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There are 8 problems.

1. (25 Points) Find all the pure strategy Nash equilibria for the bi-matrix game given by

$$\begin{pmatrix} (5, 3) & (9, 7) & (3, 5) \\ (6, 5) & (7, 8) & (1, 7) \\ (3, 7) & (5, 5) & (4, 9) \end{pmatrix}.$$

Answer:

Player one maximizes in the first column in row two, in the second column in row one, and in the third column in row three.

Player two maximizes in the first row in column two, in the second row in column two, and in the third row in column three.

The Nash equilibria are (i) row one and column two, and (ii) row three and column three.

2. (25 Points) (Use of a common resource) Three countries use the same fishing grounds. The i^{th} country catches x_i amount of fish. The net payoff for the countries (after costs) is given as follows:

$$\pi_1(x_1, x_2, x_3) = 12 \ln(x_1 + 1) - x_1^2 - \frac{1}{2}(x_2 + x_3)^2$$

$$\pi_2(x_1, x_2, x_3) = 12 \ln(x_2 + 1) - x_2^2 - \frac{1}{2}(x_1 + x_3)^2$$

$$\pi_3(x_1, x_2, x_3) = 12 \ln(x_3 + 1) - x_3^2 - \frac{1}{2}(x_1 + x_2)^2.$$

- a. How many fish should the *first* country catch to maximize its payoff? Why is it a maximum?
- b. If they form a cartel and agree to each catch the same amount, $x = x_1 = x_2 = x_3$, then the joint payoff is

$$U(x) = 3 \left[12 \ln(x + 1) - x^2 - \frac{1}{2}(2x)^2 \right].$$

How many fish x should each country catch to maximize the joint payoff?

Answer:

(a)

$$\frac{\partial \pi_1}{\partial x_1} = \frac{12}{x_1 + 1} - 2x_1 = 0$$

$$6 = x_1^2 + x_1$$

$$0 = (x_1 + 3)(x_1 - 2)$$

$$x_1 = 2.$$

The value $x_1 = 2$ is a maximum because

$$\frac{\partial^2 \pi_1}{\partial x_1^2} \Big|_{x_1=2} = -\frac{12}{x_1 + 1} - 2 \Big|_{x_1=2} = -4 - 2 < 0.$$

(b)

$$\begin{aligned}U'(x) &= 3 \left[\frac{12}{x+1} - 2x - 4x \right] = 0 \\12 &= 6x(x+1) \\0 &= x^2 + x - 2 = (x+2)(x-1).\end{aligned}$$

The value $x = 1$ is a maximum because

$$U''(x) = -\frac{36}{(x+1)^2} - 18 < 0.$$

3. (25 Points) Consider a complete information auction with 3 bidders where all the valuations are known to all the bidders. Let v_j be the valuation of the j^{th} -player and assume that $v_1 > v_2 > v_3$. The auction is a second-price sealed-bid, with payoffs to bids (b_1, b_2, b_3) of

$$u_j(b_1, b_2, b_3) = \begin{cases} 0 & \text{if } b_j < m \\ (v_j - m_{-j}) & \text{if } b_j = m \text{ and we have 1 finalists} \\ \frac{1}{r}(v_j - m_{-j}) & \text{if } b_j = m \text{ and we have } r \text{ finalists with } r \geq 2. \end{cases}$$

where $m = \max\{b_1, b_2, b_3\}$ and $m_{-j} = \max\{b_i : i \neq j\}$.

Show that the set of bids $(b_1^*, b_2^*, b_3^*) = (v_1, v_2, v_3)$ a Nash equilibrium. Check the payoffs for each player.

Answer:

For each bidder, we consider the bids which will make that player the sole winner ($b_j = m$ & $b_j > m_{-j}$), tied as a winner ($b_j = m$ & $b_j = m_{-j}$), and a loser ($b_j < m$).

(Bidder one)

$$u_1(b_1, v_2, v_3) = \begin{cases} v_1 - v_2 > 0 & \text{if } b_1 > v_2 \\ \frac{1}{2}(v_1 - v_2) > 0 & \text{if } b_1 = v_2 \\ 0 & \text{if } b_1 < v_2. \end{cases}$$

The bid $b_1^* = v_1$ falls in the first case, which has the maximum payoff.

