CHAPTER 10: EXTENSIVE GAMES WITH IMPERFECT INFORMATION

We introduce the concept of an information set through three examples.

Example 1. The strategic form of the BoS game is given by

$$\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}.$$

In the matrix the first row and first column are B and the second row and second column are S. We can give this game in extensive form as in Figure 1. Because the second player P_2 must make a choice at vertices v_1 and v_2 without knowing the choice of player P_1, we connect them with a dotted line and label the edges coming out with common labels, B_2 and S_2. Player P_2 must make the choice of the edges with the same labels at both of these vertices. This pair of vertices $\mathscr{I} = \{v_1, v_2\}$ is called an information set.

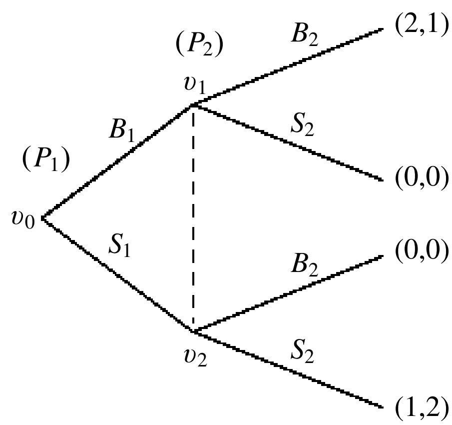

FIGURE 1. Game tree for Example 1

■

Example 2 (**315.1**). In this example, both players start by putting \$1 in the pot. One card is dealt to player P_1. There is 0.5 chance of a high card, H, and 0.5 chance of a low card, L. We represent this by saying the the root is owned by "nature" or "chance", P_0. Player P_1 can either see or raise. Since P_1 knows which card s/he has been dealt, the choices are labeled s_H and r_H at v_1 (corresponding to the card H) and s_L or r_L at v_2 (corresponding to the card L). If player P_1 sees, then P_1 gets the pot if the card is H and P_2 gets the pot of the card is L. Thus, the payoffs are $u(s_H) = (1, \text{-}1)$ and $u(s_L) = (\text{-}1, 1)$. If P_1 raises, then s/he must put in \$$k$ more dollars in the pot. Then player P_2 must decide whether to fold or meet (call) the raise. If s/he meets, then s/he must put \$$k$ more dollars in the pot. Then, the payoffs are $u(H, r_H, m) = (1 + k, \text{-}1\text{-}k)$, $u(L, r_L, m) = (\text{-}1\text{-}k, 1 + k)$, and $u(*, r_*, f) = (1, \text{-}1)$. The expected payoff for a strategy profile is the average over the payoffs averaged by the probabilities for chance, e.g., $E(u)(\{r_H, S_L\}, \{m\}) = \frac{1}{2}(1 + k, \text{-}1\text{-}k) + \frac{1}{2}(\text{-}1, 1) = (\frac{1}{2}k, \text{-}\frac{1}{2}k)$. The information set $\mathscr{I} = \{v_4, v_5\}$ is the set of vertices where P_1 has raised and P_2 does not know whether the card is H or L. Player P_2 must make the same choice at both vertices in $\mathscr{I}$. See Figure 2. ■

Example 3. Consider the extensive game with imperfect information given in Figure 3. Player P_1 owns the information sets $\mathscr{I}_1 = \{C, D\}$ and $\mathscr{I}_2 = \{E, F\}$. In the definition of an information set, we do not allow the root R to be added to either of these information sets, even though it is owned by the same player.

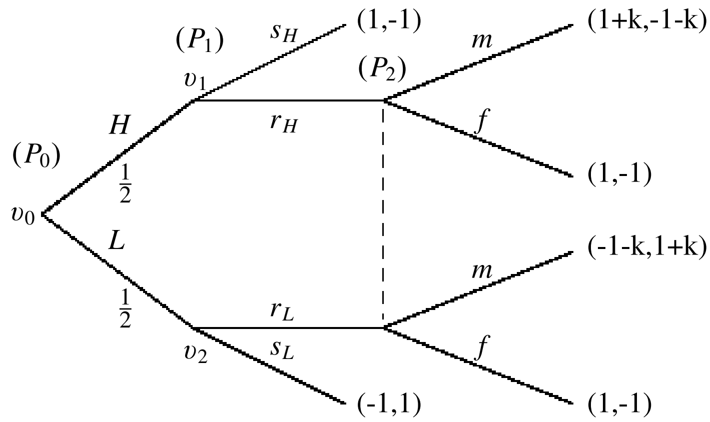

FIGURE 2. Game tree for Example 2

The problem is that vertex C (and D) in the proposed information set would be a descendant of R in the information set, and this is not allowed.

If the game has "perfect recall", then we do not allow $\{C, D, E, F\}$ to be a single information set. The problem is that C and E have have a common ancestor R that is owned by the same player P_1 as the information set. ■

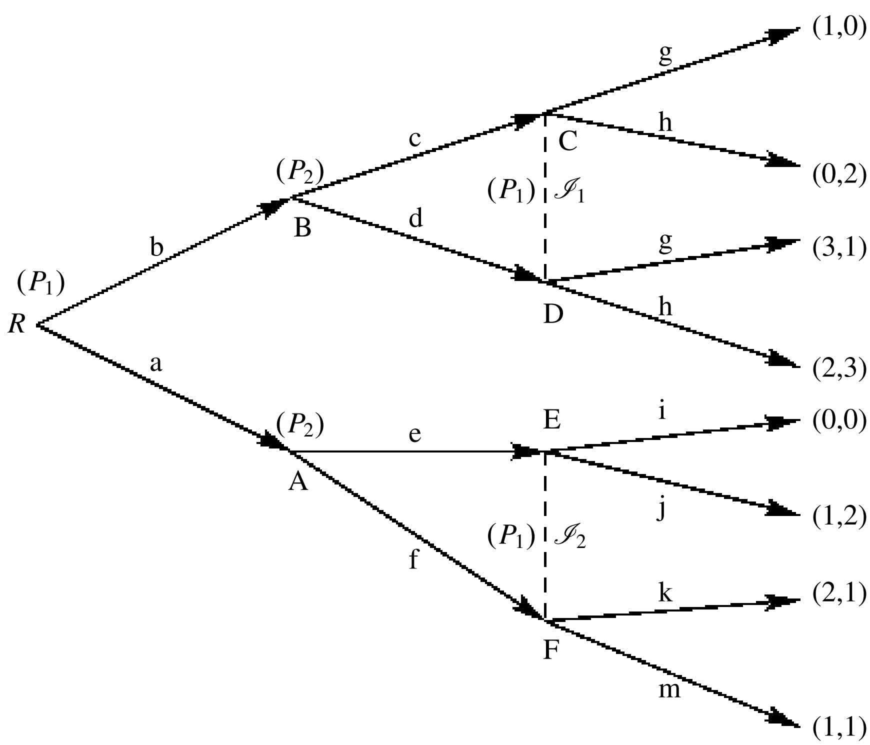

FIGURE 3. Game tree for Example 3

Definition. A set of vertices $\mathscr{I} = \{v_1, \ldots, v_m\}$ is allowable as an *information set* for player P_ℓ provided that the following conditions are satisfied.

1. All the vertices in $\mathscr{I}$ are owned by the single player P_ℓ.
2. No vertex in $\mathscr{I}$ is a descendant of any other vertex in $\mathscr{I}$.
3. All the vertices of $\mathscr{I}$ are indistinguishable for P_ℓ: That is, there is the same number k of edges coming out of each vertex v in $\mathscr{I}$, $\{e^v_1, \dots, e^v_k\}$; the edges for different vertices are naturally pair by $e^v_i \sim e^w_i$ for $v, w \in \mathscr{I}$ and $1 \le i \le k$, i.e., we can put the same label on the equivalent edges; player P_ℓ must choose the corresponding edges with the same labels e^v_ℓ at all the vertices $v \in \mathscr{I}$.

Let V_ℓ be the set of vertices owned by player P_ℓ, and V_n be all the non-terminal vertices owned by one of the players. An *information partition* is a partition of V_n by partitioning each V_ℓ into allowable information sets for each player P_ℓ. An information set for chance P_0 must be single vertex.

Definition. A game is said to have *perfect recall* provided that at each information set $\mathscr{I}$ owned by a player P_ℓ, P_ℓ knows the choices that s/he made earlier in the game, i.e., if (i) $v \neq v'$ and $v, v' \in \mathscr{I}$ owned by a player P_ℓ, and (ii) v is a descendant of a vertex $v_0 \in \mathscr{I}_0$ owned by P_ℓ, then v' is a descendant of a vertex $v'_0 \in \mathscr{I}_0$ and each choice made by P_ℓ on the path from v_0 to v corresponds to a choice with the same label made by P_ℓ on the path from v'_0 to v'. We allow $v_0 = v'_0$. (See p. 81 [3].)

Definition. An *extensive game with imperfect information and chance moves allowed* is the following.

