

1. (16 Points) For a collection of lotteries, there are three possible outcomes, a_1 , a_2 , and a_3 . The payoff function satisfies $u(a_1) = 0$, $u(a_2) = v$, and $u(a_3) = 10$. A lottery (p_1, p_2, p_3) indicates a lottery in which the probability of getting outcome a_i is p_i . The decision maker prefers the lottery $(0.1, 0.4, 0.5)$ to $(0.3, 0, 0.7)$ to $(0.275, 0.125, 0.6)$. Are these preferences consistent? And, what can you say about the value of $v = u(a_2)$? Note: $0.125 = 1/8$.

Answer:

(a) The preferences on the lotteries imply that

$$\begin{aligned} 0.4v + 5 &> 7, & 7 &> \frac{1}{8}v + 6, \\ \frac{2}{5}v &> 2, & 1 &> \frac{1}{8}v \\ v &> 5, & 8 &> v. \end{aligned}$$

Therefore, these preferences are consistent and we need $5 < v = u(a_2) < 8$.

2. (16 Points) (Cournot's duopoly) Because of the form of the costs and inverse demand function, the profit of two firms is given by

$$\pi_1(q_1, q_2) = q_1(10 - 2q_2) - 2q_1^2 \quad \text{and} \quad \pi_2(q_1, q_2) = q_2(8 - 2q_1) - q_2^2,$$

for production levels $q_1 \geq 0$ by the first firm and $q_2 \geq 0$ by the second firm. Find the Nash equilibrium.

Answer:

For a Nash equilibrium, we need

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= 10 - 2q_2 - 4q_1 = 0 & \text{and} \\ \frac{\partial \pi_2}{\partial q_2} &= 8 - 2q_1 - 2q_2 = 0. \end{aligned}$$

These two equations are equivalent to

$$5 = 2q_1 + q_2 \quad \text{and} \quad 4 = q_1 + q_2.$$

Subtracting the second equation from the first gives $q_1 = 1$. Then, $q_2 = 5 - 2 = 3$. The second partial derivatives are negative,

$$\frac{\partial^2 \pi_1}{\partial q_1^2} = -4 < 0 \quad \text{and} \quad \frac{\partial^2 \pi_2}{\partial q_2^2} = -2 < 0,$$

so the critical point is a maximum in each variable, and the point $\mathbf{q} = (1, 3)$ is a Nash equilibrium.

3. Consider the game tree with perfect information given in Figure 1.
- (14 Points) What is the subgame perfect Nash equilibrium? Give the strategy profile and payoffs for the players.
 - (14 Points) Find a strategy profile that is a Nash equilibrium but not subgame perfect. Explain why your choice (i) is not subgame perfect, and (ii) is a Nash equilibrium.

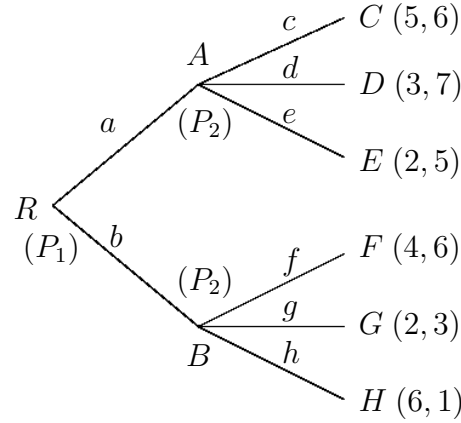


FIGURE 1

Answer:

(a) At vertex A, P_2 has possible payoffs of 6, 7, and 5, the largest of which is 7. Therefore, d is the action with highest payoff for P_2 .

At vertex B, P_2 has possible payoffs of 6, 3, and 1, the largest of which is 6. Therefore, f is the action with highest payoff for P_2 .

Given these choices for P_2 , at vertex R, P_1 has possible payoffs of 3 and 4, the largest of which is 4. Therefore, b is the action with the highest payoff for P_1 .