(Bidder two)

$$u_2(v_1, b_2, v_3) = \begin{cases} v_2 - v_1 < 0 & \text{if } b_2 > v_1 \\ \frac{1}{2}(v_2 - v_1) < 0 & \text{if } b_2 = v_1 \\ 0 & \text{if } b_2 < v_1. \end{cases}$$

The bid $b_2^* = v_2$ falls in the third case, which has the maximum payoff.

(Bidder three)

$$u_3(v_1, v_2, b_3) = \begin{cases} v_3 - v_1 < 0 & \text{if } b_3 > v_1 \\ \frac{1}{2}(v_3 - v_1) < 0 & \text{if } b_3 = v_1 \\ 0 & \text{if } b_3 < v_1. \end{cases}$$

The bid $b_3^* = v_3$ falls in the third case, which has the maximum payoff.

For each player, keeping the bids of the other players fixed and vary his or her own bid, the bid optimizing the payoff includes the bid $b_j = v_j$.

4. (25 Points) Consider a common value auction with three bidders where the probability distribution is given by $f(v_h, \omega) = \omega^2$ and $f(v_\ell, \omega) = 1 - \omega^2$. Here $v_\ell < v_h$ are the low and high valuations and ω is the signal which is uniformly distributed on $[0, 1]$. Assume the bidding rule of

$$b_j^*(\omega_j) = v_\ell + \frac{\omega_j^2}{2}(v_h - v_\ell)$$

for $j = 2, 3$. You can assume that it has been shown that

$$P(b_1 > b_2^*(\cdot) \text{ and } b_1 > b_3^*(\cdot)) = \begin{cases} 0 & \text{if } b_1 \leq v_\ell \\ \frac{2(b_1 - v_\ell)}{v_h - v_\ell} & \text{if } v_\ell \leq b_1 \leq \frac{1}{2}(v_h + v_\ell) \\ 1 & \text{if } b_1 \geq \frac{1}{2}(v_h + v_\ell), \end{cases}$$

where this expected value is averaged over the possible signals ω_2 and ω_3 . Also, you can use the fact that the expected value for player one is $E(v_1|\omega_1) = v_\ell + \omega_1^2(v_h - v_\ell)$.

Find the value of the bid b_1 which maximizes the payoff for player one,

$$E_1(b_1) = P(b_1 > b_2^*(\cdot) \text{ and } b_1 > b_3^*(\cdot)) (E(v_1|\omega_1) - b_1).$$

Answer:

The maximum occurs in the range of bids using the middle case of the probability of being the higher bidder. Therefore,

$$\begin{aligned} E_1(b_1) &= \left(\frac{2}{v_h - v_\ell} \right) [(b_1 - v_\ell)(E(v_1|\omega_1) - b_1)] \\ E'_1(b_1) &= \left(\frac{2}{v_h - v_\ell} \right) [(E(v_1|\omega_1) - b_1) - (b_1 - v_\ell)] = 0 \\ 2b_1 &= v_\ell + v_\ell + \omega_1^2(v_h - v_\ell) \\ b_1 &= v_\ell + \frac{\omega_1^2}{2}(v_h - v_\ell). \end{aligned}$$

This bid, which maximizes the payoff, satisfies the same bid rule used for the other two players.

It is easily checked that

$$E''_1(b_1) = -2 \left(\frac{2}{v_h - v_\ell} \right) < 0,$$

so this choice of b_1 is a maximum.

5. (25 Points) Consider the game tree given in Figure 1. Player P_1 owns R and the information sets $\mathcal{I}_1$ and $\mathcal{I}_2$. Player P_2 owns the nodes A and B .

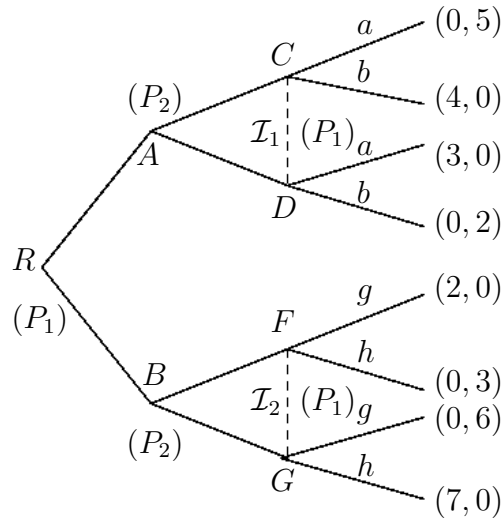


FIGURE 1

- What are the subgames of this sequential game?
- In order to have a sequential equilibrium, where must P_1 and P_2 optimize their payoffs?
- In order to have a subgame perfect Nash equilibrium, where must P_1 and P_2 optimize their payoffs?

