

MATH 364: STRATEGIC GAMES

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Office	209
Hours	M 11-noon, 1-2; W 1-2, by appointment
Grading	HW 140 (7 out of 8), Midterm 100, Final 200 = 440
Book	Osborne

Not tick-tack-toe or March madness

Overview

Strategic games: simultaneous choices of 2 or more players

- 2 firms: P_1 sets q_1 and maximizes payoff $\pi_1(q_1, q_2)$, P_2 sets q_2 and maximizes payoff $\pi_2(q_1, q_2)$.
Want simultaneous optimal choices given the other player's choice.
- Chapters 2: Finite number of choices for each player
- Chapters 3: Continuous choice of quantity for each player
- Chapter 4: Play choices probabilistically, mixed strategies
- Chapter 13: Evolutionary game theory: how characteristic dominates, aggregate population learns

Extensive games: sequence of decisions

- Example: Take over after bid. Bid/no-bid, then accept/reject.
Payoffs for both sides
- Game tree
- Chapters 5-6 with perfect information: when know the prior choices of other players
- Chapter 10 with imperfect information:
 - Not know the hand of the other player in poker
 - Not know the action that the other player has already made.
 - Not know the exact vertex in game tree

Mathematical Background for Game Theory: Maxima

In game theory, we are often trying to find the maximum of a quantity. In many situations, we merely need to find the maximum of a finite number of values. For example, the maximum of $\{2, 7, 4\}$ is 7 and the minimum is 2, $\max\{2, 7, 4\} = 7$ and $\min\{2, 7, 4\} = 2$.

In other situations, we are trying to find the extrema, maximum or minimum, of a function on a closed interval $[a, b]$. The extrema can be either (i) at a critical point in the interior where $f'(x) = 0$ or (ii) at one of the end points. We illustrate the different cases with examples.

A function like $f(x) = \left(\frac{3}{2}\right)x - x^2$ for $x \in [0, 1]$ has its maximum in the interior: $f'(x) = \frac{3}{2} - 2x = 0$ at $x^* = \frac{3}{4}$. The second derivative $f''(x) = -2 < 0$, so $x^* = \frac{3}{4}$ is a maximum. The minimum is at one of the endpoints: $f'(0) = \frac{3}{2} > 0$ and $f'(1) = -\frac{1}{2} < 0$ so the endpoints are local minima. The values at the endpoints are $f(0) = 0$ and $f(1) = \frac{1}{2}$, so $x = 0$ is the point at which f attains its minimum.

A function like $g(x) = x^2 - \left(\frac{3}{2}\right)x$ for $x \in [0, 1]$ has its maximum at an endpoint and minimum in the interior: $g'(x) = 2x - \frac{3}{2}$ and $g''(x) = 2 > 0$. The interior critical point at $x^* = \frac{3}{4}$ is a minimum. The endpoints are local maximum: $g'(0) = -\frac{3}{2} < 0$ and $g'(1) = \frac{1}{2} > 0$. The values at the endpoints are $g(0) = 0$ and $g(1) = -\frac{1}{2}$, so the maximum of g is at $x = 0$.

A function like $f(x) = (x+1)^3$ is strictly increasing on $0 \leq x \leq 1$: $f'(x) = 3(x+1)^2 > 0$ for $x \in [0, 1]$. Therefore, the maximum is at the upper endpoint $x = 1$, and the minimum is at the lower endpoint $x = 0$.

A function like $f(x) = -(x+1)^3$ is strictly decreasing on $[0, 1]$: $f'(x) = -3(x+1)^2 < 0$ for $x \in [0, 1]$. Therefore, the maximum is at the lower endpoint $x = 0$, and the minimum is at the upper endpoint $x = 1$.

Example. Wheat sells for \$4 per bushel. Fertilizer costs \$0.25 per pound. Let x be the pounds of fertilizer per acre. The yield is $\sqrt{x+30}$ bushels per acre. How many pounds of fertilizer per acre will maximize profit?

The revenue is $4\sqrt{x+30}$; expense is $0.25x = x/4$. The profit is $\pi = 4\sqrt{x+30} - x/4$. What $x \geq 0$ maximizes π ? The choice set (or domain) is $\mathcal{D} = \{x \geq 0\}$. We find the critical point as follows:

$$\pi' = 2(x+30)^{-\frac{1}{2}} - \frac{1}{4} = 0$$

$$8 = (x+30)^{\frac{1}{2}}$$

$$64 = x+30$$

$$x^* = 34.$$

The second derivative $\pi'' < 0$ so local maximum. The first derivative test is that $\pi' > 0$ for $x < x^*$ and $\pi' < 0$ for $x > x^*$, so maximum. Alternatively, comparing the values at the end points and critical point, $\pi(0) = 4\sqrt{30} \approx 21.91$ and $\pi(34) = 4\sqrt{64} - \frac{34}{4} = 23.50$. ■

Example. Let $u(x, y) = -4xy + x + 2y + 1$ be your utility when choosing $0 \leq x \leq 1$ and your opponent fixes $0 \leq y \leq 1$. Maximize u as a function of x with y as a parameter.

$$\frac{\partial u}{\partial x} = -4y + 1 = \begin{cases} > 0 & \text{for } y < \frac{1}{4} \\ \equiv 0 & \text{for } y = \frac{1}{4} \\ < 0 & \text{for } y > \frac{1}{4} \end{cases}$$

The maximum occurs at

$$x^* = \begin{cases} = 1 & \text{for } y < \frac{1}{4} \\ \in [0, 1] & \text{for } y = \frac{1}{4} \\ = 0 & \text{for } y > \frac{1}{4} \end{cases}$$

■

For “mixed strategies”, we consider functions that are averages of a finite number of values. Given numbers a_1, a_2, a_3 , assume that $f(p_1, p_2, p_3) = a_1p_1 + a_2p_2 + a_3p_3$ for $p_i \geq 0$ and $p_1 + p_2 + p_3 = 1$. It follows that $\max f(p_1, p_2, p_3) = \max\{a_1, a_2, a_3\}$.

