

1. (18 Points) The payoffs (prizes) of some lotteries are either \$0m (zero million dollars), \$1m (one million dollars), or \$5m (five million dollars). Assume the utility function u satisfies $u(\$0m) = 0$, $u(\$5m) = 5$, and $u(\$0m) = 0 < u(\$1m) < 5 = u(\$5m)$. Let U be the von Neumann-Morgenstern utility on lotteries. Consider the four lotteries

$$\begin{aligned} J &= \{ (\$0m, 0.00), (\$1m, 1.00), (\$5m, 0.00) \}, \\ K &= \{ (\$0m, 0.01), (\$1m, 0.89), (\$5m, 0.10) \}, \\ L &= \{ (\$0m, 0.89), (\$1m, 0.11), (\$5m, 0.00) \}, \quad \text{and} \\ M &= \{ (\$0m, 0.90), (\$1m, 0.00), (\$5m, 0.10) \}, \end{aligned}$$

where $(\$1m, 0.10)$ says that the payoff \$1 million occurs with probability 0.10, etc.

- a. If $U(J) > U(K)$, then what can you say about the value $u(\$1m)$?
 b. Given that $U(J) > U(K)$, which is larger between $U(L)$ and $U(M)$?

Answer:

(a)

$$\begin{aligned} U(L) &= u(1m) \\ &> U(K) = 0.01 u(0) + 0.89 u(1m) + 0.1 u(5m) \\ &= 0.89 u(1m) + 0.5 \\ 0.11 u(1m) &> 0.5 \\ u(1m) &> \frac{5}{11}. \end{aligned}$$

(b) $U(L) = 0.11 u(1m)$ and $U(M) = 0.1 u(5m) = 0.5$, so $U(L) > U(M)$.

2. (18 Points) Two firms produce slightly different products in quantities q_1 and q_2 . The (inverse) demand curves for prices are

$$\begin{aligned} p_1 &= 16 - 2q_1 - q_2 \quad \text{and} \\ p_2 &= 14 - q_1 - 2q_2. \end{aligned}$$

Suppose that the cost of producing a unit of output is 2 for the first firm and 3 for the second firm. The profit for each firm is

$$\begin{aligned} \pi_1 &= (16 - 2q_1 - q_2)q_1 - 2q_1 = (14 - 2q_1 - q_2)q_1 \quad \text{and} \\ \pi_2 &= (14 - q_1 - 2q_2)q_2 - 3q_2 = (11 - q_1 - 2q_2)q_2. \end{aligned}$$

Find the level of production for each firm which gives a Nash equilibrium for the situation. Verify that the function is maximized in the correct sense at the solution.

Answer:

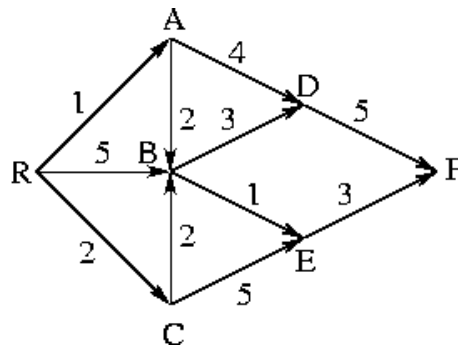
$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} &= -4q_1 + 14 - q_2 = 0 \\ \frac{\partial \pi_2}{\partial q_2} &= -4q_2 - 11 - q_1 = 0 \\ 16q_1 + 4q_2 &= 56 \\ q_1 + 4q_2 &= 11 \\ 15q_1 &= 45 \\ \hat{q}_1 &= 3 \\ \hat{q}_2 &= 14 - 4(3) = 2.\end{aligned}$$

The second derivatives are negative,

$$\begin{aligned}\frac{\partial^2 \pi_1}{\partial q_1^2} &= -4 < 0 \\ \frac{\partial^2 \pi_2}{\partial q_2^2} &= -4 < 0\end{aligned}$$

so these are maximum.

3. (18 Points) A person has the decision graph given below, where the numbers on the edges represent the cost of moving along that edge. What choice should the person make at each vertex to minimize the cost from that vertex to the terminal node F? What path should the person take to minimizing the cost from the root R to the terminal node F? What is the cost of this path from R to F? Be sure and indicated the order considering the vertices at which to make the choice.



Answer:

- D goes only to F , so it has a least cost of 5.
- E goes only to F , so it has a least cost of 3.
- B : If B goes to D , then it has a cost of $3 + 5 = 8$.
- If B goes to E , then it has a cost of $1 + 3 = 4$.
- Therefore B goes to E with a least cost of 4.

A: If *A* goes to *D*, then it has a cost of $4 + 5 = 9$.

. If *A* goes to *B*, then it has a cost of $2 + 4 = 6$.

. Therefore *A* goes to *B* with a least cost of 6.

C: If *C* goes to *B*, then it has a cost of $2 + 4 = 6$.

. If *C* goes to *E*, then it has a cost of $5 + 3 = 8$.

. Therefore *C* goes to *B* with a least cost of 6.

R: If *R* goes to *A*, then it has a cost of $1 + 6 = 7$.

. If *R* goes to *B*, then it has a cost of $5 + 4 = 9$.

. If *R* goes to *C*, then it has a cost of $2 + 6 = 8$.

. Therefore, *R* goes to *A*, with a least cost of 7.

The pay of the choice with the least cost is *RABEF* and the cost is 7.

