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1. (30 Points) Consider the game tree with perfect information given in Figure 1.
  - a. What is the subgame perfect Nash equilibrium in pure strategies? Give the complete strategy profile.
  - b. Find a strategy profile that is a Nash equilibrium but not subgame perfect.

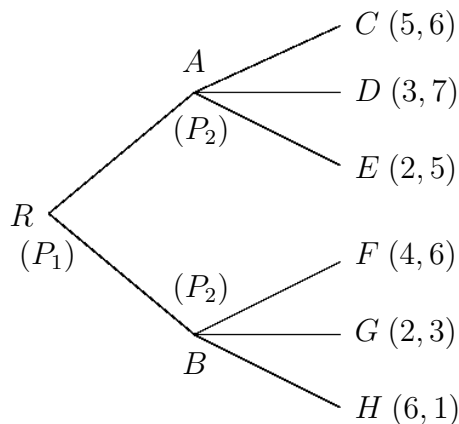


FIGURE 1

*Answer:*

(a) At A, the payoff  $\pi_2(AD) = 7$  is greater than  $\pi_2(AC) = 6$  and  $\pi_2(AE) = 5$ , so choose AD.

At B, payoff  $\pi_2(BF) = 6$  is greater than  $\pi_2(BG) = 3$  and  $\pi_2(BH) = 1$ , so choose BF.

At R,  $\pi_1(RB, AD, BF) = 4 > \pi_1(RA, AD, BF) = 3$ , so choose RB.

Therefore, the subgame perfect Nash equilibrium is  $\{RB, AD, BF\}$ .

(b) The strategy  $\{RB, AE, BF\}$  is not subgame perfect, because starting at A  $\pi_2(AD) = 7 > 5 = \pi_2(AE)$ .

This strategy is a Nash equilibrium, because

$$\begin{aligned} \pi_1(RB, AE, BF) &= 4 > \pi_1(RA, AE, BF) = 2, \\ \pi_2(RB, AE, BF) &= 6 > \pi_2(RB, A^*, BG) = 3 \\ &> \pi_2(RB, A^*, BH) = 1 \\ &= \pi_2(RB, A^*, BF), \end{aligned}$$

where  $A^*$  is any choice of AC, AD, or AE.

2. (30 Points) Consider the game given in Figure 2. Nature owns the root and has a  $\frac{1}{2}$  probability of selecting the edge to node  $A$  and  $\frac{1}{2}$  probability of selecting the edge to node  $B$ . The other behavior strategy variables are given in the figure.
- For  $(p, q) \neq (0, 0)$ , what is the consistent system of beliefs for  $P_2$  on the information set  $\mathcal{I} = \{C, D\}$ ?
  - If  $(p, q) = (0, 0)$ , show that any system of beliefs on  $\mathcal{I}$  is consistent.
  - For  $(p, q) \neq (0, 0)$ , what is the expected payoff for  $P_2$  on  $\mathcal{I}$ ? What choice of  $x$  maximizes the expected payoff? Include the case  $(p, q) = (0, 0)$  and give the answer in terms of various cases of  $p$  and  $q$ .

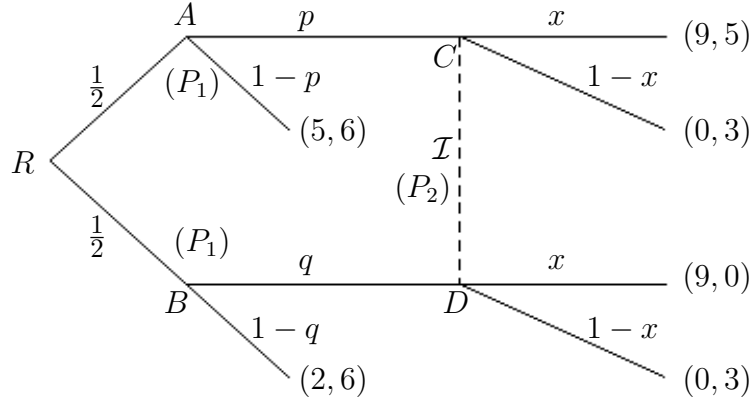


FIGURE 2

Answer:

(a)

$$\mu_2(C) = \frac{\frac{p}{2}}{\frac{p}{2} + \frac{q}{2}} = \frac{p}{p+q},$$

$$\mu_2(D) = \frac{\frac{q}{2}}{\frac{p}{2} + \frac{q}{2}} = \frac{q}{p+q}.$$

(b) Let  $0 \leq t \leq 1$  be any possible value for the belief at  $C$ . Let  $\sigma_n(AC) = \frac{t}{n}$  and  $\sigma_n(BD) = \frac{1-t}{n}$ . Then, both  $\sigma_n(AC)$  and  $\sigma_n(BD)$  converge to 0 as  $n$  goes to infinity. Also,

$$\mu_{2,n}(C) = \frac{\frac{t}{2n}}{\frac{t}{2n} + \frac{1-t}{2n}} = t \quad \text{and}$$

$$\mu_{2,n}(D) = \frac{\frac{1-t}{2n}}{\frac{t}{2n} + \frac{1-t}{2n}} = 1 - t.$$

As  $n$  goes to infinity, these converge to the beliefs  $t$  and  $1 - t$ . Since  $t$  is arbitrary, this shows the result.

(c)

$$\begin{aligned}
 E_2(\mathcal{I}) &= \frac{p}{p+q} [5x + 3(1-x)] + \frac{q}{p+q} [0x + 3(1-x)] \\
 &= \frac{p}{p+q} [3 + 2x] + \frac{q}{p+q} [3 - 3x], \\
 \frac{\partial E_2(\mathcal{I})}{\partial x} &= \frac{2p - 3q}{p+q}.
 \end{aligned}$$

For  $p = q = 0$ , the beliefs are arbitrary, and the choice of  $x$  is arbitrary.

For  $(p, q) \neq (0, 0)$ , the optimal choice of  $x$  is 0 if  $2p < 3q$ , arbitrary if  $2p = 3q$ , and 1 if  $2p > 3q$ .

3. (40 Points) Consider the sequential game with imperfect information given in Figure 3. Assume that the derivative of the expected values of the payoffs using the system of consistent beliefs on the information sets  $\mathcal{I}_1 = \{X, Y\}$ ,  $\mathcal{I}_2 = \{Z, F\}$ ,  $P_1$ , and  $\mathcal{I}_3 = \{G, H\}$  and node  $R$  as follows.

$$\begin{aligned}
 \frac{\partial E_1(R, \sigma)}{\partial x} &= p(5q - 1) - r(4q - 2) - 2, \\
 \frac{\partial E_2(\mathcal{I}_1, \sigma, \mu^\sigma)}{\partial q} &= 1 - 6r - x(4 + 2p - 6r) \equiv \Delta, \\
 \frac{\partial E_1(\mathcal{I}_2, \sigma, \mu^\sigma)}{\partial p} &= -1 + 5q, \\
 \frac{\partial E_1(\mathcal{I}_3, \sigma, \mu^\sigma)}{\partial r} &= 4q - 2,
 \end{aligned}$$

Note: The systems of beliefs on the information sets are completely determined by the behavior strategy in this game, so you do not need to worry about the beliefs being arbitrary.

- a. (25 Points) Find the sequential equilibrium with  $q = 1/2$ . Hint: Solve for the other behavior strategies in the following order:  $p$ ,  $x$ , and then  $r$ .  
b. (15 Points) Why is  $q = 1/2$ ,  $r = 1/6$ ,  $x = 0$ , and  $p = 0$  a Nash equilibrium in behavior strategies but **NOT** a sequential equilibrium? Is this second solution subgame perfect?

Answer:

(a) Because

$$\frac{\partial E_1(\mathcal{I}_2, \sigma, \mu^\sigma)}{\partial p} = -1 + 5/2 = 3/2 > 0,$$

the optimal choice of  $p$  is 1.

