

1. Consider the bi-matrix game given by

$$\begin{pmatrix} (1, 0) & (0, 2) & (0, 0) & (0, 0) \\ (0, 0) & (3, 0) & (0, 1) & (0, 0) \\ (0, 4) & (0, 0) & (6, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0, 0) & (3, 2) \end{pmatrix}.$$

- a. (15 Points) Are there any pure strategy Nash equilibria? Explain by giving the best response in pure strategies of each player to the pure strategies of the other player.
- b. (25 Points) Consider mixed strategies, where player one (who chooses the row) has a strategy (p_1, p_2, p_3, p_4) and player two (who chooses the column) has a strategy (q_1, q_2, q_3, q_4) .
- (i) For $p_4 = 0 = q_4$ and $1 \leq i \leq 4$, find the expected payoffs $E_1(r_i, (q_1, q_2, q_3, 0))$ for rows r_i and $E_2((p_1, p_2, p_3, 0), c_i)$ for columns c_i .
- (ii) Find a mixed strategy Nash equilibrium with $p_4 = 0 = q_4$ and $p_i > 0$ and $q_i > 0$ for $1 \leq i \leq 3$. Indicate why it is a Nash equilibrium.

Answer:

(a) The best responses are underlined in the following matrix:

$$\begin{pmatrix} (\underline{1}, 0) & (0, \underline{2}) & (0, 0) & (0, 0) \\ (0, 0) & (\underline{3}, 0) & (0, \underline{1}) & (0, 0) \\ (0, \underline{4}) & (0, 0) & (\underline{6}, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0, 0) & (\underline{3}, \underline{2}) \end{pmatrix}.$$

The only common entry is row 4 and column 4, which is a pure strategy Nash equilibrium.

(b) The expected payoffs for the rows and columns are as follows:

$$\begin{aligned} E_1(r_1, (q_1, q_2, q_3, 0)) &= q_1 & E_2((p_1, p_2, p_3), c_1) &= 4p_3 \\ E_1(r_2, (q_1, q_2, q_3, 0)) &= 3q_2 & E_2((p_1, p_2, p_3), c_2) &= 2p_1 \\ E_1(r_3, (q_1, q_2, q_3, 0)) &= 6q_3 & E_2((p_1, p_2, p_3), c_3) &= p_2 \\ E_1(r_4, (q_1, q_2, q_3, 0)) &= 0 & E_2((p_1, p_2, p_3), c_4) &= 0. \end{aligned}$$

To be a mixed strategy Nash equilibrium, these expected values for the first three rows must be equal:

$$\begin{aligned} q_1 &= 3q_2 = 6q_3 \geq 0 \\ 1 &= q_1 + q_2 + q_3 = q_3(6 + 2 + 1) \\ q_3 &= 1/9 \\ q_1 &= 6/9 = 2/3 \\ q_2 &= 2/9. \end{aligned}$$

To be a mixed strategy Nash equilibrium, these expected values for the first three columns must be equal:

$$\begin{aligned} 4p_3 &= 2p_1 = p_2 \geq 0 \\ 1 &= p_1 + p_2 + p_3 = p_3(2 + 4 + 1) \\ p_3 &= 1/7 \\ p_1 &= 2/7 \\ p_2 &= 4/7. \end{aligned}$$

2. (25 Points) Consider the second-price sealed-bid auction with three players with valuations $v_i > 0$. Let $b_i \geq 0$ be the bid by the i^{th} -player. Let $m_{-1} = \max\{b_2, b_3\}$ be the maximum of the bids by players two and three. The payoff for the first player is given by

$$u_1(b_1, b_2, b_3) = \begin{cases} v_1 - m_{-1} & \text{if } b_1 > m_{-1} \\ \frac{1}{r} (v_1 - m_{-1}) & \text{if } b_1 = m_{-1} \text{ with } r \text{ finalists} \\ 0 & \text{if } b_1 < m_{-1}. \end{cases}$$

The payoff functions for the other players is similar.

What is the best response function for player one in terms of the different values of m_{-1} ?

Hint: Consider $0 \leq m_{-1} < v_1$, $m_{-1} = v_1$, and $m_{-1} > v_1$.