1. There is a set of players $\mathscr{P} = \{P_0, P_1, \dots, P_n\}$, where P_0 is "chance".
2. There is a game tree $\mathscr{T}$ with non-terminal vertices $\mathbf{V}_n$ and terminal vertices $\mathbf{V}_t$.
3. There is a *player function* P that assigns a player $P(v)$ to each non-terminal vertex v, $P : \mathbf{V}_n \to \mathscr{P}$. We say that the non-terminal vertex v of the tree is owned by the player $P(v)$.
4. There is an information partition of the non-terminal vertices $\mathbf{V}_n$.
5. For a vertex v owned by P_0, chance, the edges coming out $\{e^v_1, \dots, e^v_k\}$ are assigned fixed known probabilities $p^v_i \ge 0$ for $1 \le i \le k$, with $p^v_1 + \cdots + p^v_k = 1$ that are independent of other distributions.
6. For an information set $\mathscr{I}$ owned by a player P_ℓ with $1 \le \ell \le k$, a player other than chance, there is a labeling of the vertices $\{e^v_1, \dots, e^v_k\}$ for each $v \in \mathscr{I}$ such that P_ℓ must make a choice of one set of edges with the same label, $\{e^v_i\}_{v \in \mathscr{I}}$. We say that P_ℓ chooses an action on $\mathscr{I}$.
7. For each terminal vertex v, there is a payoff vector $\mathbf{u}(v) = (u_1(v), \dots, u_n(v))$, where $u_i(v)$ is the payoff for the i^{th}-player, $\mathbf{u} : \mathbf{V}_t \to \mathbb{R}^n$. Notice that there is no payoff for P_0, chance. If there is only a game graph or infinite horizon tree, then the payoff vectors are defined for every terminal history.

Definition. A (pure) *strategy* for player P_ℓ in a extensive game with imperfect information is a choice of an action for each information set owned by P_ℓ. Thus P_ℓ must make the choice of one set of edges with the same labels on an information set. A *strategy profile* is a strategy for each player: $\mathbf{s} = (s_1, \dots, s_n)$ where each s_ℓ is a strategy for P_ℓ. The expected payoff for a strategy profile, $E(u)(\mathbf{s}) = (E_1(\mathbf{s}), \dots, E_n(\mathbf{s}))$, is the weighted average over the probabilities at the vertices owned by P_0, chance.

The definition of a Nash equilibrium is essentially the same as before.

Definition. For a two person extensive game with imperfect information, a pure strategy profile $\mathbf{s}^* = (s^*_1, s^*_2)$ is a *Nash equilibrium* provided that the expected payoffs

$$E_1(s^*_1, s^*_2) \ge E_1(s_1, s^*_2) \qquad \text{and}$$
$$E_2(s^*_1, s^*_2) \ge E_2(s^*_1, s_2)$$

for all pure strategies s_1 for P_1 and s_2 for P_2.

There is a similar definition for an n-person extensive game with imperfect information.

Once we have defined allowable subgames of an extensive game with imperfect information, the definition of a subgame perfect Nash equilibrium is the same as before.

Definition. A *subgame* of an extensive game with imperfect information is another extensive game with imperfect information such that the following conditions are hold:

1. Its game tree is a subtree $\mathscr{T}_{u_0}$ of the original game tree.

2. An information set $\mathscr{I}$ of the original game must either be completely in the subtree $\mathscr{T}_{u_0}$ or none of $\mathscr{I}$ is in $\mathscr{T}_{u_0}$; the information sets for the subgame are those information sets of the original game that are in the subtree.
3. The payoff vectors of the subgame are the same as those of the original game at the terminal vertices contained in the subtree.

Definition. A strategy profile $(s_1^*, \dots, s_n^*)$ for an extensive game with imperfect information is said to be a *subgame perfect Nash equilibrium* provided that it is a Nash equilibrium of the total game and for every subgame.

A subgame perfect equilibrium avoids threats of irrational or punitive choices at vertices of the game which are not attained by the optimal path determined by the strategy.

Example 4. Sometimes it is possible to make a choice of a pure strategy without knowing the probabilities of being at the various vertices in the information set, so we can find the Nash equilibrium by backward induction.

The extensive game with imperfect information given in Figure 4 illustrates this possibility.

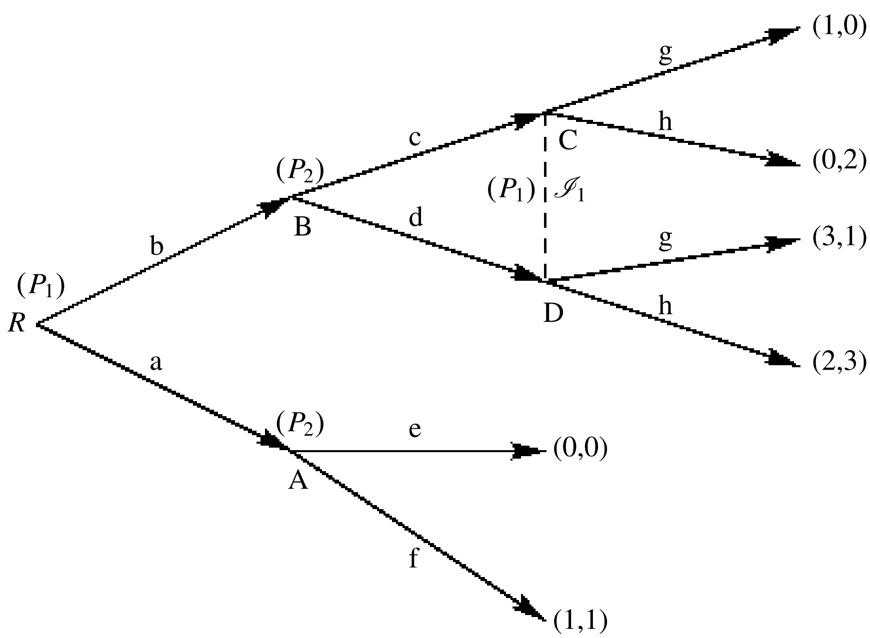

FIGURE 4. Game tree for Example 4

- On $\mathscr{I} = \{C, D\}$: $u_1(g|C) = 1 > 0 = u_2(h|C)$ and $u_1(g|D) = 3 > 2 = u_1(h|D)$. Therefore, g is better than h for both choices of P_2 and P_1 picks g.
- At A: $u_2(f) = 1 > 0 = u_2(e)$. Therefore, P_2 picks f.
- At B: $u_2(d, g) = 1 > 0 = u_2(c, g)$. Therefore, P_2 picks d.
- At R: $u_1(\{b, g\}, \{d, f\}) = 3 > 1 = u_1(\{a, g\}, \{d, f\})$. Therefore, P_1 picks b.

It can be directly checked that the strategy profile $(\{b, g\}, \{d, f\})$ is subgame perfect: it is a Nash equilibrium for the subgames starting at A and B as well as for the whole game.

There is another Nash equilibrium which is not subgame perfect. Consider the strategy profile $(\{a, g\}, \{c, f\})$. This is not "rational" because the payoff for P_2 improves starting at B by choosing d rather than c. However, it is still a Nash equilibrium for the whole game:

$$
\begin{aligned}
u_1(\{a, g\}, \{c, f\}) &= 1 \\
u_1(\{a, h\}, \{c, f\}) &= 1 \\
u_1(\{b, g\}, \{c, f\}) &= 1 \qquad \text{and} \\
u_1(\{b, h\}, \{c, f\}) &= 0.
\end{aligned}
$$

The payoff for $\{a, g\}$ is at least as large as any of the other choices for P_1. For P_2,

$$
\begin{aligned}
u_2(\{a, g\}, \{c, f\}) &= 1 \\
u_2(\{a, g\}, \{d, f\}) &= 1 \\
u_2(\{a, g\}, \{c, e\}) &= 0 \qquad \text{and} \\
u_2(\{a, g\}, \{d, e\}) &= 0.
\end{aligned}
$$

The payoff for $\{c, f\}$ is at least as large as any of the other choices for P_2. ■

Example 5. **(Simplified Poker Game)** In this very simplified poker game, there are two players which get one card each, which is either an ace (A) and or a king (K). The cards are dealt independently so it is assumed that each player has a probability of $1/2$ to receive an ace and $1/2$ to receive a king. The game tree is given in Figure 5 where the root is in the center and is owned by "nature" which deals one of the four types of hands with probability $1/4$ each. Since these vertices are owned by player one, we label them by $v^1_{c_1c_2}$ where c_i is player i's card.

Each player makes an initial ante of \$1 before the game starts. Then, the first player can either (i) fold and lose the ante or (ii) raise \$3 more to stay in the game. The player knows only his/her own hand, so must take the same action at the vertices $\mathscr{I}^1_A = \{v^1_{AA}, v^1_{AK}\}$ where s/he hold an A and the same action at the vertices $\mathscr{I}^1_K = \{v^1_{KA}, v^1_{KK}\}$ where s/he holds a K. We label the choices of raising or folding on the information set $\mathscr{I}^1_A$ by r^1_A and f^1_A and those on $\mathscr{I}^1_K$ by r^1_K and f^1_K.

If the first player decides to raise, then the second player can either (i) fold and lose the ante or (ii) call and match the \$3 bet and determine who wins the pot. Again the second player must take the same action on the information set $\mathscr{I}^2_A = \{v^2_{AA}, v^2_{KA}\}$ where s/he holds a A, which we label by c^2_A and f^2_A, and the same action on the other information set $\mathscr{I}^2_K = \{v^2_{AK}, v^2_{KK}\}$ where s/he holds a K, which we label by c^2_K and f^2_K. If both players have the same card, then they split the pot and break even. If one player has an ace and the other a king, then the player with the ace gets the whole pot and comes out \$4 ahead. The various payoffs to the two players are given in Figure 5.