- (a) If there is a single a_{i_0} that has the largest value, $a_{i_0} > a_i$ for $i \neq i_0$ (e.g., $\{7, 5, 2\}$), then the maximum of f is a_{i_0} that is attained for $p_{i_0} = 1$ and the other $p_i = 0$.
- (b) If $a_1 = a_2 > a_3$ (e.g., $\{7, 7, 2\}$), then the maximum of f is a_1 that is attained for all $(p_1, p_2, 0)$ with $p_1 + p_2 = 1$ and $p_3 = 0$.
- (c) If $a_1 = a_2 = a_3$ (e.g., $\{7, 7, 7\}$), then f has a constant value of a_1 and every choice of (p_1, p_2, p_3) gives the maximum value.

In summary, there are various type of maximum that we will consider.

1. The maximum of a finite set of numbers.
2. The maximum of a function defined on a closed interval where the maximizer can be a critical point or an end point.
3. Probabilistic combination of a finite number of values. The maximum is either at a single “vertex” or the convex combination of two or more vertices.

§2.1 Strategic Games

Example 1. This example has two players. They set prices and get payoffs (or profits). The prices are either high or low, H_i or L_i for P_i . The set of actions is $A_i = \{H_i, L_i\}$. The payoffs can be given in a bimatrix form:

$$\begin{array}{cc} & \begin{array}{cc} H_2 & L_2 \end{array} \\ \begin{array}{c} H_1 \\ L_1 \end{array} & \begin{pmatrix} (1000, 1000) & (-200, 1200) \\ (1200, -200) & (600, 600) \end{pmatrix} \end{array}$$

Thus, P_1 chooses the row and P_2 chooses the column. The first number given in each pair is the payoff for P_1 and the second is for P_2 . For example, $u_1(H_1, L_2) = -200$ and $u_2(H_1, L_2) = 1200$.

This payoff function induces a preference for P_1 with $(L_1, H_2) \succ_1 (H_1, H_2)$ and $(L_1, L_2) \succ_1 (H_1, L_2)$. Thus, in both cases, P_1 prefers L_1 . For P_2 , $(H_1, L_2) \succ_2 (H_1, H_2)$ and $(L_1, L_2) \succ_2 (L_1, H_2)$. In both cases, P_2 prefers L_2 . Therefore, we end up with the “preferred” action profile (L_1, L_2) . Notice that (H_1, H_2) has higher payoffs for both players than (L_1, L_2) , but it would require cooperation to attain. ■

Definition. A *strategic game* consists of the following: (i) A finite set of players $\{P_i\}_{i=1}^n$. (ii) For each player P_i , there is a set of actions (or choices) $\mathbf{A}_i$, with $a_i \in \mathbf{A}_i$ denoting a choice of an action. Putting the action sets for all the players together, let $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, with $\mathbf{a} = (a_1, \dots, a_n)$ in $\mathbf{A}$ an *action profile*. For a strategic game, the choices of actions are simultaneous by all the players with no knowledge of the choices by the other players. (iii) For each player P_i , there is a *preference* $\succeq_i$ on $\mathbf{A}$ that is partial ordering $\succeq_i$ such that for any two $\mathbf{a}, \mathbf{b} \in \mathbf{A}$, either $\mathbf{a} \succeq_i \mathbf{b}$ or $\mathbf{b} \succeq_i \mathbf{a}$. If $\mathbf{a} \succeq_i \mathbf{b}$ then $\mathbf{a}$ is preferred by P_i to $\mathbf{b}$. We say that two action profiles are *indifferent*, $\mathbf{a} \sim_i \mathbf{b}$, provided that $\mathbf{a} \succeq_i \mathbf{b}$ and $\mathbf{b} \succeq_i \mathbf{a}$. We also write $\mathbf{a} \succ_i \mathbf{b}$ provided that $\mathbf{a} \succeq_i \mathbf{b}$ and $\mathbf{a}$ is not indifferent to $\mathbf{b}$.

A *payoff function* or *preference indicator function* for P_i is a real valued function $u_i : \mathbf{A} \rightarrow \mathbb{R}$. On a finite set of actions, the payoff function is just a finite set of values. Such a function induces a preference by $\mathbf{a} \succeq_i \mathbf{b}$ if and only if $u_i(\mathbf{a}) \geq u_i(\mathbf{b})$. Then, two action profiles are indifferent provided that $u_i(\mathbf{a}) = u_i(\mathbf{b})$. The preferences are *ordinal preferences* because they do not depend on the values or magnitude of the payoff function but only on the order of the values. (When we treat mixed strategies, the magnitude of the payoff functions do matter.)

The theory of *rational choice* specifies that each player chooses her action to maximize her own payoff function or optimize her own preferences. See §1.2.

§2.6 Nash Equilibrium

Definition. For an action profile $(a_1^*, \dots, a_n^*)$ of an n -person game, we use the notation $\mathbf{a}_{-i}^*$ for $(a_1^*, \dots, a_{i-1}^*, a_{i+1}^*, \dots, a_n^*)$, and $(a_i, \mathbf{a}_{-i}^*)$ represents $(a_1^*, \dots, a_{i-1}^*, a_i, a_{i+1}^*, \dots, a_n^*)$. This latter notation indicates that the i^{th} entry can change and all the other entries are left the same.

An action profile $\mathbf{a}^* = (a_1^*, \dots, a_n^*)$ is a *Nash equilibrium* for an n -person strategic game provided that the following condition holds.

