

1. (16 Points) For a collection of lotteries, there are three possible outcomes, a_1 , a_2 , and a_3 . The payoff function satisfies $u(a_1) = 0$, $u(a_2) = v$, and $u(a_3) = 10$. A lottery (p_1, p_2, p_3) indicates a lottery in which the probability of getting outcome a_i is p_i . The decision maker prefers the lottery $(0.1, 0.5, 0.4)$ to $(0.3, 0, 0.7)$ to $(0.2, 0.2, 0.6)$. What choice if any for the value of v make these preferences induced by the Bernoulli payoff function induced from u ?

Answer:

For the preferences on the lotteries to be true, we need $0.5v + 4 > 7 > 0.2v + 6$. The inequalities imply the following:

$$\begin{aligned} 0.5v &> 3, \\ v &> 6, \quad \text{and} \\ 1 &> 0.2v, \\ 5 &> v. \end{aligned}$$

These two inequalities are not possible at the same time, so preferences are not determined by a Bernoulli payoff function.

2. (17 Points) Assume that two countries that impose a tariff on a product of $x \geq 0$ by country one and $y \geq 0$ by country two, and that the payoffs are $u_1(x, y) = 2000 + 60x + xy - x^2 - 90y$ and $u_2(x, y) = 2000 + 60y + xy - y^2 - 90x$. Find the Nash equilibrium.

Answer:

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= 60 + y - 2x = 0, \\ \frac{\partial u_1}{\partial y} &= 60 + x - 2y = 0. \end{aligned}$$

Since, $\frac{\partial^2 u_1}{\partial x^2} = -2 < 0$ and $\frac{\partial^2 u_2}{\partial y^2} = -2 < 0$ these are maximum. Adding twice the second equation to the first gives $0 = 180 - 3y$, or $y = 60$. Substituting in gives $x = -60 + 2y = 120 - 60 = 60$. This is the Nash equilibrium, $(x, y) = (60, 60)$.

3. (17 Points) In one form of the game of attrition, the payoff functions of two players are given by

$$u_1(t_1, t_2) = \begin{cases} 0 & \text{if } t_1 < t_2 \\ \frac{1}{2}(v_1 - t_2) & \text{if } t_1 = t_2 \\ v_1 - t_2 & \text{if } t_1 > t_2 \end{cases} \quad \text{and} \quad u_2(t_1, t_2) = \begin{cases} 0 & \text{if } t_2 < t_1 \\ \frac{1}{2}(v_2 - t_1) & \text{if } t_2 = t_1 \\ v_2 - t_1 & \text{if } t_2 > t_1, \end{cases}$$

where $t_1 \geq 0$ is the time player one waits and $t_2 \geq 0$ is the time player two waits. Assume that $0 < v_2 < v_1$. Give the best response function of each player, $B_1(t_2)$ and $B_2(t_1)$.

Answer:

The best response functions are as follows:

$$B_1(t_2) = \begin{cases} \{t_1 > t_2\} & \text{if } t_2 < v_1 \\ \{t_1 \geq 0\} & \text{if } t_2 = v_1 \\ \{t_1 < t_2\} & \text{if } t_2 > v_1 \end{cases}$$

$$B_2(t_1) = \begin{cases} \{t_2 > t_1\} & \text{if } t_1 < v_2 \\ \{t_2 \geq 0\} & \text{if } t_1 = v_2 \\ \{t_2 < t_1\} & \text{if } t_1 > v_2. \end{cases}$$

4. (25 Points) Consider the bi-matrix game given by

$$\begin{pmatrix} (8, 1) & (0, 1) & (2, 3) \\ (2, 1) & (4, 4) & (0, 0) \\ (1, 3) & (3, 0) & (3, 2) \end{pmatrix}.$$

- Are there any pure strategy Nash equilibria? Explain by giving the best response in pure strategies of each player to the pure strategies of the other player.
- Consider mixed strategies, where player one (who chooses the row) has a strategy (p_1, p_2, p_3) and player two (who chooses the column) has a strategy (q_1, q_2, q_3) .
 - For $p_2 = 0 = q_2$ and $1 \leq i \leq 3$, find the expected payoffs $E_1(r_i, (q, 0, 1 - q))$ for rows r_i and $E_2((p, 0, 1 - p), c_j)$ for columns c_j .
 - Find a mixed strategy Nash equilibrium with $p_2 = 0 = q_2$. Indicate why it is a Nash equilibrium.

Answer:

(a) $B_1(c_1) = r_1$, $B_1(c_2) = r_2$, $B_1(c_3) = r_3$, $B_2(r_1) = c_3$, $B_2(r_2) = c_2$, $B_2(r_3) = c_1$. The only choices that are in the best response for the other player are (r_2, c_2) , so this is the only Nash equilibrium in pure strategies.

(b) $E_1(r_1, (q, 0, 1 - q)) = 8q + 2(1 - q) = 6q + 2$, $E_1(r_2, (q, 0, 1 - q)) = 2q$, $E_1(r_3, (q, 0, 1 - q)) = q + 3(1 - q) = 3 - 2q$, $E_2((p, 0, 1 - p), c_1) = p + 3(1 - p) = 3 - 2p$, $E_2((p, 0, 1 - p), c_2) = p$, $E_2((p, 0, 1 - p), c_3) = 3p + 2(1 - p) = 2 + p$.

If $E_1(r_1, (q, 0, 1 - q)) = 6q + 2 = 3 - 2q = E_1(r_3, (q, 0, 1 - q))$, then $8q = 1$, $q = 1/8$, and $1 - q = 7/8$. For these values $E_1(r_1, (q, 0, 1 - q)) > E_1(r_2, (q, 0, 1 - q))$ so $p_2 = 0$.

If $E_2((p, 0, 1 - p), c_1) = 3 - 2p = 2 + p = E_2((p, 0, 1 - p), c_3)$, then $3p = 1$ or $p = 1/3$ and $1 - p = 2/3$. For these values, $E_2((p, 0, 1 - p), c_1) > E_2((p, 0, 1 - p), c_2)$ so $q_2 = 0$. This shows that $\mathbf{p} = (1/3, 0, 2/3)$ and $\mathbf{q} = (1/8, 0, 7/8)$ is a Nash equilibrium.

5. (25 Points) For an evolutionary game, members of a single population are randomly matched in pairs and have the following payoff matrix of playing row r_i against column c_j ,

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

- a. Find all the symmetric Nash equilibria.
- b. Check whether each Nash equilibrium is an ESS.

Answer:

For $\mathbf{p} = (p, 1-p)$, $E(\mathbf{e}^1, p) = 2(1-p)$ and $E(\mathbf{e}^2, p) = p$. These are equal for $2-2p = p$, $2 = 3p$, or $p = 2/3$. The best response is

$$B(p) = \begin{cases} \{1\} & \text{if } p < 2/3 \\ [0, 1] & \text{if } p = 2/3 \\ \{0\} & \text{if } p > 2/3. \end{cases}$$

The only p for which $p \in B(p)$ is $\hat{p} = 2/3$. This is the only symmetric Nash equilibrium.

For $\mathbf{q} = (q, 1-q)$, $\mathbf{q} \cdot \mathbf{A}\mathbf{q} = 3q(1-q)$ and $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} = \frac{4}{3} - q$. Is

$$\frac{4}{3} - q > 3q - 3q^2$$

$$3q^2 - 4q + \frac{4}{3} > 0$$

$$\frac{1}{3}(9q^2 - 12q + 4) > 0$$

$$(3q - 2)^2 > 0.$$

This is true for $q \neq 2/3$. This checks that $\hat{\mathbf{p}}$ is an ESS.