Since a subgame must contain a whole information set if it contains part of an information set, this game has no proper subgames.

Strategic form of this game:

We can transform this extensive game into a strategic game by calculating all the payoffs for strategy profiles. Each player has the choice of four strategies so there are sixteen strategy profiles: $(r^1_A r^1_K, c^2_A c^2_K)$, etc. For each of the four hands, we can calculate the payoffs for each of these strategy profiles. Then, the payoff for the game for the strategy profile is the average over the four possible hands. The following chart gives the calculation of the payoffs.

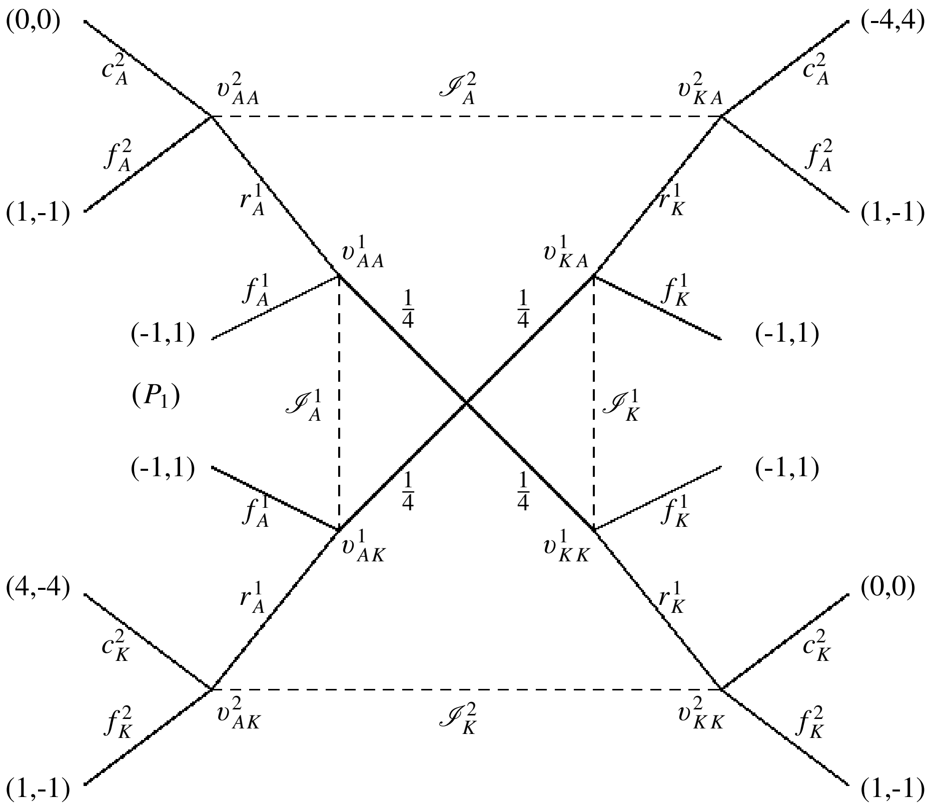

FIGURE 5. Game tree for Example 5, simplified poker game

	AA	AK	KA	KK	Payoff
$(r_A^1 r_K^1, c_A^2 c_K^2)$	(0,0)	(4,-4)	(-4,4)	(0,0)	(0,0)
$(r_A^1 r_K^1, c_A^2 f_K^2)$	(0,0)	(1,-1)	(-4,4)	(1,-1)	$(^{-1}/_2, {}^1/_2)$
$(r_A^1 r_K^1, f_A^2 c_K^2)$	(1,-1)	(4,-4)	(1,-1)	(0,0)	$(^3/_2, {}^{-3}/_2)$
$(r_A^1 r_K^1, f_A^2 f_K^2)$	(1,-1)	(1,-1)	(1,-1)	(1,-1)	(1,-1)
$(r_A^1 f_K^1, c_A^2 c_K^2)$	(0,0)	(4,-4)	(-1,1)	(-1,1)	$(^1/_2, {}^{-1}/_2)$
$(r_A^1 f_K^1, c_A^2 f_K^2)$	(0,0)	(1,-1)	(-1,1)	(-1,1)	$(^{-1}/_4, {}^1/_4)$
$(r_A^1 f_K^1, f_A^2 c_K^2)$	(1,-1)	(4,-4)	(-1,1)	(-1,1)	$(^3/_4, {}^{-3}/_4)$
$(r_A^1 f_K^1, f_A^2 f_K^2)$	(1,-1)	(1,-1)	(-1,1)	(-1,1)	(0,0)
$(f_A^1 r_K^1, c_A^2 c_K^2)$	(-1,1)	(-1,1)	(-4,4)	(0,0)	$(^{-3}/_2, {}^3/_2)$
$(f_A^1 r_K^1, c_A^2 f_K^2)$	(-1,1)	(-1,1)	(-4,4)	(1,-1)	$(^{-5}/_4, {}^5/_4)$
$(f_A^1 r_K^1, f_A^2 c_K^2)$	(-1,1)	(-1,1)	(1,-1)	(0,0)	$(^{-1}/_4, {}^1/_4)$
$(f_A^1 r_K^1, f_A^2 f_K^2)$	(-1,1)	(-1,1)	(1,-1)	(1,-1)	(0,0)
$(f_A^1 f_K^1, c_A^2 c_K^2)$	(-1,1)	(-1,1)	(-1,1)	(-1,1)	(-1,1)
$(f_A^1 f_K^1, c_A^2 f_K^2)$	(-1,1)	(-1,1)	(-1,1)	(-1,1)	(-1,1)
$(f_A^1 f_K^1, f_A^2 c_K^2)$	(-1,1)	(-1,1)	(-1,1)	(-1,1)	(-1,1)
$(f_A^1 f_K^1, f_A^2 f_K^2)$	(-1,1)	(-1,1)	(-1,1)	(-1,1)	(-1,1)

The bimatrix representation of the payoffs are

	$c_A^2 c_K^2$	$c_A^2 f_K^2$	$f_A^2 c_K^2$	$f_A^2 f_K^2$
$r_A^1 r_K^1$	(0,0)	$(^{-1}/_2, {}^1/_2)$	$(^3/_2, {}^{-3}/_2)$	(1,-1)
$r_A^1 f_K^1$	$(^1/_2, {}^{-1}/_2)$	$(^{-1}/_4, {}^1/_4)$	$(^3/_4, {}^{-3}/_4)$	(0,0)
$f_A^1 r_K^1$	$(^{-3}/_2, {}^3/_2)$	$(^{-5}/_4, {}^5/_4)$	$(^{-1}/_4, {}^1/_4)$	(0,0)
$f_A^1 f_K^1$	(-1,1)	(-1,1)	(-1,1)	(-1,1)

By checking the best response function, we can determine that the only Nash equilibrium is the strategy profile $(r_A^1 f_K^1, c_A^2 f_K^2)$, with payoffs $(^{-1}/_4, {}^1/_4)$. The reason that the second player has a higher payoff is that the first player has to fold first when they both hold kings.

Analyzing the extensive form of this game:

Player 1 on $\mathscr{I}_A^1$:
The payoff for P_1 of folding is -1. If P_1 raises, the payoff is 0 if P_2 holds an A and either 1 or 4 if P_2 holds a K. Since any of these payoffs is greater than the payoff for folding, player 1 should raise r_A^1 on $\mathscr{I}_A^1$.

Player 2:
On the information set $\mathscr{I}_A^2$, P_2 holds an A. If the player folds the payoff is -1 for both hands. If the player calls, the payoffs are 0 or 4. Since both of these payoffs are larger than -1, P_2 should call c_A^2 on $\mathscr{I}_A^2$.

On the information set $\mathscr{I}_K^2$, P_2 holds a K. If the player folds the payoff is -1 for both hands. If the player calls, the payoffs are -4 and 0, one of which is less than -1 and one of which is greater than -1. P_2 must use the probability of being at v_{AK}^2 or v_{KK}^2 to decide what choice to make. P_1 is at least as likely to bid with an A as a K, therefore, $\Pr(v_{AK}^2|\mathscr{I}_K^2) \geq \Pr(v_{KK}^2|\mathscr{I}_K^2)$ and $\Pr(v_{AK}^2|\mathscr{I}_K^2) \geq {}^1/_2$. Therefore, the expected payoff for P_2 on the information set $\mathscr{I}_K^2$ for calling satisfies

$$E_2(c_K^2|\mathscr{I}_K^2) = \Pr(v_{AK}^2|\mathscr{I}_K^2)(\text{-}4) + \Pr(v_{KK}^2|\mathscr{I}_K^2)(0) \leq \text{-}2 < \text{-}1 = E_2(f_K^2|\mathscr{I}_K^2),$$

so the payoff of matching the bid is less than the payoff of folding. Therefore P_2 should fold f_K^2 on $\mathscr{I}_K^2$.

Player 1 on $\mathscr{I}_K^1$:
The expected payoff for P_1 on the information set $\mathscr{I}_K^1$ for raising is

$$\begin{aligned} E_1(r_K^1, c_A^2 f_K^2|\mathscr{I}_K^1) &= \Pr(v_{KA}^1|\mathscr{I}_K^1)\, u_1(r_K^1, c_A^2|v_{KA}^2) + \Pr(v_{KK}^1|\mathscr{I}_K^1)\, u_1(r_K^1, f_K^2|v_{KK}^2) = \frac{1}{2}(\text{-}4) + \frac{1}{2}(1) \\ &= \text{-}\frac{3}{2} < \text{-}1 = E_1(f_K^1, c_A^2 f_K^2|\mathscr{I}_K^1), \end{aligned}$$

and P_1 should fold f_K^1 on $\mathscr{I}_K^1$.