For $n = 2$,

$$\begin{aligned} u_1(a_1^*, a_2^*) &\geq u_1(a_1, a_2^*) && \text{for all } a_1 \in A_1 && \text{and} \\ u_2(a_1^*, a_2^*) &\geq u_2(a_1^*, a_2) && \text{for all } a_2 \in A_2. \end{aligned}$$

For any $n \geq 2$,

$$u_i(\mathbf{a}^*) \geq u_i(a_i, \mathbf{a}_{-i}^*) \text{ for all } a_i \in A_i \text{ for each } 1 \leq i \leq n.$$

Example (return to 1). For the earlier example of a duopoly, the payoffs were given by the following bimatrix:

$$\begin{array}{cc} & \begin{array}{cc} H_2 & L_2 \end{array} \\ \begin{array}{c} H_1 \\ L_1 \end{array} & \begin{pmatrix} (1000, 1000) & (-200, 1200) \\ (1200, -200) & (600, 600) \end{pmatrix} \end{array}$$

Our discussion before really showed that (L_1, L_2) is a Nash equilibrium. Notice that

$$\begin{aligned} u_1(L_1, L_2) &= 600 > -200 = u_1(H_1, L_2) \quad \text{and} \\ u_2(L_1, L_2) &= 600 > -200 = u_2(L_1, H_2). \end{aligned}$$

Since the inequalities are strict, this is called a *strict Nash equilibrium*. ■

Example 2. This example is called BoS for “battle of the sexes” or “Bach versus Stravinsky”. The set of actions are $A_i = \{B_i, S_i\}$ for $i = 1, 2$, where B stands for Bach and S stands for Stravinsky. The bimatrix set of payoffs is assumed to be given as follows:

$$\begin{array}{cc} & \begin{matrix} B_2 & S_2 \end{matrix} \\ \begin{matrix} B_1 \\ S_1 \end{matrix} & \begin{pmatrix} (2, 1) & (0, 0) \\ (0, 0) & (1, 2) \end{pmatrix} \end{array}$$

The preferences satisfy

$$\begin{aligned} (B_1, B_2) \triangleright_1 (S_1, B_2) \sim_1 (B_1, S_2) \triangleleft_1 (S_1, S_2) \\ (B_1, B_2) \triangleright_2 (B_1, S_2) \sim_2 (S_1, B_2) \triangleleft_2 (S_1, S_2). \end{aligned}$$

Both (B_1, B_2) and (S_1, S_2) are more desirable for both players than a mixed choice of (S_1, B_2) or (B_1, S_2) ; but P_1 prefers (B_1, B_2) to (S_1, S_2) and P_2 has the opposite preference between these two action profiles.

Notice that $\mathbf{a}^* = (B_1, B_2)$ is a Nash equilibrium:

$$\begin{aligned} u_1(B_1, B_2) &= 2 > 0 = u_1(S_1, B_2) \quad \text{and} \\ u_2(B_1, B_2) &= 1 > 0 = u_2(B_1, S_2). \end{aligned}$$

In this game, (S_1, S_2) is a second Nash equilibrium:

$$\begin{aligned} u_1(S_1, S_2) &= 1 > 0 = u_1(B_1, S_2) \quad \text{and} \\ u_2(S_1, S_2) &= 2 > 0 = u_2(S_1, B_2). \end{aligned}$$

Example 3. This example illustrates that a Nash equilibrium of pure strategies does not always exist. It models the game of matching pennies. The set of actions are $A_i = \{H_i, T_i\}$ for $i = 1, 2$, where H stands for heads and T stands for tails. In this game, if both player get the same side after flipping the coin then P_2 pays P_1 \$1; if the players get different sides then P_1 pays P_2 \$1. The bimatrix set of payoffs is assumed to be given as follows:

$$\begin{array}{cc} & \begin{matrix} H_2 & T_2 \end{matrix} \\ \begin{matrix} H_1 \\ T_1 \end{matrix} & \begin{pmatrix} (1, -1) & (-1, 1) \\ (-1, 1) & (1, -1) \end{pmatrix} \end{array}$$

The preferences satisfy

$$(H_1, H_2) \triangleleft_2 (H_1, T_2) \triangleleft_1 (T_1, T_2) \triangleleft_2 (T_1, H_2) \triangleleft_1 (H_1, H_2).$$

Therefore, there is a cycle with no obvious choice of the best action profile.

For the matching pennies example there is no Nash equilibrium:

$$\begin{aligned} u_2(H_1, H_2) &< u_2(H_1, T_2) & \text{so } (H_1, H_2) \text{ is not a NE} \\ u_1(H_1, T_2) &< u_1(T_1, T_2) & \text{so } (H_1, T_2) \text{ is not a NE} \\ u_2(T_1, T_2) &< u_2(T_1, H_2) & \text{so } (T_1, T_2) \text{ is not a NE} \\ u_1(T_1, H_2) &< u_1(H_1, H_2) & \text{so } (T_1, H_2) \text{ is not a NE.} \end{aligned}$$

Therefore, no (pure) action profile is a Nash equilibrium. ■

§2.8 Best Response

Example 4. We introduce the idea of the best response correspondence through an example. Consider the strategic game given by the following bimatrix set of payoffs:

$$\begin{array}{c} L_2 \quad M_2 \quad R_2 \\ H_1 \quad \left((1, 1) \quad (1, 0) \quad (0, 1) \right) \\ L_1 \quad \left((1, 0) \quad (0, 1) \quad (1, 0) \right) \end{array}$$

If the action H_1 is fixed, then L_2 and R_2 give the largest payoff. Therefore, the best response to H_1 is

$$B_2(H_1) = \{L_2, R_2\}.$$

Similarly for L_1 fixed, then M_2 gives the largest payoff,

$$B_2(L_1) = \{M_2\}.$$

In the same way fixing the action of P_2 and maximizing the payoff of P_1 we get the following:

$$B_1(L_2) = \{H_1, L_1\}$$

$$B_1(M_2) = \{H_1\}$$

$$B_1(R_2) = \{L_1\}.$$

Underlining the best response of P_1 in each column and the best response of P_2 in each row, we get

$$\begin{array}{c} L_2 \quad M_2 \quad R_2 \\ H_1 \quad \left((\underline{1}, \underline{1}) \quad (\underline{1}, 0) \quad (0, \underline{1}) \right) \\ L_1 \quad \left((\underline{1}, 0) \quad (0, \underline{1}) \quad (\underline{1}, 0) \right) \end{array}$$