4. Consider the bi-matrix game given by

$$\begin{pmatrix} (0, 3) & (4, 0) & (2, 0) \\ (0, 1) & (0, 1) & (3, 2) \\ (3, 0) & (0, 5) & (1, 0) \end{pmatrix}.$$

- a. (10 Points) Find all the pure strategy Nash equilibria. Specify the strategy of the choice of the row and the column that gives the Nash equilibria. Also give the payoffs to the two players.
- b. (18 Points) Find all the interior mixed strategy Nash equilibrium. Give the strategies which give the Nash equilibrium and the payoffs at the Nash equilibrium.

Answer:

(a) The second row and third column give the only Nash equilibrium with payoffs of 3 and 2.

(b)

$$\begin{aligned} \pi_1 &= 4p_1q_2 + 2p_1(1 - q_1 - q_2) + 3p_2(1 - q_1 - q_2) \\ &\quad + 3(1 - p_1 - p_2)q_1 + (1 - p_1 - p_2)(1 - q_1 - q_2) \\ \frac{\partial \pi_1}{\partial p_1} &= 4q_2 + 2(1 - q_1 - q_2) - 3q_1 - (1 - q_1 - q_2) \\ &= 1 - 4q_1 + 3q_2 = 0 \\ \frac{\partial \pi_1}{\partial p_2} &= 3(1 - q_1 - q_2) - 3q_1 - 1 + q_2 \\ &= 2 - 5q_1 - 2q_2 = 0. \end{aligned}$$

The solution is

$$\hat{q}_1 = \frac{8}{23}, \quad \hat{q}_2 = \frac{3}{23}, \quad \hat{q}_3 = \frac{12}{23}.$$

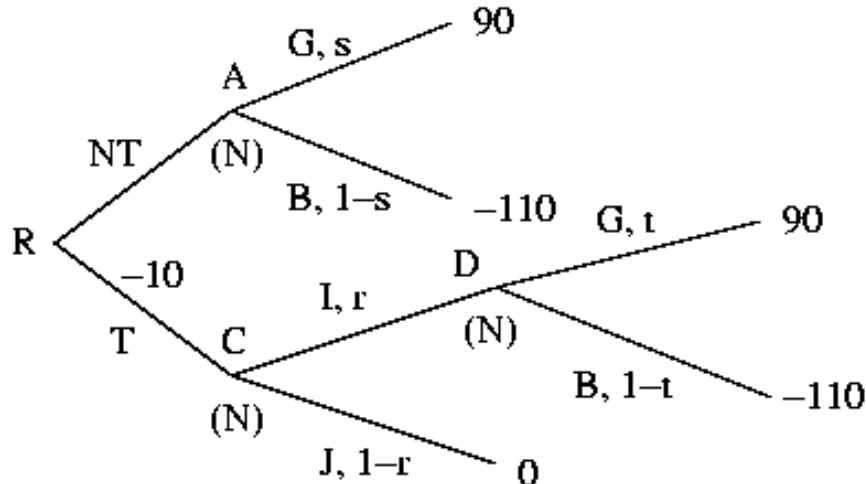
$$\begin{aligned}
\pi_2 &= 3p_1q_1 + p_2q_1 + p_2q_2 \\
&\quad + 2p_2(1 - q_1 - q_2) + 5(1 - p_1 - p_2)q_2 \\
\frac{\partial \pi_2}{\partial q_1} &= 3p_1 + p_2 - 2p_2 \\
&= 3p_1 - p_2 = 0 \\
p_2 &= 3p_1 \\
\frac{\partial \pi_2}{\partial q_2} &= p_2 - 2p_2 + 5(1 - p_1 - p_2) \\
&= 5 - 5p_1 - 6p_2 = 0 \\
&= 5 - 5p_1 - 18p_1 = 5 - 23p_1 \\
\hat{p}_1 &= \frac{5}{23} \\
\hat{p}_2 &= 3 \cdot \frac{5}{23} = \frac{15}{23} \\
\hat{p}_3 &= 1 - \frac{5}{23} - \frac{15}{23} = \frac{3}{23}.
\end{aligned}$$

The solution is

$$\hat{p}_1 = \frac{5}{23}, \quad \hat{p}_2 = \frac{15}{23}, \quad \hat{p}_3 = \frac{3}{23}.$$

5. (18 Points) Should the person decide to market test or not in the following decision tree, where the market test costs 10 units? Let I be a positive market test and J a negative market test, with $r = P(I) = \frac{4}{7}$ and $P(J) = 1 - r = \frac{3}{7}$. If there a negative market test, then the product is not marketed and there is a payoff of 0. Let G be a good reception of the product and B a bad reception. Let $t = P(G|I) = 0.9$, and $P(G|J) = 0.2$. Note that with no market test $s = 0.9(\frac{4}{7}) + 0.2(\frac{3}{7}) = 0.6$. The payoffs at the terminal nodes are indicated in the figure.

- Calculate the expected payoffs at the nodes owned by “nature”, A , D , and C .
- Given that the test costs 10 units, should the person decide to test (T) or not test (NT) at vertex R ?



Answer:

$$\begin{aligned} E(A) &= 0.6(90) + 0.4(-110) = 54 - 44 \\ &= 10 \end{aligned}$$

$$\begin{aligned} E(D) &= 0.9(90) + 0.1(-110) = 81 - 11 \\ &= 70 \end{aligned}$$

$$\begin{aligned} E(C) &= \frac{4}{7}(70) + \frac{3}{7}(0) \\ &= 40 \end{aligned}$$

$$E(T) = E(C) - 10 = 30$$

$$E(NT) = E(A) = 10.$$

Since $E(T) = 30 > 10 = E(NT)$, it is better to test.