Expected payoffs for the Nash equilibrium ($\mathbf{r_A^1 f_K^1, r_A^2 f_K^2}$):
The Nash equilibrium is $(r_A^1 f_K^1, c_A^2 f_K^2)$, where (i) P_1 raises on $\mathscr{I}_A^1$ and folds on $\mathscr{I}_K^1$ and (ii) P_2 calls on $\mathscr{I}_A^2$ and folds on $\mathscr{I}_K^2$. The expected payoffs for the players are

$$\begin{aligned} E_1(r_A^1 f_K^1, c_A^2 f_K^2) &= \Pr(v_{AA}^1) E_1(r_A^1 f_K^1, c_A^2 f_K^2|v_{AA}^1) + \Pr(v_{KA}^1) E_1(r_A^1 f_K^1, c_A^2 f_K^2|v_{KA}^1) \\ &\quad + \Pr(v_{AK}^1) E_1(r_A^1 f_K^1, c_A^2 f_K^2|v_{AK}^1) + \Pr(v_{KK}^1) E_1(r_A^1 f_K^1, c_A^2 f_K^2|v_{KK}^1) \\ &= \frac{1}{4}(0) + \frac{1}{4}(\text{-}1) + \frac{1}{4}(1) + \frac{1}{4}(\text{-}1) = \text{-}\frac{1}{4} \quad \text{and} \\ E_2(r_A^2 f_K^2, c_A^2 f_K^2) &= \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}(\text{-}1) + \frac{1}{4}(1) = \frac{1}{4}. \end{aligned}$$

The expected payoffs are not equal, but P_2 comes out ahead. The reason for the difference is payoffs is that on deals of (K,K) P_1 must fold first, so P_2 wins. This gives player 2 an advantage. ■

10.4 Belief Systems and Behavioral Strategies

Definition. A *belief system* $\mu = (\mu_1, \dots, \mu_n)$ for an extensive game with imperfect information is the following: For each information set $\mathscr{I}$ owned by a player P_ℓ, P_ℓ chooses a probability distribution over the vertices in $\mathscr{I}$, $\sum_{v \in \mathscr{I}} \mu_\ell(v) = 1$. If $v \in \mathscr{I}$, then $\mu_\ell(v)$ is the belief by P_ℓ that we are at v given that P_ℓ knows we are in $\mathscr{I}$. For a vertex v owned by P_ℓ, we write $\mu(v)$ to mean $\mu_\ell(v)$, with the understanding that we use the beliefs of player P_ℓ on a vertex owned by P_ℓ.

For the simplified game of poker, P_2 inferred a "belief system" for the information set $\mathscr{I}^2_K$ that $\mu(v^2_{AK}) = \Pr(v^2_{AK} | \mathscr{I}^2_K) \geq 1/2 \geq \Pr(v^2_{KK} | \mathscr{I}^2_K) = \mu(v^2_{KK})$. This belief was used to calculate the expected payoff of c_K. Player P_1 used the belief system that $\mu(v_{KA}) = 1/2 = \mu(v_{KK})$ on $\mathscr{I}^1_K$. $\mu(v_{AA}) = 1/2 = \mu(v_{AK})$ on $\mathscr{I}^1_A$.

A mixed strategy for a player P_ℓ in an extensive game is a probability distribution over that players complete pure strategies. For the simplified game of poker, P_1 would assign a probability distribution over the four strategies $r^1_A r^1_K$, $r^1_A f^1_K$, $f^1_A r^1_K$, and $f^1_A f^1_K$. Similarly for P_2. It is easier to assign probability distributions over the choices at each information set. Such a choice is called a behavioral strategy.

Definition. A *behavioral strategy* of player P_ℓ in an extensive game with imperfect information is a choice of a probability distribution over the edges coming out of each information set owned by P_ℓ. A *behavioral strategy profile* is a behavioral strategy for each player.

A behavioral strategy profile is said to be *completely mixed* provided that every choice at every information set is taken with a positive probability.

For the BoS game given in Example 1, we could put $p = \beta(B_1)$ for the behavioral strategy on B_1, $1 - p = \beta(S_1)$ for the behavioral strategy on S_1, $q = \beta(B_2)$, and $1 - q = \beta(S_2)$. If we have the belief system μ on the information set $\mathscr{I}$ for P_2, then the expected payoff, based on the probability given by the belief systems, is given by

$$E_2(\mathscr{I}; q, \mu) = \mu_2(v_1)\ [q(1) + (1-q)(0)] + \mu_2(v_2)\ [q(0) + (1-q)(2)]\,.$$

Definition. An *assessment* in an extensive game is a pair (β, μ) consisting of a behavioral strategy profile $\beta = (\beta_n, \dots, \beta_n)$ and a belief system $\mu = (\mu_1, \dots, \mu_n)$.

For two vertices v_0 and v, with path $(v_0, v_1, \dots, v_{k-1}, v)$ from v_0 to v, the behavioral strategy profile determines a probability of the path by

$$\beta(v_0, v) = \beta(v_0, v_1)\beta(v_1, v_2) \cdots \beta(v_{k-1}, v).$$

If a vertex v_i is owned by chance P_0, then we use the given probability of the edge for $\beta(v_i, v_{i+1})$. This quantity is the probability of taking this path from v_0 to v with the given behavioral strategy given that the path. If v_0 is the root R, then this gives the probability of getting to the vertex v, and we write

$$\Pr^\beta(v) = \beta(R, v)$$

for the probability of v determined by β. The probability of an information set determined by β is given by

$$\Pr^\beta(\mathscr{I}) = \sum_{v \in \mathscr{I}} \Pr^\beta(v).$$

For the belief system to be rational, we require that it is consistent with the conditional probability on the information sets induced by the behavioral strategy profile, β.

A conditional probability is the probability of E given that F is true, and is denoted by $\Pr(E|F)$. It is given by the formula

$$\Pr(E|F) = \frac{\Pr(E \,\&\, F)}{\Pr(F)}.$$

For a behavioral strategy profile β and an information set $\mathscr{I}$ with $\Pr^\beta(\mathscr{I}) > 0$ and a vertex $v_0 \in \mathscr{I}$,

$$\Pr^\beta(v_0|\mathscr{I}) = \frac{\Pr^\beta(v_0 \,\&\, \mathscr{I})}{\Pr^\beta(\mathscr{I})} = \frac{\Pr^\beta(v_0)}{\sum_{v \in \mathscr{I}} \Pr^\beta(v)}.$$

Definition. A behavioral belief system is *weakly consistent* with a behavioral strategy profile β provided that if $\Pr^{\beta}(\mathscr{I}) > 0$ and $v \in \mathscr{I}$ then

$$\mu(v) = \frac{\Pr^{\beta}(v)}{\Pr^{\beta}(\mathscr{I})} = \Pr^{\beta}(v_0|\mathscr{I}).$$

We denote a belief system that is weakly consistent with a behavioral strategy profile β by μ^{β}.

To be *consistent*, we also need to say something about the case when $\Pr^{\beta}(\mathscr{I}) = 0$ (when β is not a completely mixed behavioral strategy). In this case, we require that a sequence of completely mixed behavioral strategies β^n which converge to β and the corresponding consistent belief systems μ^{β^n} converge to μ^{β}. This process determines a system of beliefs on information sets that are not accessible by the original behavioral strategy profile. Different choices of the β^n can lead to different limit belief systems μ^{β}. The behavioral strategies that converges to β are thought of as a "shaking hand" of the selection process. We will not check whether our belief systems are more than just weakly consistent.

Definition. Let $X_1, \dots, X_k$ be the terminal vertices and $u_j(X_i)$ be the payoff for player P_j at one of the terminal vertices. For a strategy profile β, the *expected payoff* for P_j at any vertex v is given by

$$E_j(v; \beta) = \sum_{i=1}^{k} \beta(v, X_i)\, u_j(X_i).$$

It considers only payoffs at terminal vertices which can be reached by path from the vertex v, and it takes the sum of the payoffs at these terminal vertices weighted with the probability of getting from the vertex v to the terminal vertex determined by the behavioral strategy profile. We combine these payoffs for different players to form the *expected payoff vector* at v for β, $E(v; \beta) = (E_1(v; \beta), \dots, E_n(v; \beta))$.

To determine the *expected payoff on an information set* $\mathscr{I}$, we need to the belief system as well as the behavioral strategy profile in order to weight the payoffs at the different vertices:

$$E_j(\mathscr{I}; \beta, \mu) = \sum_{v \in \mathscr{I}} \mu(v) \sum_{i=1}^{k} \beta(v, X_i)\, u_j(X_i).$$

Example 6. We consider the extensive game given in Figure 6. Player P_2 cannot make an easy choice between c and d on the information set $\mathscr{I} = \{A, B\}$ because d is better starting at A and c is better starting at B.