The Nash equilibria are those profiles for which each choice is the best response to the choice of the other player, i.e., those profiles that are underlined for both players. In this example the only Nash equilibrium is (H_1, L_2) : it is the only profile that is underlined by both players, $H_1 \in B_1(L_2)$ and $L_2 \in B_2(H_1)$. This is a *non-strict* Nash equilibrium because $u_1(H_1, L_2) = 1 = u_1(B_1, L_2)$ and $u_2(H_1, L_2) = u_2(H_1, R_2) = 1$. ■

Definition. For an n -person strategic game, the *best response correspondence* is defined by

$$B_i(\mathbf{a}_{-i}) = \{a_i \in A_i : u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a'_i, \mathbf{a}_{-i}) \text{ for all } a'_i \in A_i\}.$$

Because the best response is set valued, we call it a correspondence rather than a function.

Proposition (36.1). *The action profile $\mathbf{a}^*$ is a Nash equilibrium for a strategic game if and only if $a_i^* \in B_i(\mathbf{a}_{-i}^*)$ for $1 \leq i \leq n$.*

Example 5. Assume there are two firms that can set prices of H (high), M (medium), or L (low). The payoffs are 12, 10, or 8 where the lower prices gets all and the firms split the profit if the prices are the same. Thus, the bimatrix payoff are as follows:

$$\begin{array}{c} H_2 \quad M_2 \quad L_2 \\ H_1 \quad \left((6, 6) \quad (0, 10) \quad (0, 8) \right) \\ M_1 \quad \left((10, 0) \quad (5, 5) \quad (0, 8) \right) \\ L_1 \quad \left((8, 0) \quad (8, 0) \quad (4, 4) \right) \end{array}$$

The best responses are as follows and then underlining the best responses:

$$B_1(H_2) = \{M_1\}$$

$$B_2(H_1) = \{M_2\}$$

$$B_1(M_2) = \{L_1\}$$

$$B_2(M_1) = \{L_2\}$$

$$B_1(L_2) = \{L_1\}$$

$$B_2(L_1) = \{L_2\}.$$

$$\begin{array}{c} H_2 \quad M_2 \quad L_2 \\ H_1 \quad \left((6, 6) \quad (0, \underline{10}) \quad (0, 8) \right) \\ M_1 \quad \left((\underline{10}, 0) \quad (5, 5) \quad (0, \underline{8}) \right) \\ L_1 \quad \left((8, 0) \quad (\underline{8}, 0) \quad (\underline{4}, \underline{4}) \right) \end{array}$$

The profile (L_1, L_2) is the only profile that is underlined by both players with $L_1 \in B_1(L_2)$ and $L_2 \in B_2(L_1)$. For this profile, neither firm has an incentive to change, so it is a Nash equilibrium. ■

Example (39.1). This is a synergistic relationship of two players where a_i is the effort of the i^{th} player and is assumed to be $a_i \geq 0$. For the payoff u_i , let a_j be the effort for the other player. Assume that

$$u_i = a_i(c + a_j - a_i) \quad \text{where } c > 0.$$

For u_i , let a_j be the effort for the other player. At the maximum

$$\begin{aligned} \frac{\partial u_i}{\partial a_i} &= c + a_j - a_i - a_i = 0, & \text{or} \\ a_i &= \frac{c + a_j}{2}. \end{aligned}$$

This is a maximum, because it is the unique critical point of the single variable a_i and $\frac{\partial^2 u_i}{\partial a_i^2} = -2 < 0$.

Therefore, $B_i(a_j) = \{(c + a_j)/2\}$. To find the Nash equilibrium, we need to solve

$$\begin{aligned} c &= 2a_1 - a_2 \\ c &= -a_1 + 2a_2. \end{aligned}$$

Adding twice the first equation to the second equation, we get $3c = 3a_1$ or $a_1^* = c$. Substituting in, we get $a_2^* = c$. The only Nash equilibrium is $(a_1^*, a_2^*) = (c, c)$. ■

Parables for Nash equilibria

The following motivations for Nash equilibria are based on the book by Dutta, [1]. There are various other ways in which the Nash equilibrium concept in game theory has been motivated. These motivations are parables in the sense that we will only give a verbal description of each one. Some of these motivations have been precisely worked out in mathematical models; some others have turned out to be simple and intuitive verbally but virtually impossible to analyze formally. In either case, the parables are worth telling because Nash equilibrium will be the most widely used solution concept in this (and every other) game theory text. Hopefully, these parable will convince you even more about the reasonableness of this solution concept.

Play Prescription

One can think of a Nash Equilibrium as a *prescription* for play. If this action profile is proposed to the players, then it is a stable prescription in the sense that no one has an incentive to play otherwise. By playing an alternative action, a player would simply lower her payoffs, if she thinks the others are going to follow the prescription of the proposed action profile.

Rational Introspection

A related motivation for following a Nash equilibrium is *rational introspection*: Each player could ask himself what he expects will be the outcome to a game. Other possible action profiles appear unreasonable in that there is at least one player who could do better than he is doing; that is, some player would not be choosing a best response to the other actions. The only profiles for which no player appears to be making a mistake is when each is playing a best response, that is, when we are at a Nash equilibrium.

Focal Point

Another explanation for playing a Nash equilibrium was first advanced by Thomas Schelling in 1960 in his book *The Strategy of Conflict*. A *focal point* stands out from the other action profiles because of some distinguishing characteristics. A Nash equilibrium action profile has the distinguishing characteristic that each player plays a best response under that strategy vector.