Answer:

(i) For $0 \leq m_{-1} < v_1$,

$$u_1(b_1, m_{-1}) = \begin{cases} v_1 - m_{-1} & \text{if } b_1 > m_{-1} \\ \frac{1}{r} (v_1 - m_{-1}) & \text{if } b_1 = m_{-1} \\ 0 & \text{if } b_1 < m_{-1}, \end{cases}$$

so $B_1(m_{-1}) = (m_{-1}, \infty)$.

(ii) For $m_{-1} = v_1$,

$$u_1(b_1, m_{-1}) = \begin{cases} v_1 - m_{-1} = 0 & \text{if } b_1 > m_{-1} \\ \frac{1}{r} (v_1 - m_{-1}) = 0 & \text{if } b_1 = m_{-1} \\ 0 & \text{if } b_1 < m_{-1}, \end{cases}$$

so $B_1(m_{-1}) = [0, \infty)$.

(iii) For $m_{-1} > v_1$,

$$u_1(b_1, m_{-1}) = \begin{cases} v_1 - m_{-1} < 0 & \text{if } b_1 > m_{-1} \\ \frac{1}{r} (v_1 - m_{-1}) < 0 & \text{if } b_1 = m_{-1} \\ 0 & \text{if } b_1 < m_{-1}, \end{cases}$$

so $B_1(m_{-1}) = [0, m_{-1})$.

3. (25 Points) For a collection of lotteries, there are three possible outcomes, a_1 , a_2 , and a_3 . The payoff function satisfies $u(a_1) = 0$, $u(a_2) = v$, and $u(a_3) = 100$. A lottery (p_1, p_2, p_3) indicates a lottery in which the probability of getting outcome a_i is p_i . The decision maker strictly prefers the lottery $(0.4, 0.2, 0.4)$ to $(0, 1, 0)$ to $(0.3, 0.6, 0.1)$. Are these preferences consistent? And, what can you say about the value of $v = u(a_2)$?

Answer:

The payoffs for the lotteries are as follows:

$$U(0.4, 0.2, 0.4) = 0.2v + 0.4(100) = \frac{v}{5} + 40$$

$$U(0, 1, 0) = v$$

$$U(0.3, 0.6, 0.1) = 0.6v + 0.1(100) = \frac{3v}{5} + 10.$$

Therefore, we need

$$\begin{aligned} \frac{v}{5} + 40 &> v \\ 40 &> \frac{4v}{5} \\ 50 &> v \end{aligned}$$

and

$$\begin{aligned} v &> \frac{3v}{5} + 10 \\ \frac{2v}{5} &> 10 \\ v &> 25. \end{aligned}$$

These are consistent, and we need $25 < v < 50$.

4. (30 Points) Consider the infinitely repeated game with discount $0 < \delta < 1$ of the Prisoner's Dilemma with one time payoffs given as follows:

$$\begin{array}{cc} & \begin{array}{cc} C_2 & D_2 \end{array} \\ \begin{array}{c} C_1 \\ D_1 \end{array} & \begin{pmatrix} (10, 10) & (0, y) \\ (y, 0) & (1, 1) \end{pmatrix}, \end{array}$$

where $10 < y$. Assume that player one strictly prefers (i) the infinitely repeated choice of (C_1, C_2) to

(ii) the infinitely alternating choices of (D_1, C_2) and (C_1, D_2) starting with (D_1, C_2) .

a. With these assumptions, what inequality must y and δ satisfy?

b. Given that $\delta < 1$, why must $y < 20$?

c. What range of δ are compatible with $y = 16$?

Answer:

(a) The payoffs are

$$\begin{aligned} U_1((C_1, C_2)^\infty) &= 10 + 10\delta + 10\delta^2 + \dots \\ &= \frac{10}{1 - \delta} \quad \text{and} \\ U_1([(D_1, C_2), (C_1, D_2)]^\infty) &= y + 0 \cdot \delta + y\delta^2 + 0 \cdot \delta^3 + y\delta^4 + \dots \\ &= \frac{10}{1 - \delta^2}. \end{aligned}$$

Therefore, we need

$$\begin{aligned} \frac{10}{1 - \delta} &> \frac{10}{1 - \delta^2} \\ 10 + 10\delta &> y. \end{aligned}$$

(b) Thus, $y < 10 + 10\delta < 20$.