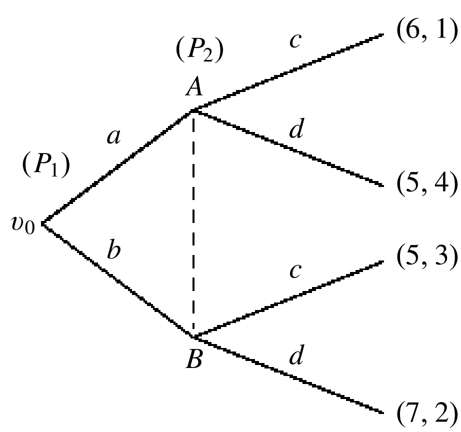

FIGURE 6. Game tree for Example 6

On the whole game, if P_1 chooses a then P_2 would choose d. Then, P_1 has an incentive to switch to b since $7 > 5$. However, if P_1 chooses b then P_2 would choose c. Then, P_1 has an incentive to switch to a since $6 > 5$. Therefore, there is no pure strategy profile which gives a Nash equilibrium on the whole game.

Rather than considering mixed strategies on the corresponding strategic form game, we use behavioral strategies. Let p and q be the behavioral strategies $p = \beta(a)$ and $q = \beta(c)$, so $1 - p = \beta(b)$ and $1 - q = \beta(d)$. Then $\Pr^{(p,q)}(A) = p$, $\Pr^{(p,q)}(B) = 1 - p$, and $\Pr^{(p,q)}(\mathscr{I}) = p + 1 - p = 1 > 0$. To be consistent, we need $\mu_2(A) = p/1 = p$ and $\mu_2(B) = 1 - p$. Thus,

$$E_2(c|\mathscr{I}) = p[1] + (1-p)[3] = 3 - 2p \qquad \text{and}$$
$$E_2(d|\mathscr{I}) = p[4] + (1-p)[2] = 2 + 2p.$$

To have a Nash equilibrium with $q \neq 0$ and $1 - q \neq 0$, these two payoffs have to be equal, $3 - 2p = 2 + 2p$, $1 = 4p$, and $p^* = 1/4$.

For P_1 at v_0,

$$E_1(a) = q[6] + (1-q)[5] = 5 + q,$$
$$E_1(b) = q[5] + (1-q)[7] = 7 - 2q.$$

To have a Nash equilibrium with $p \neq 0$ and $1 - p \neq 0$, these two payoffs have to be equal, $5 + q = 7 - 2q$, $3q = 2$, and $q^* = 2/3$. Thus, the behavioral strategy profile $p^* = 1/4$ and $q^* = 2/3$ is a Nash equilibrium for the game.

The weakly consistent belief system for this behavioral strategy is $\mu^*(A) = 1/4$ and $\mu^*(B) = 3/4$.

The solution process is similar to method we used to finding mixed strategy equilibrium for a strategic game. The strategies at the two levels must be solved simultaneously. ■

Sequential Equilibria

For the equilibrium in behavioral strategy profiles, we want it to be "rational" not only on the whole game but also on parts of the game tree. We could require it to be subgame perfect, but in the game just considered there are no subgames. Instead, we require that it is optimal or rational on all the information sets. Since we are considering a behavioral strategy that is optimal in the part of the game that follows each information set, the belief system is fixed on the information set.

Definition. A *weak sequential equilibrium* for a n-person a extensive game with imperfect information is an assessment (β^*, μ^*) such that, (i) the belief system μ^* is weakly consistent with β^* and (ii) on any information set $\mathscr{I}$ (which could be a single vertex) owned by a player P_ℓ, the expected payoff

$$E_\ell(\mathscr{I}; \beta^*, \mu^*) = \max_{\beta_\ell} E_\ell(\mathscr{I}; \beta_1^*, \dots, \beta_\ell, \dots, \beta_n^*, \mu^*),$$

where the maximum is taken by changing their behavioral strategy β_ℓ of P_ℓ while keeping fixed the behavioral strategies β for $i \neq \ell$ of the other players and the system of beliefs μ^*.

Theorem (Kreps-Wilson). *Every extensive game with imperfect information and perfect recall has a sequential equilibrium.*

We have defined various types of Nash equilibria for extensive games

1. Nash equilibria for pure strategy profiles.
2. Nash equilibrium in mixed strategies.
3. Subgame perfect equilibria.
4. Nash equilibrium in behavioral strategy profiles.
5. Sequential equilibria that are rational on all the information sets. For this equilibrium, we need a system of beliefs in addition to a behavioral strategy profile.

Example 7. **Sequential equilibrium for a simplified poker game.** This is a variation of the earlier game of poker considered. There are two players who each get one card that can be either an ace A, a king K, or a queen Q. Between the two players there are nine possible hands each one occurring with probability $1/9$. We assume the ante is 4 and the bet is 6.

If either player holds an A, the worst payoff for raising or calling is 0 while the payoff for folding is -4. Therefore, both players will not fold with an ace. If either player holds a Q, the probability of the other player holding an A is at least as high as a Q. Thus, the payoff for raising or calling is worse than folding. Therefore, both players will fold with a Q.

Figure 7 shows the game tree. We have dotted the edges for P_1 to raise with a Q or fold with an A since these are not chosen. We have omitted the edges for P_2 to call with a Q or fold with an A.

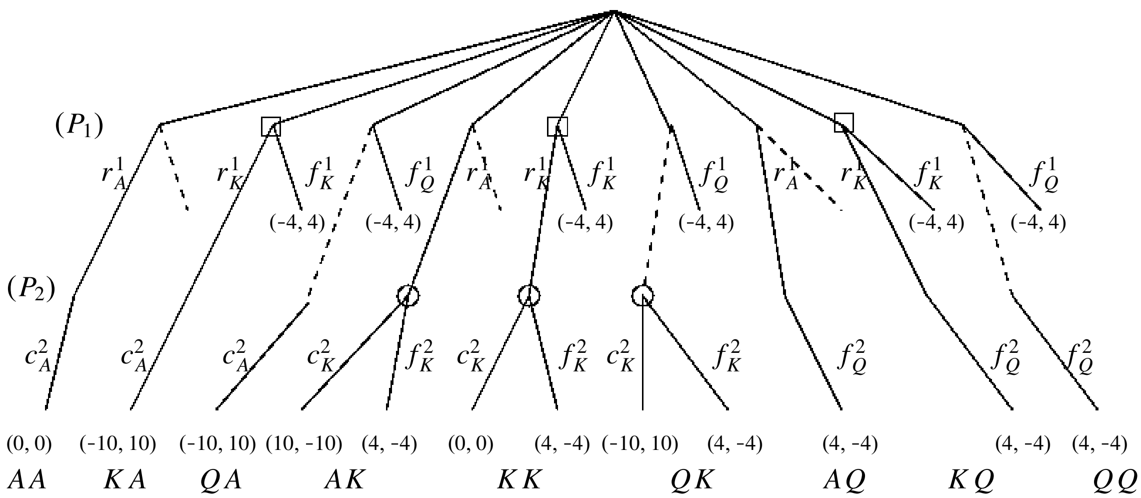

FIGURE 7. Example 7: Game of poker

Let p be the behavioral strategy of P_1 raising with a king and q be the behavioral strategy of P_2 calling with a king.

The information set $\mathscr{I}_K^2$ where P_2 holds a king is the set of vertices which are circled in the figure. The probabilities are $\Pr^p(v_{AK}^2) = {}^1\!/\!_9$, $\Pr^p(v_{KK}^2) = {}^p\!/\!_9$, $\Pr^p(v_{QK}^2) = 0$, and $\Pr^p(\mathscr{I}_K^2) = {}^{(1+p)}\!/\!_9 > 0$. The consistent beliefs are

$$\mu_2(AK|\mathscr{I}_K^2) = \frac{1/9}{1/9 + (p/9) + 0} = \frac{1}{1+p}$$
$$\mu_2(KK|\mathscr{I}_K^2) = \frac{(p/9)}{(1/9)\,(1+p)} = \frac{p}{1+p}$$
$$\mu_2(QK|\mathscr{I}_K^2) = \frac{0}{(1/9)\,(1+p)} = 0.$$

Using the consistent belief system, we can calculate the expected payoff for P_2 on $\mathscr{I}_K^2$ for c_K^2 and f_K^2:

$$E_2(f_K^2|\mathscr{I}_K^2) = \text{-}4 \qquad \text{and}$$
$$E_2(c_K^2|\mathscr{I}_K^2) = \frac{1}{1+p}(\text{-}10) + \frac{p}{1+p}(0) + 0(10) = \frac{\text{-}10}{1+p} \leq \text{-}5 < \text{-}4.$$

Since $E_2(c_K^2|\mathscr{I}_K^2) < E_2(f_K^2|\mathscr{I}_K^2)$ for all p, P_2 always folds with a K, $q^* = 0$.

Now, consider the information set $\mathscr{I}_K^1$ where P_1 holds a king. These vertices are surrounded by squares in the figure. All the hands in $\mathscr{I}_K^1$ are equally likely, so P_1 should give them a belief of ${}^1\!/\!_3$ probability each. Then,

$$E_1(f_K^1|\mathscr{I}_K^1) = \text{-}4 \qquad \text{and}$$
$$E_1(r_K^1|\mathscr{I}_K^1) = \frac{1}{3}\,[\text{-}10 + 4 + 4] = \text{-}\frac{2}{3} > \text{-}4.$$

Thus, player P_1 always bids with a king, $p^* = 1$.

Combining, the sequential equilibrium has a strategy profile (r_K^1, f_K^2) and a belief system of $\mu_2(AK|\mathscr{I}_K^2) = 1/2$, $\mu_2(KK|\mathscr{I}_K^2) = 1/2$, and $\mu_2(QK|\mathscr{I}_K^2) = 0$.