Preplay Communication

How would the players in a game find their way to a Nash equilibrium? One answer that has been proposed

is that they could agree on mutually acceptable actions through preplay communications. An action that is not a Nash equilibrium would not be a credible negotiated choice because at least one player would cheat against such an agreement to increase her payoff.

Trial and Error

If players started by playing a strategy vector that is not a Nash equilibrium, somebody would discover that she could do better. If she changes her strategy choice, and we are still not in a Nash equilibrium, somebody else might want to change his strategy. This process of trial and error would go on until such time as we reach a Nash equilibrium – and then nobody has the incentive to change her strategy choice. This reasoning is persuasive but not entirely correct because there is no guarantee that this process would ever lead to a stable situation. Moreover, it is easy to construct examples in which this process could leave us trapped in cycles in which players keep changing their strategies in search of higher payoffs but nowhere is everyone satisfies simultaneously.

Evolutionary Game Theory

“In dynamic games that model an evolutionary process whereby successful strategies drive out unsuccessful ones over time, stationary states are, with some obvious and uninteresting exception, Nash equilibria.” See Gintis [2].

§3.1.3 Cournot Duopoly

In this model, we assume that there are two players (firms). The action by P_i is a choice of the production level $q_i \in A_i = [0, \infty)$ and there is a continuous set of possible actions. The total production by the two firms is $Q = q_1 + q_2$. The price is assumed to be determined by the total production:

$$p = p(Q) = \begin{cases} a - bQ & \text{if } 0 \leq Q \leq a/b \\ 0 & \text{if } Q \geq a/b, \end{cases}$$

where $a, b > 0$ are fixed parameters. Because the price is determined by the quantity, this function is called the inverse demand function. (The demand function gives the quantity demanded by the consumers as a function of the price.) The revenue of P_i is $q_i p(Q)$, and the cost is assumed to be $c_i q_i$, where $c_i > 0$ are fixed parameters of marginal cost. The profit is the revenue minus the cost, so the profit of P_i is

$$\pi_i(q_1, q_2) = \begin{cases} q_i [a - b q_1 - b q_2 - c_i] & \text{if } 0 \leq q_1 + q_2 \leq a/b \\ -c_i q_i & \text{if } a/b \leq q_1 + q_2. \end{cases}$$

The quantities that maximize the profit for P_1 satisfy

$$\begin{aligned} 0 &= \frac{\partial \pi_1}{\partial q_1} = -2b q_1 - b q_2 + a - c_1 \quad \text{or} \\ 2 q_1 + q_2 &= \frac{a - c_1}{b}. \end{aligned}$$

Notice that $\frac{\partial^2 \pi_1}{\partial q_1^2} = -2b < 0$, so the critical point maximizes π_1 as a function of q_1 .

The quantities that maximize the profit for P_2 satisfy

$$\begin{aligned} 0 &= \frac{\partial \pi_2}{\partial q_2} = -b q_1 - 2b q_2 + a - c_2 \quad \text{or} \\ q_1 + 2 q_2 &= \frac{a - c_2}{b}. \end{aligned}$$

Again, $\frac{\partial^2 \pi_2}{\partial q_2^2} = -2b < 0$, so the critical point maximizes π_2 as a function of q_2 .

If we solve these two simultaneous equations, we get

$$q_1^* = \frac{1}{3b} (a + c_2 - 2c_1) \quad \text{and}$$

$$q_2^* = \frac{1}{3b} (a + c_1 - 2c_2).$$

Notice that $q_1^* + q_2^* = (2a - c_1 - c_2)/3b < a/b$, so we do not need to worry about the second expression in the definition of the profit.

See Figure 1 for the range of marginal costs where both firms produce positive quantities.

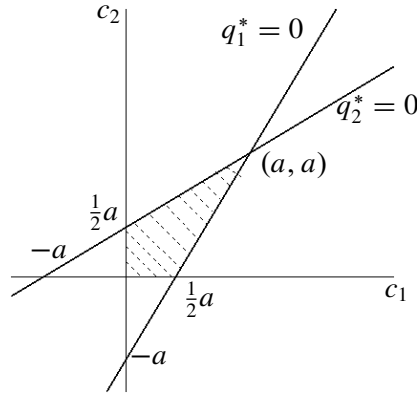


FIGURE 1. Allowable marginal costs of the Cournot Duopoly

Cournot Oligopoly. Assume there are n firms with all the same marginal costs $c_i = c > 0$. The cost of producing q_i by P_i is cq_i . The total amount produced is $Q = q_1 + \dots + q_n$. The inverse demand function that determines the price in terms of the amount produced Q is assumed to be of the form

$$p = p(Q) = k Q^{-\epsilon},$$

where $0 < \epsilon < 2n-1$ is a constant and $k > 0$. (This inverse demand function has a constant elasticity of demand because

$$\frac{dp}{dQ} = -\epsilon k Q^{-\epsilon-1} = \frac{-\epsilon p}{Q}.$$

Thus, the marginal rate of change equals a constant times the average price to quantity ratio.) The profit of P_i is

$$\pi_i = pq_i - cq_i = q_i [k(q_1 + \dots + q_n)^{-\epsilon} - c].$$

The critical occurs for

$$\begin{aligned} \frac{\partial \pi_i}{\partial q_i} &= k(q_1 + \dots + q_n)^{-\epsilon} - c - q_i \epsilon k (q_1 + \dots + q_n)^{-\epsilon-1} = 0 \\ \epsilon k q_i Q^{-\epsilon-1} &= k Q^{-\epsilon} - c \\ q_i^* &= \frac{1}{\epsilon} Q^* - \frac{c}{\epsilon k} (Q^*)^{\epsilon+1} \end{aligned}$$

which is the same for all i . So,

$$\begin{aligned} Q^* &= q_1^* + \dots + q_n^* = n q_i^* \\ q_i^* &= \frac{Q^*}{n}. \end{aligned}$$

