

1. Consider the bi-matrix game given by

$$\begin{pmatrix} (5, 5) & (2, 3) & (1, 3) \\ (3, 0) & (2, 2) & (7, 0) \\ (1, 3) & (6, 5) & (5, 7) \end{pmatrix}.$$

- a. (15 Points) Are there any pure strategy Nash equilibria? Explain by giving the best response in pure strategies of each player to the pure strategies of the other player.
- b. (30 Points) Consider mixed strategies, where player one (who chooses the row) has a strategy (p_1, p_2, p_3, p_4) and player two (who chooses the column) has a strategy (q_1, q_2, q_3, q_4) .
- (i) Find the expected payoff $E_1(r_i, (q_1, q_2, q_3))$ for rows r_i and $E_2((p_1, p_2, p_3), c_j)$ for columns c_j .
- (ii) Find a completely mixed strategy Nash equilibrium. Indicate why it is a Nash equilibrium.

Answer:

(a) $B_1(c_1) = r_1$, $B_1(c_2) = r_3$, $B_1(c_3) = r_2$, $B_2(r_1) = c_1$, $B_2(r_2) = c_2$, and $B_2(r_3) = c_3$. The only matches for a Nash equilibrium in pure strategies is (r_1, c_1) .

(b) Using $q_3 = 1 - q_1 - q_2$,

$$E_1(r_1, \mathbf{q}) = 5q_1 + 2q_2 + q_3 = 4q_1 + q_2 + 1,$$

$$E_1(r_2, \mathbf{q}) = 3q_1 + 2q_2 + 7q_3 = -4q_1 - 5q_2 + 7,$$

$$E_1(r_3, \mathbf{q}) = q_1 + 6q_2 + 5q_3 = -4q_1 + q_2 + 5.$$

Setting $E_1(r_1, \mathbf{q}) = E_1(r_3, \mathbf{q})$, $8q_1 = 4$, and $q_1^* = 1/2$. Setting $E_1(r_2, \mathbf{q}) = E_1(r_3, \mathbf{q})$, $6q_2 = 2$, and $q_2^* = 1/3$. Finally $q_3^* = 1 - q_1^* - q_2^* = 1/6$.

In the same way, using $p_2 = 1 - p_1 - p_3$,

$$E_2(\mathbf{p}, c_1) = 5p_1 + 0p_2 + 3p_3 = 5p_1 + 3p_3,$$

$$E_2(\mathbf{p}, c_2) = 3p_1 + 2p_2 + 5p_3 = p_1 + 3p_3 + 2,$$

$$E_2(\mathbf{p}, c_3) = 3p_1 + 0p_2 + 7p_3 = 3p_1 + 7p_3.$$

Setting $E_2(\mathbf{p}, c_1) = E_2(\mathbf{p}, c_3)$, $2p_1 = 4p_3$, and $p_1 = 2p_3$. Setting $E_2(\mathbf{p}, c_1) = 13p_3 = E_2(\mathbf{p}, c_2) = 2 + 5p_3$, $8p_3 = 2$, and $p_3^* = 1/4$. Then $p_1^* = 1/2$, and $p_2^* = 1 - p_1^* - p_3^* = 1/4$.

This is a Nash equilibrium because the expected payoff of each row and column is the same.

2. (30 Points) One version of the game of attrition is follows: Assume that $0 < v_2 < v_1$ are the values to the two players and the payoff functions for the two player are

$$u_1(t_1, t_2) = \begin{cases} 0 & \text{if } t_1 < t_2 \\ v_1 - t_2 & \text{if } t_1 \geq t_2 \end{cases} \quad \text{and} \quad u_2(t_1, t_2) = \begin{cases} 0 & \text{if } t_2 \leq t_1 \\ v_2 - t_1 & \text{if } t_2 > t_1, \end{cases}$$

where $t_1 \geq 0$ is the time player one waits and $t_2 \geq 0$ is the time player two waits.

- a. Determine the best response function $B_1(t_2)$ for P_1 by separating the values of t_2 into the three cases (i) $0 \leq t_2 < v_1$, (ii) $t_2 = v_1$, and (iii) $t_2 > v_1$. Sketch this best response function in the (t_1, t_2) -plane.

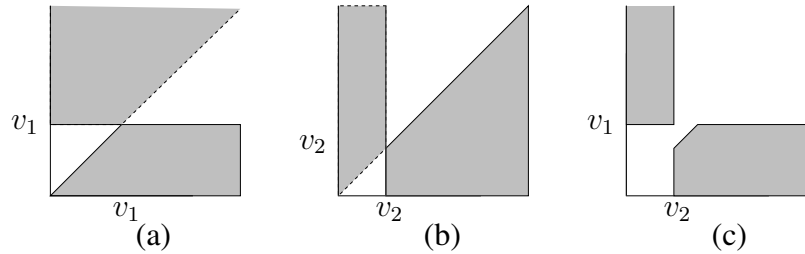
- b. Determine the best response function $B_2(t_1)$ for P_2 by separating the values of t_1 into the three cases (i) $0 \leq t_1 < v_2$, (ii) $t_1 = v_2$, and (iii) $t_1 > v_2$. Sketch this two best response function in a separate copy of the (t_1, t_2) -plane.
- c. Using the sketches of the best response functions, determine all the Nash equilibria.