The reader can check that the expected payoff for this game is (-6, 6). ■

Signaling Games

Example 8. **Quiche or Beer: A Signaling Game** (See Binmore [2] pages 463-6.) There is a $1/3$ chance that player P_1 is a "tough" guy T and $2/3$ chance that he is a "wimp" W. The type is known to P_1 but not to P_2. Player P_1 decides whether to eat quiche or drink beer; the tough guys prefer beer and the wimps prefer quiche. (Tough guys don't eat quiche.) Then P_2 decides whether to bully P_1 or defer. A bullying P_2 does better against a wimp than a tough guy. The behavioral strategies for P_2 are $x = \beta_2(\text{bully}|\mathscr{I}_Q)$, $1 - x = \beta_2(\text{defer}|\mathscr{I}_Q)$, $y = \beta_2(\text{bully}|\mathscr{I}_B)$, and $1 - y = \beta_2(\text{defer}|\mathscr{I}_B)$. See Figure 8.

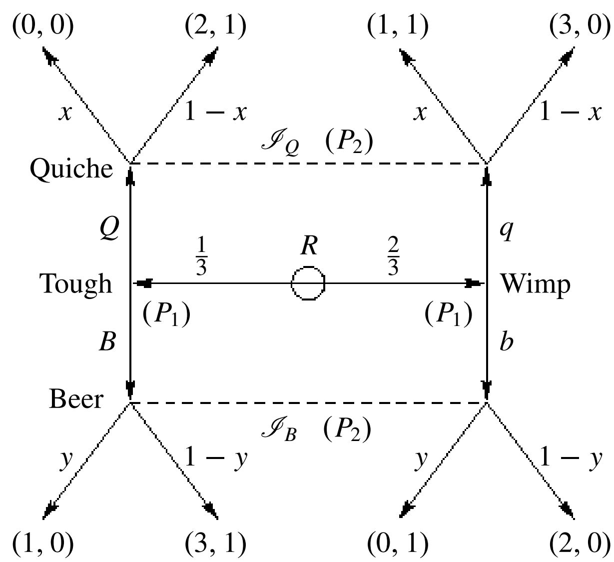

FIGURE 8. Game tree for Example 8: quiche or beer

The behavioral strategies for P_1 are labeled in the figure with $Q+B = 1$ and $q+b = 1$. If $(Q, q) \neq (0, 0)$, then the consistent belief for P_2 on the information set $\mathscr{I}_Q$ is

$$\mu_2(T|\mathscr{I}_Q) = \frac{\frac{1}{3}(Q)}{\frac{1}{3}(Q) + \frac{2}{3}(q)} = \frac{Q}{Q+2q} \qquad \text{and}$$

$$\mu_2(W|\mathscr{I}_Q) = \frac{\frac{2}{3}(q)}{\frac{1}{3}(Q) + \frac{2}{3}(q)} = \frac{2q}{Q+2q},$$

In the same way, if $(B, b) \neq (0, 0)$, then

$$\mu_2(T|\mathscr{I}_B) = \frac{B}{B+2b} \quad \text{and} \quad \mu_2(W|\mathscr{I}_B) = \frac{2b}{B+2b}.$$

The expected payoffs for P_2 on $\mathscr{I}_Q$ are

$$E_2(\text{bully}|\mathscr{I}_Q) = \frac{Q}{Q+2q}(0) + \frac{2q}{Q+2q}(1) = \frac{2q}{Q+2q} \qquad \text{and}$$

$$E_2(\text{defer}|\mathscr{I}_Q) = \frac{Q}{Q+2q}(1) + \frac{2q}{Q+2q}(0) = \frac{Q}{Q+2q}.$$

Therefore, the value of x which maximizes $E_2(\mathscr{I}_Q)$ is

$$x = \begin{cases} \text{arbitrary} & \text{if } (Q, q) = (0, 0) \\ 0 & \text{if } 2q < Q \\ \text{arbitrary} & \text{if } 2q = Q \\ 1 & \text{if } 2q > Q. \end{cases}$$

In the same way, on $\mathscr{I}_B$,

$$E_2(\text{bully}|\mathscr{I}_B) = \frac{B}{B+2b}(0) + \frac{2b}{B+2b}(1) = \frac{2b}{B+2b} \qquad \text{and}$$
$$E_2(\text{defer}|\mathscr{I}_B) = \frac{B}{B+2b}(1) + \frac{2b}{B+2b}(0) = \frac{B}{B+2b}.$$

Therefore, the value of y which maximizes $E_2(\mathscr{I}_B)$ is

$$y = \begin{cases} \text{arbitrary} & \text{if } (B, b) = (0, 0) \\ 0 & \text{if } 2b < B \\ \text{arbitrary} & \text{if } 2b = B \\ 1 & \text{if } 2b > B. \end{cases}$$

Turning to P_1,

$$E_1(Q) = x(0) + (1-x)2 = 2 - 2x,$$
$$E_1(B) = y(1) + (1-y)3 = 3 - 2y,$$
$$E_1(q) = x(1) + (1-x)3 = 3 - 2x, \qquad \text{and}$$
$$E_1(b) = y(0) + (1-y)2 = 2 - 2y.$$

We next combine the cases above to find the sequential equilibrium.

1. Assume $Q \geq 2q$. Then $1 - B = Q \geq 2q = 2(1-b)$ so $2b \geq B + 1 > B$ and $y = 1$. Then, $E_1(b) = 0 < 3 - 2x = E_1(q)$, so $b = 0$ and $q = 1$. This would imply that $Q \geq 2q = 2$, which is impossible. Thus, there is not solution with $Q \geq 2q$.

2. Assume that $Q < 2q$. Then, $x = 1$. Since $1 - B = Q < 2q = 2 - 2b$, $2b < B + 1$. We can still have (i) $2b > B$, (ii) $2b < B$, or (iii) $2b = B$.

(i) Assume $2b > B$ and $Q < 2q$, so $x = 1$ and $y = 1$. Then, $E_1(Q) = 0 < 1 = E_1(B)$, so $Q = 0$ and $B = 1$; $E_1(q) = 1 > 0 = E_1(b)$, so $q = 1$ and $b = 0$. Then, $2b$ is not greater than B. This contradiction implies there is no solution.

(ii) Assume $2b < B$ and $Q < 2q$, so $x = 1$ and $y = 0$. Then, $E_1(Q) = 0 < 3 = E_1(B)$, so $Q = 0$ and $B = 1$. Thus, $b < 1/2(B) = 1/2$. But, $E_1(q) = 1 < 2 = E_1(b)$, so $q = 0$ and $b = 1$. This is a contradiction.

(iii) Assume $2b = B$ and $Q < 2q$, so $x = 1$ and y is arbitrary. Then, $E_1(Q) = 0 < 3 - 2y = E_1(B)$, so $Q = 0$ and $B = 1$. Thus, $b = 1/2(B) = 1/2$. Also, $E_1(q) = 1$ and $E_1(b) = 2 - 2y$. For $b = 1/2$ to be feasible, we need $1 = 2 - 2y$ and $y = 1/2$. Thus, we have found a solution $x = 1$, $y = 1/2$, $B = 1$, and $b = 1/2$.

Thus, the only equilibrium is $x = 1$, $y = 1/2$, $B = 1$, and $b = 1/2$: player P_2 always bullies the person who eats quiche and bullies the beer drinker half of the time, while a tough guy always drinks beer and a wimp drinks beer half of the time. ■

Example 9. **Example 332.1: Entry as a signaling game** The game tree for this example is given in Figure 9. The tree is the same as the last example but the payoffs have been changed. The challenger is P_1 and the incumbent is P_2. There are two types of challengers, strong s or weak w. The probability of a strong player is some fixed (known) parameter $0 < p < 1$, and the probability of a weak player is $1 - p$. The challenger may either "ready" herself r for the battle or remain "unready" u. The incumbent can either "fight" F or "acquiesce" A.

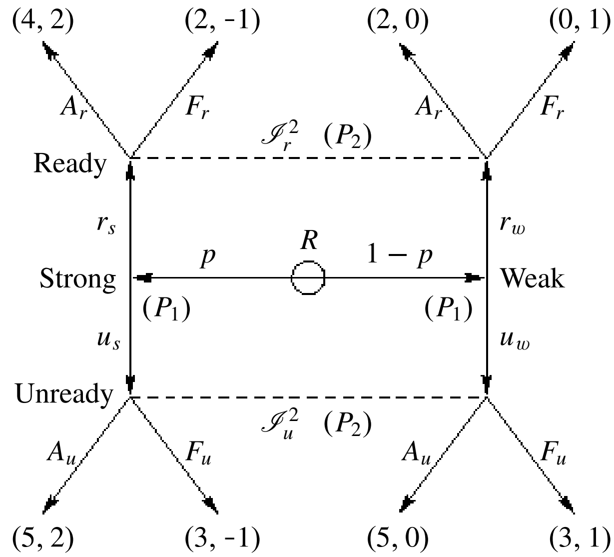

FIGURE 9. Game tree for Example 9: challenger and incumbent

Weak for P_1: Each of the possible payoffs for being unready is higher than any of the possible payoffs for being ready, so $u_w = 1$ and $\beta(r_w) = 0$ is always the best choice.