Letting $Q = Q^*$ and $q_i^* = \frac{Q}{n}$, we have from above that

$$\begin{aligned}\frac{Q}{n} &= \frac{1}{\epsilon} Q - \frac{c}{\epsilon k} Q^{\epsilon+1} \\ \frac{\epsilon k}{cn} &= \frac{k}{c} - Q^\epsilon \\ Q^\epsilon &= \frac{k}{c} \left(1 - \frac{\epsilon}{n}\right) \\ Q^* &= \left[\frac{k}{c} \left(1 - \frac{\epsilon}{n}\right) \right]^{\frac{1}{\epsilon}}.\end{aligned}$$

As n increases, the last expression for Q^* increases and limits to $(k/c)^{\frac{1}{\epsilon}}$.

The second derivative at the critical point, q_j^* for all j , satisfies

$$\begin{aligned}\frac{\partial^2 \pi_i}{\partial q_i^2} &= -2\epsilon k Q^{-\epsilon-1} + q_i k \epsilon (\epsilon + 1) Q^{-\epsilon-2} \\ &= k \epsilon Q^{-\epsilon-1} \left[\frac{\epsilon + 1}{n} - 2 \right] < 0\end{aligned}$$

provided that $\epsilon + 1 < 2n$. Therefore with this restriction on ϵ , π_i attains a maximum at $q_i = q_i^*$ while the other q_{-i} held fixed. Therefore $\mathbf{q}^*$ is a Nash equilibrium.

§3.2.2 Bertrand Duopoly

In this model, each of the two firms sets a price $p_i \geq 0$ for the common product. Here we follow the book and assume that $C_i(q_i) = cq_i$ with the same marginal cost for both firms. In this model, we assume that the total quantity consumed is a linear decreasing function of price $Q = D(p) = a - p$. (This is the demand function.) We assume that $a > c$. The profit for P_i is $\pi_i = p_i q_i - cq_i = (p_i - c)q_i$. Let $j \neq i$ be the other firm. If $p_i < p_j$, then $p = p_i$, P_i gets the total market $q_i = Q = a - p_i$, and

$$\pi_i = (p_i - c)(a - p_i).$$

If $p_i = p_j$, then the firms split the market and

$$\pi_i = \frac{1}{2}(p_i - c)(a - p_i).$$

If $p_i > p_j$, then P_i gets no market share and $\pi_i = 0$. Therefore,

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(a - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(a - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j. \end{cases}$$

Consider the function $f(p) = (p - c)(a - p)$. Then $\frac{df}{dp} = a - p - p + c = a + c - 2p$, $\frac{d^2 f}{dp^2} = -2 < 0$, and the critical point $\bar{p} = \frac{a + c}{2}$ is a maximum.

Following the book, we use the best response correspondence to find the Nash equilibrium. For P_1 , we break the analysis into cases depending on $p_2 < c, = c, > c$.

(i) Assume $p_2 < c$. Then,

$$\pi_1(p_1, p_2) \begin{cases} < 0 & \text{if } p_1 < p_2 \\ < 0 & \text{if } p_1 = p_2 \\ = 0 & \text{if } p_1 > p_2. \end{cases}$$

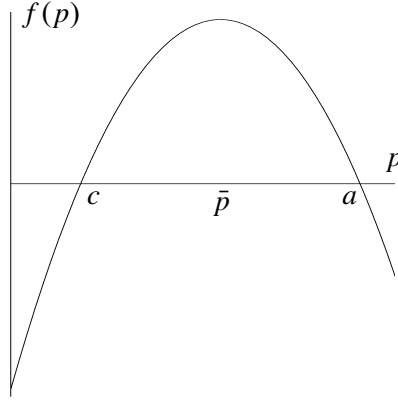


FIGURE 2. Function related to profit for Bertrand duopoly

Therefore, in this case,

$$B_1(p_2) = \{p_1 : p_1 > p_2\}.$$

(ii) Assume $p_2 = c$. Then,

$$\pi_1(p_1, p_2) \begin{cases} < 0 & \text{if } p_1 < p_2 \\ = 0 & \text{if } p_1 = p_2 \\ = 0 & \text{if } p_1 > p_2. \end{cases}$$

Therefore, in this case,

$$B_1(p_2) = \{p_1 : p_1 \geq p_2\}.$$

(iii) Assume $c < p_2 \leq \bar{p}$. Then π_1 does not attain a maximum. The values limits on $f(p_2)$ as p_1 approaches p_2 , but $\pi_1(p_2, p_2) = \frac{1}{2}f(p_2)$. Therefore, in this case,

$$B_1(p_2) = \emptyset.$$

(iv) Finally, assume $\bar{p} < p_2$. Then,

$$\pi_1(p_1, p_2) \begin{cases} = (p_1 - c)(a - p_1) & \text{if } p_1 < p_2 \\ = \frac{1}{2}(p_1 - c)(a - p_1) & \text{if } p_1 = p_2 \\ = 0 & \text{if } p_1 > p_2. \end{cases}$$

Therefore, in this case,

$$B_1(p_2) = \{p_1 : p_1 = \bar{p}\}.$$

If we plot the best response correspondences in a single figure, we have Figure 3. The only common point in both sets is $(p_1^*, p_2^*) = (c, c)$. Therefore, this is the only Nash equilibrium. Notice, for these prices, both firms have zero profit.

Common Property Resources

(cf. §3.1.5) This example shows that the individual incentive can work against the common interest.