Answer:

(a,b) The best responses are given in panes (a) and (b) of the figure and as follows:

$$B_1(t_2) = \begin{cases} [t_2, \infty) = \{t_1 \geq t_2\} & \text{if } 0 \leq t_2 < v_1 \\ [0, \infty) = \{t_1 \geq 0\} & \text{if } t_2 = v_1 \\ [0, t_2) = \{0 \leq t_1 < t_2\} & \text{if } t_2 > v_1 \end{cases}$$

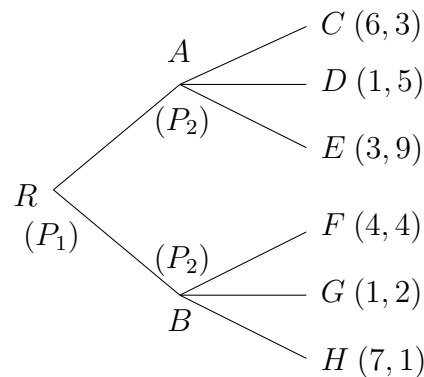
$$B_2(t_1) = \begin{cases} (t_1, \infty) = \{t_2 > t_1\} & \text{if } 0 \leq t_1 < v_2 \\ [0, \infty) = \{t_2 \geq 0\} & \text{if } t_1 = v_2 \\ [0, t_1] = \{0 \leq t_2 \leq t_1\} & \text{if } t_1 > v_2 \end{cases}$$



(c) The intersection is given in pane (c) of the figure and is the union of the sets $\{(t_1, t_2) : 0 \leq t_1 \leq v_2, t_2 \geq v_1\}$ and $\{(t_1, t_2) : 0 \leq t_2 \leq v_1, t_1 \geq t_2, t_1 \geq v_2\}$.

3. (35 Points) Consider the game with perfect information whose game tree is given in the following figure.

- a. Give the strategy profile for the subgame perfect Nash equilibrium in pure strategies?
- b. Find a strategy profile that is a Nash equilibrium but not subgame perfect. Explain why your choice (i) is not subgame perfect, and (ii) is a Nash equilibrium.

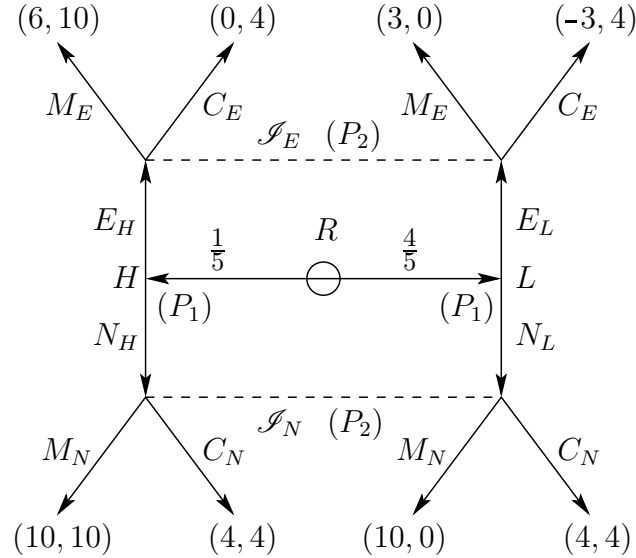


Answer:

(a) The same perfect game is $(RB, (AE, BF))$.

(b) There are two Nash equilibria that are not subgame perfect: $(RB, (AD, BF))$ and $(RA, (AE, BG))$.

4. An individual P_1 is either of high ability H with probability $1/5$ or low ability L with probability $4/5$. The ability level is known to the individual but not the firm P_2 . The individual either decides to get educational training E or not N . The firm observes the level of education and hires the individual for a managerial job M or a clerical job C . The payoffs are given in the following figure, with individual's payoff first and the firm's second.



- a. (30 Points) Show that there is a separating equilibrium with $\beta_1(E_H) = 1$ and $\beta_1(N_L) = 1$ as follows:
- What is the consistent belief system on J_E ?
 - On J_E , is M_E or C_E the better choice for the firm P_2 ? Explain why.
 - What is the consistent belief system on J_N ?
 - On J_N , is M_N or C_N the better choice for the firm P_2 ? Explain why.
 - Why is $\beta(E_H) = 1$ a rational choice for P_1 who is of high ability?
 - Why is $\beta(N_L) = 1$ a rational choice for P_1 who is of low ability?
 - What is the complete assessment for this separating equilibrium?