Then,

$$\frac{\partial E_1(R, \sigma)}{\partial x} = p(5q - 1) - r(4q - 2) - 2 = 1(3/2) - r(2 - 2) - 2 = -1/2 < 0,$$

so the optimal choice of  $x$  is 0.

In order for  $q = 1/2$  to be an allowable choice, we need  $0 = \Delta = 1 - 6r - 0 = 1 - 6r$ .

Therefore,  $r$  must equal  $1/6$ .  $\frac{\partial E_1(\mathcal{I}_3, \sigma, \mu^\sigma)}{\partial r} = 0$  so  $r = 1/6$  is allowable.

This gives all the values of the behavior strategies for a sequential equilibrium.

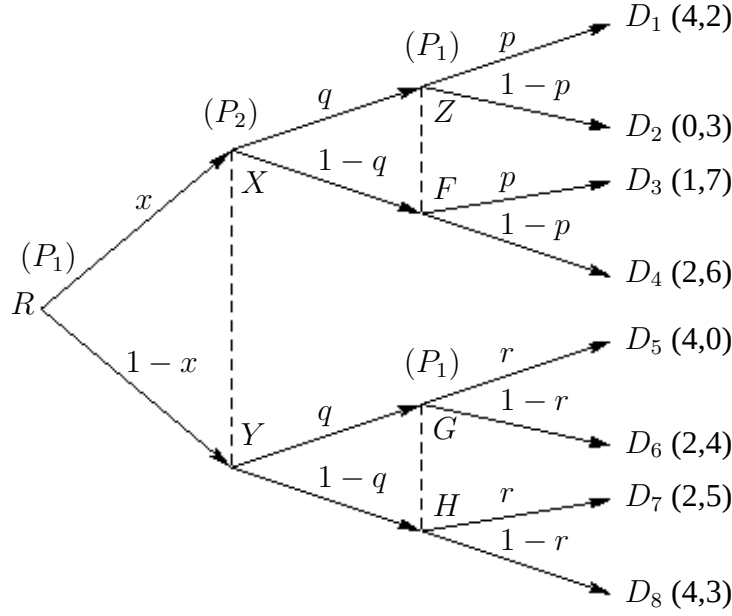


FIGURE 3

(b) On the information set  $\mathcal{I}_2$  the choice  $p = 0$  is not optimal. Therefore, this is not a sequential equilibrium.

Now we check that the values given satisfy the conditions to be a Nash equilibrium. Fix the choices of  $P_1$ :  $x = 0$ ,  $r = 1/6$ , and  $p = 0$ . The derivative  $\frac{\partial E_2(\mathcal{I}_1, \sigma, \mu^\sigma)}{\partial q} = 0$ , so the payoff is independent of the choice of  $q$ . Thus, the choice of  $q = 1/2$  is optimal for  $P_2$  given the choices of  $P_1$ .

Next, fix the choice  $q = 1/2$  of  $P_2$ . Then,

$$\frac{\partial E_1(R, \sigma)}{\partial x} = \frac{3p}{2} - 0 - 2 < 0,$$

so  $x = 0$  is the optimal choice of  $P_1$  at  $R$  for any choice of  $p$  by  $P_1$  on  $\mathcal{I}_2$ . Once  $x = 0$ , then the value of  $p$  does not affect the payoff for  $P_1$  at  $R$  so  $p = 0$  is an optimal choice on the whole game. When  $q = 1/2$ , then  $\frac{\partial E_1(\mathcal{I}_3, \sigma, \mu^\sigma)}{\partial r} = 0$ , so the choice of  $r$  does not affect the payoff for  $P_1$ . Therefore,  $r = 1/6$  is an optimal choice for  $P_1$ .

We have checked that these choices satisfy the conditions to be a Nash equilibrium.

There are no subgames, so these choices are a subgame perfect Nash equilibrium.