We first consider a non-cooperative equilibrium. Assume there are $N + 1$ players that use the same resource (e.g., fishing grounds). Let $r_i \geq 0$ be the amount of resource used by P_i ; r_i is the choice of the action. Let

$$R = \sum_{i=1}^{N+1} r_i \quad \text{and}$$

$$R_{-i} = R - r_i = \sum_{j \neq i} r_j.$$

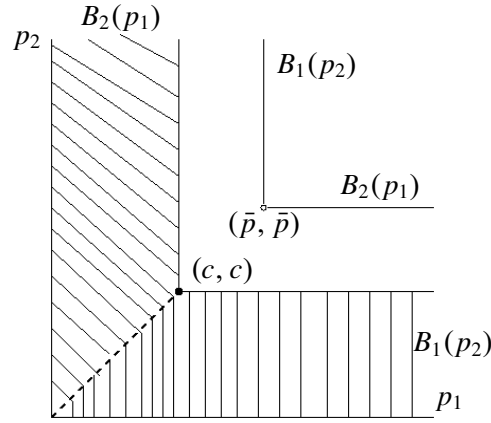


FIGURE 3. Best response functions for Bertrand duopoly

The cost function for P_i , $C : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, is assumed to be of the form

$$C(r_i, R_{-i}) = k(r_i) + K(R_{-i}),$$

where the functions $k(r)$ and $K(R)$ are assumed to be the same for all players. We assume that

$$\begin{aligned} k'(r_i) &= \frac{\partial C}{\partial r_i} > 0, \\ k''(r_i) &= \frac{\partial^2 C}{\partial r_i^2} > 0, \\ K'(R_{-i}) &= \frac{\partial C}{\partial R_{-i}} > 0, \quad \text{and} \\ K''(R_{-i}) &= \frac{\partial^2 C}{\partial R_{-i}^2} > 0. \end{aligned}$$

The first and second inequalities are the assumptions that there are positive marginal costs that increase with increasing use of the resource by P_i . The third inequality is the assumptions that there are positive marginal costs in terms of the use of the resource by other players. This marginal cost is also assumed to increase with increasing use of the resource by the other players.

The utility $u(r_i)$ for P_i before the costs is assumed to satisfy

$$\begin{aligned} u'(r) &> 0, \\ u''(r) &< 0, \\ u'(0) &> k'(0), \quad \text{and} \\ u'(r) &< k'(r) \quad \text{for all } r \geq r_0, \end{aligned}$$

This third assumption is that the marginal utility is greater than the marginal cost for small values of r_i . The fourth assumption is that for large use of the resource, the marginal utility is less than the marginal cost.

The payoff function is

$$\pi_i(r_1, \dots, r_{N+1}) = u(r_i) - k(r_i) - K(R_{-i}).$$

If we hold r_j fixed for all $j \neq i$, then R_{-i} is fixed and

$$\begin{aligned} \frac{\partial \pi_i}{\partial r_i} &= u'(r_i) - k'(r_i) \quad \text{and} \\ \frac{\partial^2 \pi_i}{\partial r_i^2} &= u''(r_i) - k''(r_i) < 0. \end{aligned}$$

The critical point is thus a maximum. The critical point is the point $r_i = r^*$ where $u'(r^*) = k'(r^*)$. See Figure 4. This value is the same for all players. Thus, $R^* = (N + 1)r^*$ and $r^* = \frac{R^*}{N + 1}$. This is a Nash equilibrium.

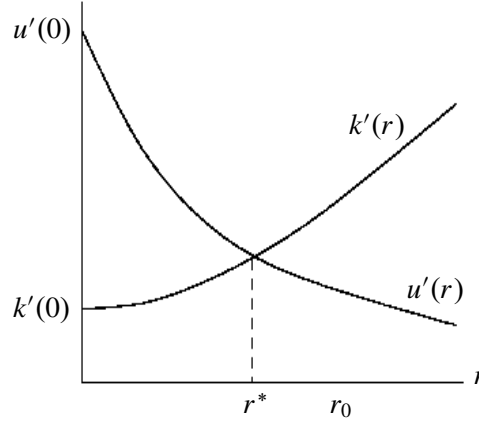


FIGURE 4. Graphs of marginal utility and cost for common property

Social Optimum. In this situation, we maximize the sum of the payoffs for the individual players or joint payoff function V . Assume that $r = r_i = \frac{R}{N + 1}$ is the same for each player. Since the payoff functions are all the same,

$$V(r) = (N + 1) [u(r) - k(r) - K(Nr)].$$

The critical point satisfies

$$\begin{aligned} V'(r) &= (N + 1) [u'(r) - k'(r) - N K'(Nr)] = 0, \\ u'(r) &= k'(r) + N K'(Nr). \end{aligned}$$

We need $u'(0) > k'(0) + (N + 1) K'(0)$ for there to be a point $r = \rho^*$ giving this equality.

$$V''(r) = (N + 1) [u''(r) - k''(r) - N^2 K''(Nr)] < 0,$$

so ρ^* maximizes V . Since $k'(r) + N K'(Nr) > k'(r)$ and $u'(r)$ has negative slope, $\rho^* < r^*$ as can be seen in Figure 5. For the maximum of V , the total resource used is $(N + 1)\rho^*$ which is less than $(N + 1)r^*$. Since ρ^* maximizes V , $V(\rho^*) > V(r^*)$. Thus, by using the social optimum rather than the non-cooperative optimum, there is greater good for the total (and each player) while using less of the resource.