Answer:

- On J_E , $\mu(E_H) = 1$ and $\mu(E_L) = 0$.
- Therefore, the firm P_2 should choose based on E_H and so choose M_E .
- On J_N , $\mu(N_L) = 1$ and $\mu(N_H) = 0$.
- Therefore, P_2 should choose based on N_L and so choose C_N .
- For H , $E_1(E_H) = 6 > 4 = E_1(N_H)$, so P_1 should choose E_H .
- $E_1(E_L) = 3 < 4 = E_1(N_L)$, so P_1 should choose N_L .
- The assessment is $E_H, N_L, M_E, C_N, \mu(E_H) = 1$ and $\mu(N_L) = 1$.

- b. (30 Points) Show that there is a pooling equilibrium with $\beta_1(N_H) = 1$ and $\beta_1(N_L) = 1$ as follows:
- If $\beta_1(N_H) = 1$ and $\beta_1(N_L) = 1$, then what is the consistent belief system on J_N ?
 - On J_N , is M_N or C_N a better choice for P_2 ? Explain why.
 - For a person of high ability, what is the expected value of $E_1(H, E_H, \beta_2(M_E))$ in terms of $\beta_2(M_E)$ and $\beta_2(C_E) = 1 - \beta_2(M_E)$? Using the value of $\beta_2(M_N)$ found in part (ii), what is the expected value of $E_1(H, N_H, \beta_2(M_N))$? What must $\beta_2(M_E)$

be for $\beta_1(N_H) = 1$ to be a rational choice for P_1 ? If P_2 makes a pure choice on $\mathcal{J}_E$ for this pooling equilibrium, does it have to be M_E or C_E ?

(iv) What are the consistent belief system on $\mathcal{J}_E$ for P_2 ? What restrictions on the belief system for P_2 are needed to make rational the pure choice of M_E or C_E on $\mathcal{J}_E$ found in the last part? Explain why.

(v) What is the complete assessment for this pooling equilibrium?

Answer:

(i) $\mu(N_H) = 1/5$ and $\mu(N_L) = 4/5$.

(ii) $E_2(M_N) = \frac{1}{5}(10) + \frac{4}{5}(0) = 2 < 4 = \frac{1}{5}(4) + \frac{4}{5}(4) = E_2(C_N)$. So P_2 should choose C_N .

(iii) $E_1(H, E_H, \beta_2(M_E)) = 6\beta_2(M_E) + (1 - \beta_2(M_E))(0) = 6\beta_2(M_E)$.

$E_1(H, N_H, \beta_2(M_N)) = 4$. Therefore, we need $4 \geq 6\beta_2(M_E)$ or $\beta_2(M_E) \leq 2/3$, $\beta_2(C_E)1 - \beta_2(M_E) \geq 1/3$.

(iv) Any belief system is weakly consistent on $\mathcal{J}_E$ since the probability of reaching it is 0. For $\beta_2(M_E) < 1$, P_2 need to have $E_2(M_E) = 10\mu_2(E_H) \leq 4 = E_2(C_E)$. Thus, need $\mu_2(E_H) \leq 2/5$.

(v) The assessment is $((C_E, C_N), (N_H, N_L))$, $\mu_2(E_H) \leq 2/5$, $\mu_2(N_H) = 1/5$, and $\mu_2(N_L) = 4/5$.

5. (30 Points) Consider the infinitely repeated game with discount $0 < \delta < 1$ of the Prisoner's Dilemma with one time payoffs given as follows:

$$\begin{array}{cc} & \begin{array}{cc} C_2 & D_2 \end{array} \\ \begin{array}{c} C_1 \\ D_1 \end{array} & \begin{pmatrix} (8, 8) & (0, 9) \\ (9, 0) & (3, 3) \end{pmatrix} \end{array}$$

Assume that player P_2 plays the tit-for-tat strategy where the action by P_2 is C_2 in the first stage and

$$a_{2,t} = \begin{cases} C_2 & \text{if } a_{1,t-1} = C_1 \text{ and} \\ D_2 & \text{if } a_{1,t-1} = D_1 \end{cases}$$

in the t^{th} -stage.

a. What inequality must δ satisfy for P_1 to prefer (i) always choosing C_1 (i.e. following the tit-for-tat strategy) to (ii) always defecting and always choosing D_1 ? Start the choices at stage $t = 1$.

b. What inequality must δ satisfy for P_1 to prefer (i) $a_{1,1} = a_{1,2} = C_1$ to (ii) $a_{1,1} = D_1$ and $a_{1,2} = C_1$? That is, follow the tit-for-tat strategy for the first two stages versus defecting for $t = 1$ and cooperating for $t = 2$.

Answer:

(a) We need $\frac{8}{1-\delta} \geq 9 + \frac{3\delta}{1-\delta}$, $8 \geq 9(1-\delta) + 3\delta = 9 - 6\delta$, $6\delta \geq 1$, or $\delta \geq 1/6$.

(b) We need $8 + 8\delta \geq 9$, $8\delta \geq 1$, and $\delta \geq 1/8$. For both to be true, we need $\delta \geq 1/6$.