The method of backward induction gives the strategy profile $(b, \{d, f\})$, which must be the subgame perfect equilibrium.

(b) There are two strategy profiles that are Nash equilibria but not subgame perfect: $(b, \{e, f\})$ and $(a, \{d, g\})$. We consider the strategy profile $(b, \{e, f\})$. On the subgame starting at A, this strategy does not give the largest payoff for P_2 , so it is not a Nash equilibrium on this subgame. Therefore, the strategy profile is not a subgame perfect equilibrium. However, it is a Nash equilibrium, since

$$\begin{aligned} u_1(b, \{e, f\}) &= 4 > 2 = u_1(a, \{e, f\}), \\ u_2(b, \{e, f\}) &= 6 = u_2(b, \{*, f\}) > 3 = u_2(b, \{*, g\}) > 1 = u_2(b, \{*, h\}). \end{aligned}$$

4. (16 Points) In a game of attrition, the payoff functions of two players are given by

$$u_1(t_1, t_2) = \begin{cases} -t_1 & \text{if } t_1 < t_2 \\ 5 - t_2 & \text{if } t_1 = t_2 \\ 10 - t_2 & \text{if } t_1 > t_2 \end{cases} \quad \text{and} \quad u_2(t_1, t_2) = \begin{cases} -t_2 & \text{if } t_2 < t_1 \\ 6 - t_1 & \text{if } t_2 = t_1 \\ 12 - t_1 & \text{if } t_2 > t_1, \end{cases}$$

where $t_1 \geq 0$ is the time player one waits and $t_2 \geq 0$ is the time player two waits. What is the best response function of player one, $B_1(t_2)$.

Hint: Consider the three cases: (i) $t_2 < 10$, (ii) $t_2 = 10$, and (iii) $t_2 > 10$.

Answer:

(i) Assume $t_2 < 10$. Then $u_1(t_1, t_2) \leq 0$ if $0 \leq t_1 < t_2$, $u_1(t_1, t_2) = 5 - t_2 < 10 - t_2$ if $t_1 = t_2$, and $u_1(t_1, t_2) = 10 - t_2 > 0$ if $t_1 > t_2$. Therefore, the maximum is realized in the third case: $B_1(t_2) = \{t_1 : t_1 > t_2\}$.

(ii) Assume $t_2 = 10$, Then $u_1(0, t_2) = 0$, $u_1(t_1, t_2) < 0$ if $0 < t_1 < t_2$, $u_1(t_2, t_2) = 5 - 10 < 0$ if $t_1 = t_2$, and $u_1(t_1, t_2) = 10 - 10 = 0$ if $t_1 > t_2$. Therefore, the maximum value is zero, and it is attained when $t_1 = 0$ and $t_1 > t_2 = 10$: $B_1(t_2) = \{0\} \cup \{t_1 : t_1 > t_2\}$.

(iii) Assume If $t_2 > 10$. Then $u_1(0, t_2) = 0$, $u_1(t_1, t_2) < 0$ if $0 < t_1 < t_2$, $u_1(t_1, t_2) = 5 - t_2 < 5 - 10 < 0$ if $t_1 = t_2$, and $u_1(t_1, t_2) = 10 - t_2 < 0$ if $t_1 > t_2$. Therefore, the maximum value is zero, and it is attained when $t_1 = 0$: $B_1(t_2) = \{0\}$. Summarizing,

$$B_1(t_2) = \begin{cases} \{t_1 : t_1 > t_2\} & \text{if } t_2 < 10 \\ \{0\} \cup \{t_1 : t_1 > t_2\} & \text{if } t_2 = 10 \\ \{0\} & \text{if } t_2 > 10. \end{cases}$$

Notice that the best response function takes on set values of ranges for the t_1 variable.