Since μ is determined on $\mathscr{I}_r^2$ when $\beta(r_s) > 0$ but not when $\beta(r_s) = 0$, we split the rest of the analysis into the cases when (1) $\beta(r_s) > 0$ and (2) $\beta(r_s) = 0$.

Case 1: Assume $\beta(r_s) > 0$.
Since $\beta(r_w) = 0$, $\mu_2(r_s) = 1$ and $\mu_2(r_w) = 0$. Thus,

$$E_2(F_r) = \text{-}1, \qquad E_2(A_r) = 2,$$

so $\beta(A_r) = 1$.

Strong for P_1: The payoffs for a strong challenger are

$$E_1(r_s|\beta(A_r) = 1) = 4$$
$$E_1(u_s) = \beta(A_u)\,(5) + \beta(F_u)\,(3) = 3 + 2\,\beta(A_u).$$

$\beta(r_s) > 0$ implies that $E_1(r_s) \geq E_1(u_s)$, so

$$4 \geq 3 + 2\,\beta(A_u)$$
$$1 \geq 2\,\beta(A_u)$$
$$1/2 \geq \beta(A_u) \qquad \text{and} \quad \beta(F_u) \leq 1/2.$$

(1a) Assume $1/2 > \beta(A_u)$ and $\beta(r_s) > 0$. Then, $E_1(r_s) > E_1(u_s)$, $\beta(r_s) = 1$, and $\beta(u_s) = 0$. The consistent belief systems on $\mathscr{I}_u^2$ is $\mu_2(u_s) = 0$ and $\mu_2(u_w) = 1$, so

$$E_2(F_u) = 1, \quad \text{and} \quad E_2(A_u) = 0,$$

so $\beta(F_u) = 1$ and $\beta(A_u) = 0$. Thus, we have a sequential equilibrium:

$$\beta(r_s) = 1, \quad \beta(u_w) = 1, \quad \beta(F_u) = 1, \quad \beta(A_r) = 1,$$
$$\mu_2(r_s) = 1, \quad \mu_2(r_w) = 0, \quad \mu_2(u_s) = 0, \quad \mu_2(u_w) = 1.$$

It is *separating*, because the first player different actions for the two types and second player can distinguish the type by the actions taken.

(1b) Assume $1/2 = \beta(A_u)$ and $\beta(r_s) > 0$. Since $0 < \beta(A_u) < 1$, $E_2(A_u) = E_2(F_u)$. This equality has an implication about the belief system $\mu = \mu(u_s)$ and $1 - \mu = \mu(u_w)$ on $\mathscr{I}_u^2$.

$$\begin{aligned}
E_2(F_u) &= \mu\,(-1) + (1-\mu)(1) = 1 - 2\mu, \\
E_2(A_u) &= \mu\,(2) + (1-\mu)(0) = 2\mu, \qquad \text{so} \\
2\mu &= 1 - 2\mu, \\
4\mu &= 1, \\
\mu &= 1/4.
\end{aligned}$$

Then, $\mu(u_w) = 1 - 1/4 = 3/4 = 3\mu(u_s)$. However, the belief system is determined by the behavioral strategy as follows: $\Pr(u_s) = p\beta(u_s)$, $\Pr(u_w) = 1 - p$, $\Pr(\mathscr{I}_u^2) = 1 - p + p\beta(u_s) > 0$, and

$$\begin{aligned}
\mu(u_s) &= \frac{p\beta(u_s)}{\Pr(\mathscr{I}_u^2)} \\
\mu(u_w) &= \frac{1-p}{\Pr(\mathscr{I}_u^2)}, \qquad \text{so} \\
3p\beta(u_s) &= 1 - p, \\
\beta(u_s) &= \frac{1-p}{3p}, \\
\beta(r_s) &= 1 - \beta(u_s) = \frac{4p-1}{3p} > 0, \\
4p &> 1, \qquad \text{and} \\
p &> \frac{1}{4}.
\end{aligned}$$

For $p > \frac{1}{4}$, this is a separating sequential equilibrium with

$$\begin{aligned}
&\beta(A_u) = 1/2 = \beta(F_u), \quad \beta(A_r) = 1 \\
&\beta(u_w) = 1, \quad \beta(r_s) = \frac{4p-1}{3p} > 0, \quad \beta(u_s) = \frac{1-p}{3p} > 0, \\
&\mu(r_s) = 1, \quad \mu(r_w) = 0, \quad \mu(u_s) = 3/4, \quad \mu(u_w) = 1/4.
\end{aligned}$$

Case 2: Assume $\beta(r_s) = 0$.

Since $\beta(r_w) = 0$ also, the belief system μ is not determined on $\mathscr{I}_r^2$, and $\beta(A_r)$ is not directly determined. Since $\beta(u_s) = 1 = \beta(u_w)$, the belief system on $\mathscr{I}_u^2$ is determined to be $\mu(u_s) = p$ and $\mu(u_w) = 1 - p$.

Consider the payoffs for P_2 on $\mathscr{I}_u^2$.

$$\begin{aligned}
E_2(A_u) &= p(2) + (1-p)(0) = 2p \\
E_2(F_u) &= p(\text{-}1) + (1-p)(1) = 1 - 2p.
\end{aligned}$$

We consider the following subcases: (a) If $p < 1/4$, then $E_2(A_u) < E_2(F_u)$ and $\beta(F_u) = 1$. (b) If $p > 1/4$, then $E_2(A_u) > E_2(F_u)$ and $\beta(A_u) = 1$. (c) If $p = 1/4$, then $E_2(A_u) = E_2(F_u)$ and $\beta(F_u)$ and $\beta(A_u)$ are arbitrary.

(2a) Assume $p < 1/4$, $\beta(F_u) = 1$, and $\beta(r_s) = 0$.
Considering a strong P_1, $\beta(r_s) = 0$ so $E_1(r_s) \le E_1(u_s)$. But,

$$\begin{aligned}
E_1(r_s) &= 4\beta(A_r) + 2\beta(F_r) = 2 + 2\beta(A_r) \\
E_1(u_s) &= 3, \qquad \text{so} \\
2 + 2\beta(A_r) &\le 3 \\
\beta(A_r) &\le 1/2.
\end{aligned}$$

Since $\beta(A_r) \leq 1/2 < 1$, $E_2(A_r) \leq E_2(F_r)$. For the belief system $\nu = \mu_2(r_s)$ and $1 - \nu = \mu_2(r_w)$ on $\mathscr{I}_r^2$,

$$E_2(A_r) = \nu(2) + (1-\nu)(0) = 2\nu,$$
$$E_2(F_r) = \nu(-1) + (1-\nu)(1) = 1 - 2\nu,$$

so $\nu = \mu(r_s) \leq 1/4$ always works. This is a sequential equilibrium with

$$\beta(u_s) = 1 = \beta(u_w), \quad \beta(F_u) = 1, \quad \beta(A_r) \leq 1/2 \leq \beta(F_r),$$
$$\mu(r_s) \leq 1/4, \quad \mu(r_w) \geq 3/4, \quad \mu(u_s) = p, \quad \mu(u_w) = 1 - p, \quad \text{with } p < 1/4.$$

It is called *pooling* because the first player takes the same action for the two types.

(2b) $p > 1/4$, $\beta(A_u) = 1$, and $\beta(r_s) = 0$. Comparing the payoffs of P_1 for strong,

$$E_1(r_s) = 2 + 2\beta(A_r)$$
$$E_1(u_s) = 5.$$

Since $\beta(r_s) = 0$, $E_1(r_s) \leq E_1(u_s)$; but, $E_1(r_s) \leq 4 < 5 = E_1(u_s)$, so this is satisfied. The behavioral strategy of P_2 on $\mathscr{I}_r^2$ is determined by the choice of a belief system, which is not determined. This is a pooling sequential equilibrium with

$$\beta(u_s) = 1 = \beta(u_w), \quad \beta(A_u) = 1,$$
$$\mu_2(u_s) = p, \quad \mu_2(u_w) = 1 - p, \quad \text{with } p > 1/4,$$
$$\beta(A_r) = \begin{cases} 0 & \text{if } \mu(r_s) < 1/4 \ \& \ \mu(r_w) > 3/4, \\ \text{arbitrary} & \text{if } \mu(r_s) = 1/4 \ \& \ \mu(r_w) = 3/4, \\ 1 & \text{if } \mu(r_s) > 1/4 \ \& \ \mu(r_w) < 3/4. \end{cases}$$

(2c) $p = 1/4$, $\beta(F_u)$ and $\beta(A_u)$ are arbitrary, and $\beta(r_s) = 0$. Comparing the payoffs of P_1 for strong,

$$E_1(r_s) = 2 + 2\beta(A_r)$$
$$E_1(u_s) = 3 + 2\beta(A_u).$$

Since $\beta(r_s) = 0$, $E_1(r_s) \leq E_1(u_s)$, $2 + 2\beta(A_r) \leq 3 + 2\beta(A_u)$, and $\beta(A_r) \leq 1/2 + \beta(A_u)$. Also, This is a pooling sequential equilibrium with

$$\beta(u_s) = 1 = \beta(u_w), \quad \beta(A_r) \leq 1/2 + \beta(A_u), \quad p = 1/4$$
$$\mu_2(u_s) = p, \quad \mu_2(u_w) = 1 - p,$$
$$\mu(r_s) = 1 - \mu(r_w) \quad \begin{cases} \leq 1/4 & \text{if } \beta(A_r) = 0 \\ = 1/4 & \text{if } 0 < \beta(A_r) < 1 \\ \geq 1/4 & \text{if } \beta(A_r) = 1. \end{cases}$$

■

Example 10. **§10.7 Education as a Signaling Game:** This is the example of Section 10.7 in Osborne. The worker is P_3 and the firms are P_1 and P_2. The probability of a high ability worker is $\pi > 0$ and of a low ability worker is $1 - \pi > 0$. The worker chooses education levels e_H and e_L. The two firms offer wages (w_1, w_2). The worker can accept the offer from firm one or firm two. The values of the worker to the firms are V for a high ability worker and v for a low ability worker, with $V > v > 0$. The payoffs are given in Figure 10.

