Example
We assume that the population has two types,

A: Aggressive (hawk)
P: Passive (dove)

The payoff matrix gives the payoff to the first type of individual playing against the second type is given as follows:

\[
\begin{pmatrix}
A & P \\
P & \frac{G-C}{2} & \frac{G}{2}
\end{pmatrix},
\]

where \(G < C\) so \(G - C < 0\).

Let \(W(I, Q)\) be the payoff in fitness for an individual of type \(I\) meeting a population of type \(Q\). If the population has a frequency \(q\) of aggressive type and a frequency \(1 - q\) of passive type, then

\[
W(A, q) = \left(\frac{G - C}{2}\right)q + G(1 - q) \quad \text{and}
\]

\[
W(P, q) = \frac{G}{2}(1 - q).
\]

Thus, if a combination of individuals of frequency \(p\) of type \(A\) plays against the population of frequency \(q\) of type \(A\), then the payoff fitness for the individuals of frequency \(p\) is

\[
W(p, q) = \left(\frac{G - C}{2}\right)pq + Gp(1 - q) + \frac{G}{2}(1 - p)(1 - q).
\]

A frequency \(\hat{q}\) is called a Nash equilibrium if \(W(p, \hat{q}) \leq W(\hat{q}, \hat{q})\) for all \(p\). Thus, no other frequency of types has a larger payoff playing against the equilibrium frequency. If it is an interior equilibrium, then it must be the case that \(W(p, \hat{q}) = W(\hat{q}, \hat{q})\) for all \(p\).

For an interior Nash equilibrium \(\hat{q}\) in the example being considered,

\[
W(p, \hat{q}) = p \left[ \left(\frac{G - C}{2}\right)\hat{q} + G(1 - \hat{q}) \right] + (1 - p) \frac{G}{2}(1 - \hat{q})
\]

must be constant in \(p\), so

\[
\left(\frac{G - C}{2}\right)\hat{q} + G(1 - \hat{q}) = \frac{G}{2}(1 - \hat{q})
\]

\[
\frac{G}{2}(1 - \hat{q}) = \left(\frac{-G + C}{2}\right)\hat{q}
\]

\[
G = C \hat{q}
\]

\[
\hat{q} = \frac{G}{C}.
\]
Assume a small amount of a new frequency is introduced into the population. The Nash equilibrium $\hat{q}$ is called an evolutionarily stable strategy if the fitness of new population $p$ played against $\epsilon p + (1 - \epsilon)\hat{q}$ has less fitness payoff than the Nash equilibrium played against this population,

$$W(p, \epsilon p + (1 - \epsilon)\hat{q}) < W(\hat{q}, \epsilon p + (1 - \epsilon)\hat{q})$$

or

$$\epsilon W(p, p) + (1 - \epsilon)W(p, \hat{q}) < \epsilon W(\hat{q}, p) + (1 - \epsilon)W(\hat{q}, \hat{q}).$$

Since $W(p, \hat{q}) = W(\hat{q}, \hat{q})$, we need

$$W(p, p) < W(\hat{q}, p)$$

for all $p \neq \hat{q}$.

For the example being considered, $\hat{q} = G/C$ and $1 - \hat{q} = (C - G)/C$, so

$$W(\hat{q}, p) - W(p, p) = \frac{G - C}{2} \left( \frac{G}{C} \right) p + G \left( \frac{G}{C} \right) (1 - p) + \frac{G}{2} \left( \frac{C - G}{C} \right) (1 - p)$$

$$- \left[ \left( \frac{G - C}{2} \right) p^2 + Gp(1 - p) + \left( \frac{G}{2} \right) (1 - p)^2 \right]$$

$$= p^2 \left[ - \left( \frac{G - C}{2} \right) + G - \frac{G}{2} \right]$$

$$+ p \left[ \left( \frac{G - C}{2} \right) \left( \frac{G}{C} \right) \right] - \frac{G^2}{2} - \left( \frac{G}{2} \right) \left( \frac{C - G}{C} \right) - G + G$$

$$+ \left[ \frac{G^2}{C} + \left( \frac{G}{2} \right) \left( \frac{C - G}{C} \right) - \frac{G}{2} \right]$$

$$= p^2 \left( \frac{C}{2} \right) - p G + \frac{G^2}{2C}$$

$$= \frac{1}{2C} (Cp - G)^2$$

$$> 0$$

for all $p \neq G/C = \hat{q}$. This checks that $\hat{q}$ satisfies the conditions to be an evolutionarily stable strategy.

The general definitions of a Nash equilibrium and evolutionarily stable strategy are the following. A frequency $\hat{q}$ is called a Nash equilibrium if

$$W(p, \hat{q}) \leq W(\hat{q}, \hat{q})$$

for all $p$.

A frequency $\hat{q}$ is called an evolutionarily stable strategy if it is a Nash equilibrium and

$$W(p, p) < W(\hat{q}, p)$$

for all $p \neq \hat{q}$ with $W(p, \hat{q}) = W(\hat{q}, \hat{q})$. 