(c) If $y = 16$, then

$$\begin{aligned} 10 + 10\delta &> y = 16 \\ 10\delta &> 6 \\ \delta &> \frac{3}{5}. \end{aligned}$$

5. (Aficionado versus fan) A person, P_1 , attempting to buy a rare document can be either a mere fan F or an aficionado A (super-fan). Assume that the proportion of aficionados is $1/3$ and that of fans is $2/3$. The value of the document to an aficionado is 12 and to a mere fan is 4. The potential buyer sends a signal which can be either "I am an aficionado" (s_A or s'_A) or "I am only a fan" (s_F or s'_F).

The holder of the document, P_2 , does not know if the person attempting to buy the document is an aficionado or a fan but only the signal sent by the potential buyer, i.e., only if the node is in the information set $\mathcal{I}_A = \{s_A, s'_A\}$ (for a buyer who claims to be an aficionado) or $\mathcal{I}_F = \{s_F, s'_F\}$ (for a

buyer who claims to be only a fan). He decides that he will either (H) offer the document at the high price of $12/2 = 6$, or (L) offer the buyer a 50% chance of buying the document at the low price of 2. The payoffs are given in the game tree in Figure 1.

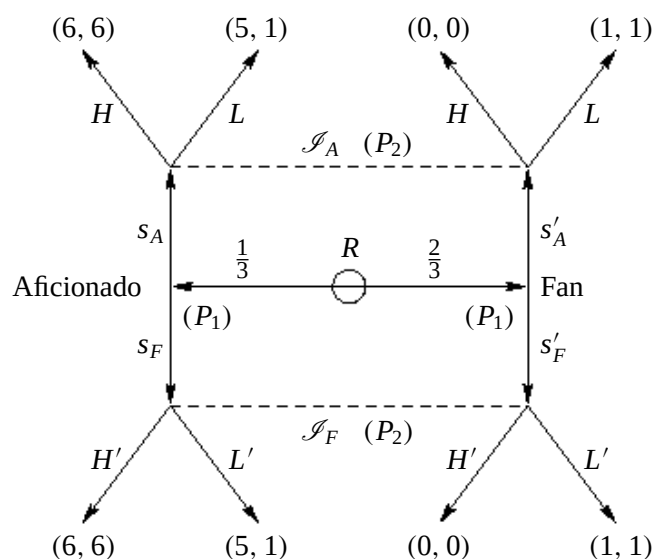


FIGURE 1. Game tree for aficionado versus fan

- a. (25 Points) Show that there is a separating equilibrium with $\beta_1(s_A) = 1$ and $\beta_1(s'_F) = 1$ as follows:
- What is the consistent belief system on $\mathcal{I}_A$?
 - On $\mathcal{I}_A$, is H or L the better choice for P_2 ? Explain why.
 - What is the consistent belief system on $\mathcal{I}_F$?
 - On $\mathcal{I}_F$, is H' or L' the better choice for P_2 ? Explain why.
 - Why is $\beta(s_A) = 1$ a rational choice for P_1 who is an aficionado?
 - Why is $\beta(s'_F) = 1$ a rational choice for P_1 who is a fan?
 - What is the complete assessment for this separating equilibrium?
- b. (25 Points) Show that there is a pooling equilibrium with $\beta_1(s_F) = 1$ and $\beta_1(s'_F) = 1$ as follows:
- What is the consistent belief system on $\mathcal{I}_F$?
 - On $\mathcal{I}_F$, is H' or L' a better choice for P_2 ? Explain why.
 - What are the expected values of $u_1(s'_F)$ and $u_1(s'_A)$ in terms of $\beta_2(H)$ and $\beta_2(L)$?
What must $\beta_2(H)$ and $\beta_2(L)$ be for $\beta_1(s'_F) = 1$ to be a rational choice for P_1 ?
 - What are the consistent belief system on $\mathcal{I}_A$ for P_2 ? What restrictions on the belief system are needed to make the values of $\beta_2(H)$ rational for P_2 on $\mathcal{I}_A$. Explain why.
 - What is the complete assessment for this pooling equilibrium?