Answer:

- The proper subgames start at the nodes A and B . In addition, there is the whole game.
- For a sequential equilibrium, P_1 maximizes the payoff on the information sets $\mathcal{I}_1$ and $\mathcal{I}_2$ and at the root R . Player P_2 maximizes the payoff at the nodes A and B .
- For a subgame perfect Nash equilibrium, both players maximize their payoffs for the subgames starting at the nodes A , B , and R .

6. (25 Points) Consider the game tree with perfect information given in Figure 2.
- What is the subgame perfect Nash equilibrium in pure strategies? Give the complete strategy profile.
 - Find a strategy profile that is a Nash equilibrium but not subgame perfect.

Answer:

(a) At node A , P_2 chooses AD : $u_2(AD) = 7$, which is greater than $u_2(AC) = 6$ and $u_2(AE) = 5$.

At node B , P_2 chooses BF : $u_2(BF) = 6$, which is greater than $u_2(BG) = 3$ and $u_2(BH) = 1$.

At node R , P_1 chooses RA : $u_1(RA, AD, BF) = 4$, which is greater than $u_1(RB, AD, BF) = 3$.

(b) The strategy profile (RA, AD, BG) is a Nash equilibrium which is not subgame perfect.

It is not subgame perfect, because $u_2(BG) = 3 < u_2(BF) = 6$. Thus, it is not a Nash equilibrium on the subgame starting at B .

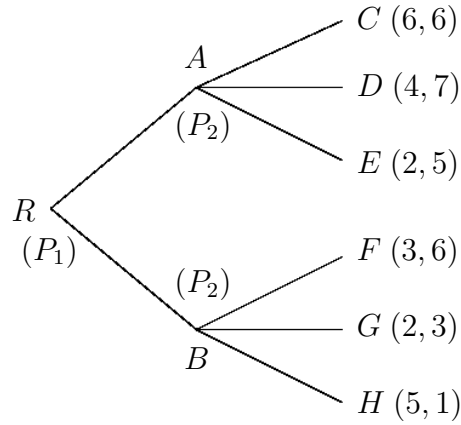


FIGURE 2

It is a Nash equilibrium because

$$u_1(RA, AD, BG) = 4 > u_1(RB, AD, BG) = 2 \quad \text{and}$$

$$u_2(RA, AD, BG) = 7 \begin{cases} > u_2(RA, AC, B^*) = 6 \\ = u_2(RA, AD, B^*) = 7 \\ > u_2(RA, AE, B^*) = 5. \end{cases}$$

These are the conditions necessary for the strategy profile to be a Nash equilibrium on the whole game.

7. (25 Points) (Selten's horse) Consider the game tree with three players given in Figure 3.
- For $x = y = 1$, show that $\mu_3(N_2) = 1/3$ and $\mu_3(N_3) = 2/3$ is a consistent system of beliefs on $\mathcal{I} = \{N_2, N_3\}$. Show this very explicitly using the definition of a consistent system of beliefs.
 - For the system of beliefs $\mu_3(N_2) = 1/3$ and $\mu_3(N_3) = 2/3$, what values of z are optimal for P_3 on $\mathcal{I}$?
 - Consider the node N_1 owned by P_2 . What values of z allow $y = 1$ to be optimal?
 - Consider the root N_0 owned by P_1 . Assume that $y = 1$. What values of z allow $x = 1$ to be optimal?
 - For $x = y = 1$, which values of z give a sequential equilibrium?