5. Consider the bi-matrix game given by

$$\begin{pmatrix} (0, 0) & (8, -4) & (8, -4) \\ (2, -2) & (0, 0) & (2, -2) \\ (1, -1) & (1, -1) & (0, 0) \end{pmatrix}.$$

- a. (8 Points) Are there any pure strategy Nash equilibria? Explain by giving the best response in pure strategies of each player to the pure strategies of the other player.
- b. (16 Points) Consider mixed strategies, where player one (who chooses the row) has a strategy (p_1, p_2, p_3) and player two (who chooses the column) has a strategy (q_1, q_2, q_3) .
 - (i) For $p_3 = 0 = q_3$ and $1 \leq i \leq 3$, find the expected payoffs $E_1(r_i, (q_1, q_2, 0))$ for rows r_i and $E_2((p_1, p_2, 0), c_i)$ for columns c_i .
 - (ii) Find a mixed strategy Nash equilibrium with $p_3 = 0 = q_3$. Indicate why it is a Nash equilibrium and determine the payoff for each player.

Answer:

(a) The best responses are

$$\begin{aligned} B_1(c_1) &= r_2 & B_1(c_2) &= r_1 & B_1(c_3) &= r_1, \\ B_1(r_1) &= c_1 & B_1(r_2) &= c_2 & B_1(r_3) &= c_3. \end{aligned}$$

There is no pair that give back the other one, so there are no Nash equilibria of pure strategies.

(b) (i) In calculating the expected payoffs to mixed strategies we must consider r_3 and c_3 and show that they have smaller payoff. The payoffs by pure strategies in reaction to mixed strategies of the other player are as follows:

$$\begin{aligned} E_1(r_1, (q_1, q_2, 0)) &= 8q_2, \\ E_1(r_2, (q_1, q_2, 0)) &= 2q_1, \\ E_1(r_3, (q_1, q_2, 0)) &= q_1 + q_2, \\ E_2((p_1, p_2, 0), c_1) &= -2p_2, \\ E_2((p_1, p_2, 0), c_2) &= -4p_1, \\ E_2((p_1, p_2, 0), c_3) &= -4p_1 - 2p_2. \end{aligned}$$

(ii) For the mixed strategy profile $((p_1, p_2, 0), c_1), (q_1, q_2, 0))$ to be a Nash equilibrium, we need

$$\begin{aligned} 8q_2 &= 2q_1 > q_1 + q_2 & \text{and} \\ -2p_2 &= -4p_1 > -4p_1 - 2p_2. \end{aligned}$$

The first equality gives that $q_1 = 4q_2$, so

$$\begin{aligned} 1 &= q_2 (4 + 1 + 0) \\ q_2 &= 1/5 \\ q_1 &= 4/5. \end{aligned}$$

For these values, the expected payoff

$$E_1(r_3, (4/5, 1/5, 0)) = 4/5 + 1/5 = 1 < 8/5 = E_1(r_1, (4/5, 1/5, 0)),$$

so that inequality is satisfied.

The second equality gives that $p_2 = 2p_1$, so

$$\begin{aligned} 1 &= p_1 (1 + 2 + 0) \\ p_1 &= 1/3 \\ p_2 &= 2/3. \end{aligned}$$

For these values, the expected payoff

$$E_2((1/3, 2/3, 0), c_3) = -4(1/3) - 2(2/3) = -8/3 < -4/3 = E_2((1/3, 2/3, 0), c_1),$$

so that inequality is satisfied.

The payoffs are

$$\begin{aligned} U_1((1/3, 2/3, 0), (4/5, 1/5, 0)) &= \left(\frac{1}{3}\right) \left(\frac{1}{5}\right) 8 + \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) 2 = \frac{8}{5}, \\ U_2((1/3, 2/3, 0), (4/5, 1/5, 0)) &= \left(\frac{1}{3}\right) \left(\frac{1}{5}\right) (-4) + \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) (-2) = -\frac{4}{3}. \end{aligned}$$