_1 - q_2 - c_2 - q_2 = 0,$$

$$q_2^*(q_1) = \frac{1}{2}(A - q_1 - c_2),$$

$$\frac{\partial^2 \pi_2}{\partial q_2^2} = -2 < 0$$

so the value is a maximum.

$$\begin{aligned}
\pi_1(q_1, q_2^*(q_1)) &= q_1 \left(A - q_1 - \frac{1}{2}(A - q_1 - c_2) - c_1 \right) \\
&= q_1 \left(\frac{1}{2}A - \frac{1}{2}q_1 + \frac{1}{2}c_2 - c_1 \right) \\
\frac{\partial \pi_1}{\partial q_1} &= \frac{1}{2}A - \frac{1}{2}q_1 + \frac{1}{2}c_2 - c_1 - \frac{1}{2}q_1 \\
&= \frac{1}{2}A + \frac{1}{2}c_2 - c_1 - q_1 = 0, \\
\frac{\partial^2 \pi_1}{\partial q_1^2} &= -2 < 0, \\
q_1^* &= \frac{1}{2}(A + c_2 - 2c_1), \\
q_2^* &= q_2^*(q_1^*) = \frac{1}{4}(A - 3c_2 + 2c_1).
\end{aligned}$$