Answer:

- (a) Take the values x_n and y_n satisfying $1 - x_n = 1/n$ and $1 - y_n = 2/n$. Then, the consistent

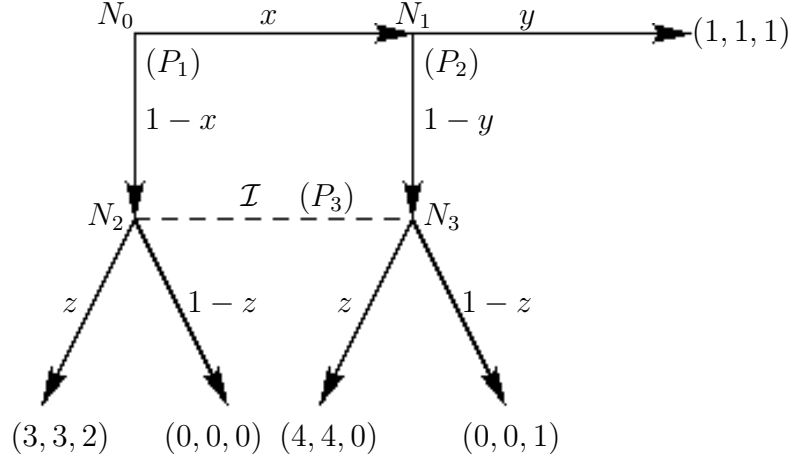


FIGURE 3

system of beliefs is

$$\begin{aligned}\mu_n(N_2) &= \frac{\frac{1}{n}}{\frac{1}{n} + \left(1 - \frac{1}{n}\right) \left(\frac{2}{n}\right)} \\ &= \frac{1}{1 + 2 - \frac{2}{n}} = \frac{1}{3 - \frac{2}{n}}, \\ \mu_n(N_3) &= \frac{\left(1 - \frac{1}{n}\right) \left(\frac{2}{n}\right)}{\frac{1}{n} + \left(1 - \frac{1}{n}\right) \left(\frac{2}{n}\right)} \\ &= \frac{2 - \frac{2}{n}}{1 + 2 - \frac{2}{n}} = \frac{2 - \frac{2}{n}}{3 - \frac{2}{n}}.\end{aligned}$$

Then, $\mu_n(N_2)$ converges to $1/3$ and $\mu_n(N_3)$ converges to $1/3$ as n goes to infinity.

(b)

$$E_2(\mathcal{I}) = \frac{1}{3}(2z + (1-z)(0)) + \frac{2}{3}(z(0) + (1-z)1) = \frac{2}{3}$$

is independent of z , so any value of z is optimal.

(c)

$$\begin{aligned}E_2(N_1) &= y(1) + (1-y)(4z + (1-z)(0)) = 4z + y(1-4z), \\ \frac{\partial E_2(N_1)}{\partial y} &= 1 - 4z,\end{aligned}$$

so $y = 1$ is optimal for $z \leq 1/4$.

(d)

$$\begin{aligned}E_1(N_0) &= x(1)(1) + (1-x)(3z + (1-z)(0)) = 3z + x(1-3z), \\ \frac{\partial E_1(N_0)}{\partial x} &= 1 - 3z,\end{aligned}$$

so $x = 1$ is optimal for $z \leq 1/3$.

(e) For $x = y = 1$ to be the optimal choices at N_0 and N_1 , and z to be the optimal choice on $\mathcal{I}$, we need z to satisfy the conditions of parts (b), (c), and (d), so $z \leq 1/4$.

8. (25 Points) (Aficionado versus fan) A person, P_1 , attempting to buy a rare document can be either a mere fan F or an aficionado A (super-fan). Assume that the proportion of aficionados and fans are both $1/2$. The value to an aficionado is b and to a mere fan is a , with $0 < a < b$. The potential buyer sends a signal which can be either “I am an aficionado” (s_H or s'_H) or “I am only a fan” (s_L or s'_L).

The holder of the document, P_2 , does not know if the person attempting to buy the document is an aficionado or a fan but only the signal sent by the potential buyer, i.e., only if the node is in the information set $\mathcal{I}_H = \{A_H, F_H\}$ (for a buyer who claims to be an aficionado) or $\mathcal{I}_L = \{A_L, F_L\}$ (for a buyer who claims to be only a fan). He decides that he will either (H) offer the document at the high price of $b/2$, or (L) offer the buyer a 50% chance of buying the document at the low price of $a/2$. The payoffs are given in the game tree in Figure 4.

Assume that $b > 2a > 0$, and that the behavior strategies satisfy $\sigma(s_H) = 1$, $\sigma(s_L) = 0$, $\sigma(s'_H) < 1$, and $\sigma(s'_L) > 0$.