Answer:

(a) (i)

$$\mu_2(s_A) = \frac{\frac{1}{3}}{\frac{1}{3} + 0} = 1 \quad \text{and}$$

$$\mu_2(s'_A) = \frac{0}{\frac{1}{3} + 0} = 0.$$

(ii) The payoffs are

$$E_2(H|\mathcal{I}_A) = 1(6) + 0(0) = 6 \quad \text{and} \\ E_2(L|\mathcal{I}_A) = 1(1) + 0(1) = 1.$$

Therefore H is a better choice, $\beta_2(H) = 1$.

(iii)

$$\mu_2(s_F) = \frac{0}{0 + \frac{2}{3}} = 0 \quad \text{and} \\ \mu_2(s'_F) = \frac{\frac{2}{3}}{0 + \frac{2}{3}} = 1.$$

(iv) The payoffs are

$$E_2(H'|\mathcal{I}_F) = 0(6) + 1(0) = 0 \quad \text{and} \\ E_2(L'|\mathcal{I}_F) = 0(1) + 1(1) = 1.$$

Therefore L' is a better choice, $\beta_2(L') = 1$.

(v) The expected payoffs for P_1 who is an aficionado are

$$E_1(s_A; \beta_2) = u_1(s_A, H) = 6 \quad \text{and} \\ E_1(s_F; \beta_2) = u_1(s_F, L') = 5.$$

Since $6 > 5$, s_A is the better choice, $\beta_1(s_A) = 1$.

(vi) The expected payoffs for P_1 who is a fan are

$$E_1(s'_A; \beta_2) = u_1(s'_A, H) = 0 \quad \text{and} \\ E_1(s'_F; \beta_2) = u_1(s'_F, L') = 1.$$

Since $1 > 0$, s'_F is the better choice, $\beta_1(s'_F) = 1$.

(vii) The complete assessment is

$$\begin{aligned} \beta_1(s_A) = 1 = \beta_1(s'_F) & \quad \beta_1(s_F) = 0 = \beta_1(s'_A) \\ \beta_2(H) = 1 = \beta_2(L') & \quad \beta_2(L) = 0 = \beta_2(H') \\ \mu_2(s_A) = 1 = \mu_2(s'_F) & \quad \mu_2(s_F) = 0 = \mu_2(s'_A). \end{aligned}$$

(b) (i) On $\mathcal{I}_F$,

$$\mu_2(s_F) = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{2}{3}} = \frac{1}{3} \quad \text{and} \\ \mu_2(s'_F) = \frac{\frac{2}{3}}{\frac{1}{3} + \frac{2}{3}} = \frac{2}{3}.$$

(ii) The payoffs are

$$E_2(H'|\mathcal{I}_F) = \frac{1}{3}(6) + \frac{2}{3}(0) = 2 \quad \text{and} \\ E_2(L'|\mathcal{I}_F) = \frac{1}{3}(1) + \frac{2}{3}(1) = 1.$$

Therefore H' is a better choice, $\beta_2(H') = 1$.

(iii) The expected payoffs for P_1 who is a fan are

$$E_1(s'_A; \beta_2) = \beta_2(H)(0) + \beta_2(L)(1) = \beta_2(L) \quad \text{and} \\ E_1(s'_F; \beta_2) = u_2(s'_F, H') = 0.$$

In order for $\beta_1(s'_A)$ to be rational, we need $0 \geq \beta_2(L)$ so $\beta_2(L) = 0$. It follows that $\beta_2(H) = 1$.

(iv) Since $\Pr(\mathcal{J}_A) = 0$, the belief system $\mu = \mu_2(s_A)$ and $1 - \mu = \mu_2(s'_A)$ is arbitrary. The expected payoffs on $\mathcal{J}_A$ are

$$\begin{aligned} E_2(H|\mathcal{J}_A) &= \mu(6) + (1 - \mu)(0) = 6\mu \quad \text{and} \\ E_2(L|\mathcal{J}_A) &= \mu(1) + (1 - \mu)(1) = 1. \end{aligned}$$

For $\beta_2(H) = 1$, we need $6\mu \geq 1$ or $\mu \geq 1/6$.