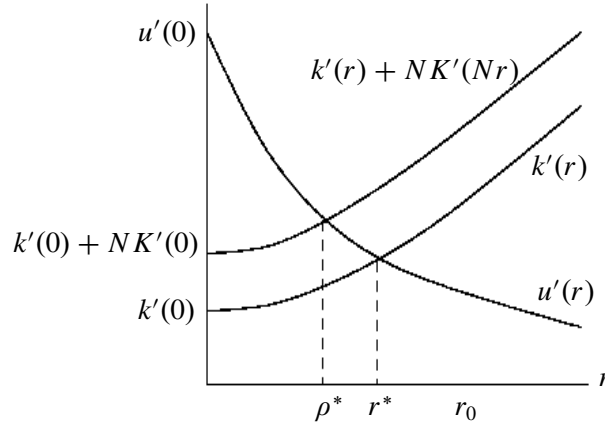


FIGURE 5. Graphs of marginal utility and cost for common property

§3.5 Second-price Sealed-bid Auctions

We consider an n -players with valuations $v_i \geq 0$ for all $1 \leq i \leq n$. (We DO NOT assume the ordering $v_1 > v_2 > \dots > v_n > 0$ as is made in the book.) The valuations are known by everyone and fixed. Each player submits a bid $b_i \geq 0$, with b_i known only to the person submitting it, sealed bids. The profile of all bids is denoted by $\mathbf{b} = (b_1, \dots, b_n)$. The highest bidder wins and pays the second highest bid submitted, hence the name *second-price sealed-bid auction*. In the case when $r > 1$ bidders submit the same highest price (are finalists), we use a different payoff than the book: we assume that they have an equal chance of getting the item auctioned. We give the payoff in terms of the function

$$M(\mathbf{b}_{-i}) = \max\{b_j : j \neq i\}.$$

Notice that $M(\mathbf{b}_{-i})$ depends on the bids and not the valuations. Then, the payoff functions are given by

$$u_i(b_1, \dots, b_n) = \begin{cases} v_i - M(\mathbf{b}_{-i}) & \text{if } b_i > M(\mathbf{b}_{-i}) \text{ \& } P_i \text{ wins} \\ \frac{1}{r}[v_i - M(\mathbf{b}_{-i})] & \text{if } b_i = M(\mathbf{b}_{-i}) \text{ \& } P_i \text{ is one of } r > 1 \text{ finalists} \\ 0 & \text{if } b_i < M(\mathbf{b}_{-i}) \text{ \& } P_i \text{ loses} \end{cases}$$

We consider various cases to determine the best response correspondence.

(a) $v_i > M(\mathbf{b}_{-i})$

- (i) $b_i > M(\mathbf{b}_{-i})$: P_i wins, $u_i(b_i, \mathbf{b}_{-i}) = v_i - M(\mathbf{b}_{-i}) > 0$.
- (ii) $b_i = M(\mathbf{b}_{-i})$: P_i ties, $u_i(b_i, \mathbf{b}_{-i}) = \frac{1}{r}[v_i - M(\mathbf{b}_{-i})] > 0$.
- (iii) $b_i < M(\mathbf{b}_{-i})$: P_i loses, $u_i(b_i, \mathbf{b}_{-i}) = 0$.

The best response correspondence is

$$B_i(\mathbf{b}_{-i}) = \{b_i : b_i > M(\mathbf{b}_{-i})\}.$$

(b) $v_i = M(\mathbf{b}_{-i})$

- (i) $b_i > M(\mathbf{b}_{-i}) = v_i$: P_i wins, $u_i(b_i, \mathbf{b}_{-i}) = v_i - M(\mathbf{b}_{-i}) = 0$.
- (ii) $b_i = M(\mathbf{b}_{-i}) = v_i$: P_i ties, $u_i(b_i, \mathbf{b}_{-i}) = \frac{1}{r}[v_i - M(\mathbf{b}_{-i})] = 0$.
- (iii) $b_i < M(\mathbf{b}_{-i})$: P_i loses, $u_i(b_i, \mathbf{b}_{-i}) = 0$.

The best response correspondence is

$$B_i(\mathbf{b}_{-i}) = \{b_i : b_i \geq 0\}.$$

(c) $v_i < M(\mathbf{b}_{-i})$

- (i) $b_i > M(\mathbf{b}_{-i}) > v_i$: P_i wins, $u_i(b_i, \mathbf{b}_{-i}) = v_i - M(\mathbf{b}_{-i}) < 0$.
- (ii) $b_i = M(\mathbf{b}_{-i}) > v_i$: P_i ties, $u_i(b_i, \mathbf{b}_{-i}) = \frac{1}{r}[v_i - M(\mathbf{b}_{-i})] < 0$.
- (iii) $b_i < M(\mathbf{b}_{-i})$: P_i loses, $u_i(b_i, \mathbf{b}_{-i}) = 0$.

The best response correspondence is

$$B_i(\mathbf{b}_{-i}) = \{b_i : b_i < M(\mathbf{b}_{-i})\}.$$

Since $v_i \in B_i(\mathbf{b}_{-i})$ for each i , $(b_1, \dots, b_n) = (v_1, \dots, v_n)$ is one Nash equilibrium.

However, there are more than one Nash equilibrium.

Consider the case where $v_1 > v_i$ for $i \geq 2$. If $b_1 > M(\mathbf{v}_{-1})$, then $\{b_i : b_i \leq M(\mathbf{v}_{-1})\} \subset B_i(b_1)$. Therefore any bid profile $(b_1, \dots, b_n)$ for which $b_1 > M(\mathbf{v}_{-1})$ and $b_i \leq M(\mathbf{v}_{-1})$ for $i \geq 2$ is also a Nash equilibrium.

The book notes that there is yet another Nash equilibrium when $v_1 > v_2 \geq v_i$ for $i \geq 3$ that is less obvious and less plausible (less “natural” or less “logical”). Assume that a bid profile $(b_1^*, b_2^*, \dots, b_n^*)$ satisfies $b_2^* \geq v_1 > v_2$, and $b_i^* \leq v_2$ for $i \neq 2$. A direct check shows that this is a Nash equilibrium with P_2 receiving a payoff of $v_2 - M(\mathbf{b}_{-2}) \geq 0$ and every player other receiving a payoff of zero.

REFERENCES

- [1] P. Dutta, **Strategies and Games**, MIT Press, Cambridge MA and London UK, 1999
- [2] H. Gintis, **Game Theory Evolving**, Princeton University Press, Princeton NJ, 2000.