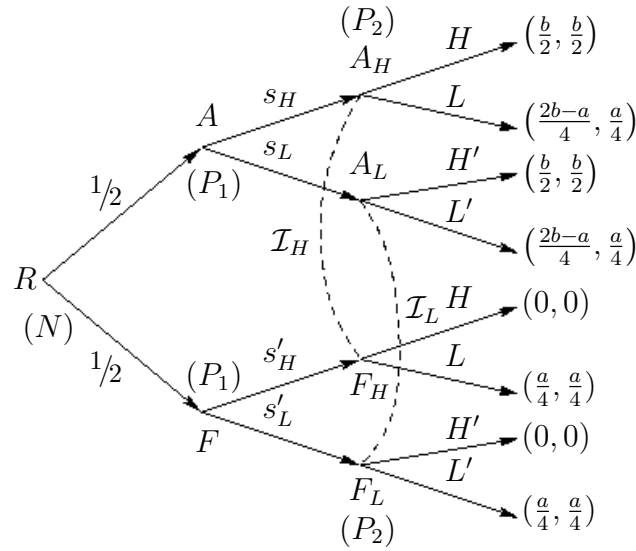


FIGURE 4

- What is a consistent system of beliefs on $\mathcal{I}_L$?
- On $\mathcal{I}_L$, which is the better choice by P_2 between H' and L' ?
- What can you say about a consistent system of beliefs on $\mathcal{I}_H$?
- What is the expected payoff for P_2 on $\mathcal{I}_H$ for H and L ? On $\mathcal{I}_H$, which is the better choice by P_2 between H and L ?
- At F , which gives the higher payoff for P_1 between s'_H and s'_L ?

Answer:

(a)

$$\begin{aligned}\mu(F_L) &= \frac{\frac{1}{2}\sigma(s'_L)}{\frac{1}{2}\sigma(s'_L) + \frac{1}{2}\sigma(s_L)} \\ &= \frac{\sigma(s'_L)}{\sigma(s'_L)} = 1, \\ \mu(A_L) &= \frac{\frac{1}{2}\sigma(s_L)}{\frac{1}{2}\sigma(s'_L) + \frac{1}{2}\sigma(s_L)} \\ &= \frac{0}{\sigma(s'_L)} = 0.\end{aligned}$$

(b)

$$\begin{aligned}E_2(H') &= \mu(F_L)(0) + \mu(A_L)\left(\frac{b}{2}\right) = 0 \quad \text{and} \\ E_2(L') &= \mu(F_L)\left(\frac{a}{4}\right) + \mu(A_L)\left(\frac{2b-a}{4}\right) = \frac{a}{4}.\end{aligned}$$

Since $a/4 > 0$, L' is a better choice.

(c)

$$\begin{aligned}\mu(A_H) &= \frac{\frac{1}{2}\sigma(s_H)}{\frac{1}{2}\sigma(s_H) + \frac{1}{2}\sigma(s'_H)} \\ &= \frac{1}{1 + \sigma(s'_H)} > \frac{1}{2}, \\ \mu(F_H) &= \frac{\frac{1}{2}\sigma(s'_H)}{\frac{1}{2}\sigma(s_H) + \frac{1}{2}\sigma(s'_H)} \\ &= \frac{\sigma(s'_H)}{1 + \sigma(s'_H)} < \frac{1}{2}, \\ \mu(A_H) &> \mu(F_H).\end{aligned}$$

(d)

$$\begin{aligned}E_2(H) &= \mu(A_H)\left(\frac{b}{2}\right) + \mu(F_H)(0) = \mu(A_H)\left(\frac{b}{2}\right) \\ E_2(L) &= \mu(A_H)\left(\frac{a}{4}\right) + \mu(F_H)\left(\frac{a}{4}\right) = \frac{a}{4}.\end{aligned}$$

Since $\mu(A_H)\left(\frac{b}{2}\right) > \frac{b}{4} > \frac{a}{4}$, H is the optimal choice.

(e)

$$\begin{aligned}E_1(s'_H, F) &= 0 \\ E_1(s'_L, F) &= \frac{a}{4},\end{aligned}$$

so s'_L is the optimal choice.