(v) The complete assessment is

$$\begin{aligned} \beta_1(s_F) &= 1 = \beta_1(s'_F) & \beta_1(s_A) &= 0 = \beta_1(s'_A) \\ \beta_2(H) &= 1 = \beta_2(H') & \beta_2(L) &= 0 = \beta_2(L') \\ \mu_2(s_F) &= \frac{1}{3} & \mu_2(s'_F) &= \frac{2}{3} \\ \mu_2(s_F) &\geq \frac{1}{6} & \mu_2(s'_A) &= 1 - \mu_2(s_F). \end{aligned}$$

6. (30 Points) (A variation on the education level example) Assume that there are two types of individuals with equal proportion in the population, H of high ability and L of low ability. The individual P_I knows her own ability level but the firm P_F hiring does not. The firm P_F picks a value of $m \geq 0$ that determines a wage scale $w(e) = me$ based on the level of education of the individual. Then, the individual P_I chooses a level of education $e_H \geq 0$ or $e_L \geq 0$ depending on the type of individual. The payoff are given in the game tree in Figure 2 where the first player is the firm and the second player is the individual. **What** is the sequential equilibrium? Give the assessment for this equilibrium.

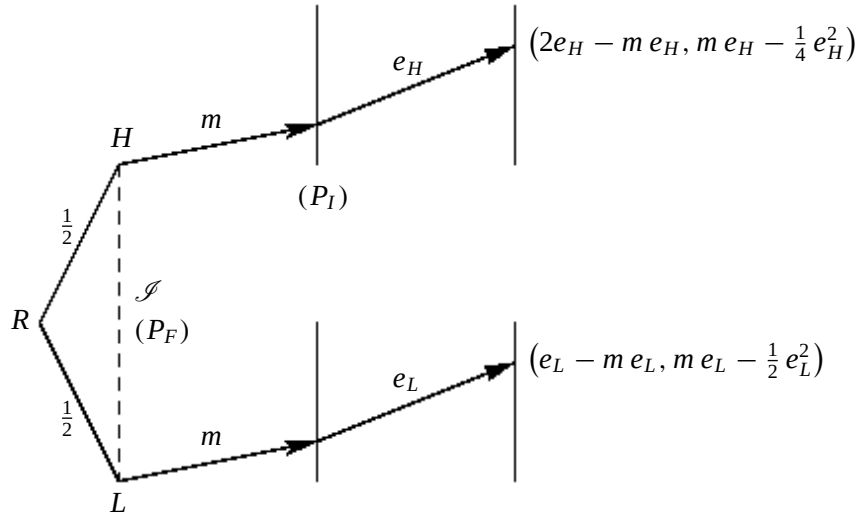


FIGURE 2. Game tree for the education problem

Answer:

For an individual of type H ,

$$\begin{aligned} u_I(e, m) &= me - \frac{1}{4}e^2 \\ \frac{\partial u_I}{\partial e} &= m - \frac{1}{2}e \\ e_H^* &= 2m. \end{aligned}$$

For an individual of type L ,

$$\begin{aligned}u_I(e, m) &= me - \frac{1}{2}e^2 \\ \frac{\partial u_I}{\partial e} &= m - e \\ e_L^* &= m.\end{aligned}$$

Thus, the expected payoff for the firm is

$$\begin{aligned}E_F(m|e_H^*, e_L^*) &= \frac{1}{2} [2(2m) - m(2m)] + \frac{1}{2} [m - m(m)] \\ &= \frac{1}{2} [4m - 2m^2 + m - m^2] \\ \frac{\partial E_F}{\partial m} &= \frac{1}{2} [4 - 4m + 1 - 2m] \\ &= \frac{1}{2} [5 - 6m].\end{aligned}$$

This equals zero when $m^* = 5/6$. Since $\frac{\partial^2 E_F}{\partial m^2} < 0$, this is a maximum. Thus, the sequential equilibrium is $e_H^* = 2(5/6) = 5/3$, $e_L^* = 5/6$, and $m^* = 5/6$.