We check that the following strategy profile is a separating sequential equilibrium:

- The worker P_3 chooses the higher wage that is offered at both C_H and C_L.
- The worker P_3 selects $e_H = e^* = v(V - v)$ and $e_L = 0$. (This value of e^* is the one that works in the following calculations.)

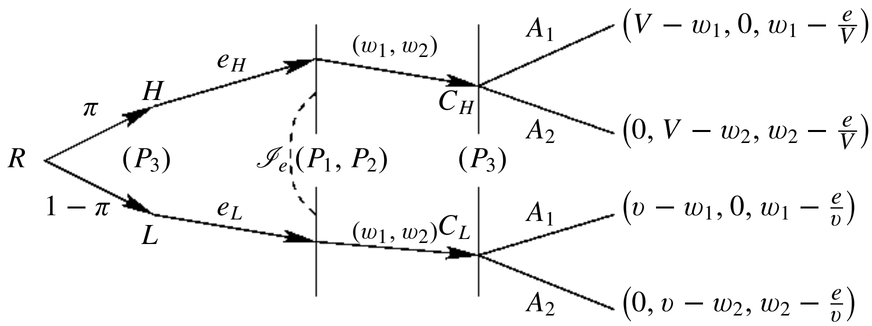

FIGURE 10. Game tree for Example 10

- On the information set $\mathscr{I}_e$, the firms P_1 and P_2 set the wages by

$$w_1^* = w_2^* = \begin{cases} V & \text{if } e \ge e^* \\ v & \text{if } 0 \le e < e^*. \end{cases}$$

The choices at C_H and C_L are clearly correct for a sequential equilibrium.

For the worker at L, if $0 \le e_L < e^*$, then $E_3(L, w_1^*, w_2^*, e_L) = v - \frac{e_L}{v}$, which has a maximum of v for $e_L = 0$. If $e_L \ge e^*$, then $E_3(L, w_1^*, w_2^*, e_L) = V - \frac{e_L}{v}$, which is maximal for $e_L = e^*$. For the best choice to be $e_L = 0$, we need

$$v \ge V - \frac{e^*}{v}, \qquad \text{or} \qquad e^* \ge v(V - v).$$

For the worker at H, if $e_H \ge e^*$, then $E_3(H, w_1^*, w_2^*, e_H) = V - \frac{e_H}{V}$ which attains a maximum for $e_H = e^*$. The maximum payoff is $V - \frac{e^*}{V}$. For $0 \le e_H < e^*$, $E_3(H, w_1^*, w_2^*, e_H) = v - \frac{e_H}{V}$ which attains a maximum of v at $e_H = 0$. For the best choice to be $e_H = e^*$, we need

$$v \le V - \frac{e^*}{V}$$
$$e^* \le V(V - v).$$

For $e_L = 0$ and $e_H = e^*$, we need $v(V - v) \le e^* \le V(V - v)$. For the maximal payoff at H, we need e^* as small as possible, so $e^* = v(V - v)$.

On the information set $\mathscr{I}_e$, the firms have a belief

$$\mu(H|\mathscr{I}_e) = \begin{cases} 1 & \text{if } e \ge e^* \\ 0 & \text{if } 0 \le e < e^* \end{cases} \qquad \text{and} \qquad \mu(L|\mathscr{I}_e) = \begin{cases} 0 & \text{if } e \ge e^* \\ 1 & \text{if } 0 \le e < e^* \end{cases}.$$

With this system of beliefs and the strategy given above, each firm gets a payoff of 0. If they raise their offer, they would get a negative payoff. If they lower their offer, they would still get 0, so they have nothing to gain. Therefore, this is a sequential equilibrium. ■

Example 11. **Alternative Education as a Signaling Game:** This is a different example than treated in Osborne and is a modification of an example in [1]. Assume that chance makes an individual of high ability H with probability $1/2$ and an individual of low ability L with probability $1/2$. Then the firm P_1 sets a pay scale $w(e) = me + 0.1$ where m is a parameter. The firm does not know whether the person is of high or low ability, but must set the wage scale as a function of the education level, $m\,2 + 0.1$, where m is a parameter. The individual P_2 knows the ability type and chooses an education level $e_H \ge 0$ for a high ability and $e_L \ge 0$ for a low ability. The firm then makes an offer based on the salary structure. We do not show a vertex for this because it does not involve a choice. Finally, P_2 accept or rejects the offer depending on whether the payoffs are positive or negative. See Figure 11.

For the final choice, P_2 chooses to accept as long as the payoff is non-negative.

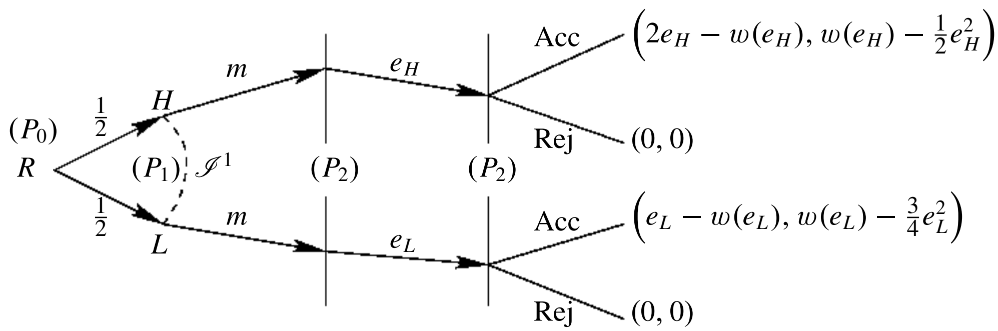

FIGURE 11. Game tree for Example 11

For H, P_2 chooses $e = e_H$ to maximize $u_2(H, e, m) = w(e) - \frac{1}{2}e^2 = me + 0.1 - \frac{1}{2}e^2$. The partial derivative $\dfrac{\partial u_2}{\partial e} = m - e = 0$ for $e_H^*(m) = m$. Note that $u_2(H, m, m) = \frac{1}{2}m^2 + 0.1 > 0$. Since $\dfrac{\partial^2 u_2}{\partial e^2} = -1 < 0$, this is the maximum.

For L, P_2 choose $e = e_L$ to maximize $u_2(L, e, m) = me + 0.1 - \frac{3}{4}e^2$. The partial derivative $\dfrac{\partial u_2}{\partial e} = m - \frac{3}{2}e = 0$ for $e_L^*(m) = \frac{2}{3}m$. Note that $u_2(L, \frac{2}{3}m, m) = \frac{1}{3}m^2 + 0.1 > 0$. This is the maximum since $\frac{\partial^2 u_2}{\partial e^2} = -\frac{3}{2} < 0$.

On the information set $\mathscr{I}^1$, P_1 has expected payoff of

$$\begin{aligned}
E_1(m) &= \frac{1}{2}\left[2e_H^*(m) - me_H^*(m) - 0.1\right] + \frac{1}{2}\left[e_L^*(m) - me_L^*(m) - 0.1\right] \\
&= \frac{1}{2}\left[2m - m^2 - 0.1 + \frac{2}{3}m - \frac{2}{3}m^2 - 0.1\right] \\
&= \frac{1}{2}\left[\frac{8}{3}m - \frac{5}{3}m^2 - 0.2\right], \\
\frac{d}{dm}E_1(m) &= \frac{1}{2}\left[\frac{8}{3} - \frac{10}{3}m\right] = 0, \\
m^* &= \frac{8}{10} = 0.8, \\
\frac{d^2}{dm^2}E_1(m) &= -\frac{5}{3} < 0.
\end{aligned}$$

The sequential equilibrium is $m^* = 0.8$, $e_L^* = e_L^*(m^*) = \frac{2}{3}0.8$, $e_H^* = e_H^*(m^*) = 0.8$, and

$$s^*(H, e, m) = \begin{cases} \text{Accept} & \text{if } me + 0.1 - \frac{1}{2}e^2 \geq 0 \\ \text{Reject} & \text{if } me + 0.1 - \frac{1}{2}e^2 < 0, \end{cases}$$

$$s^*(L, e, m) = \begin{cases} \text{Accept} & \text{if } me + 0.1 - \frac{3}{4}e^2 \geq 0 \\ \text{Reject} & \text{if } me + 0.1 - \frac{3}{4}e^2 < 0. \end{cases}$$

This is a separating sequential equilibrium which distinguishes the type of sender by the level of education, the signal. The individual with less ability gets less education than the high ability person. This "signals" the ability to the firm.

The expected payoffs are

$$\begin{aligned} E_1(0.8) &= \frac{1}{2}\left[\frac{8}{3}\cdot\frac{8}{10} - \frac{5}{3}\cdot\cdot\frac{16}{25} - 0.2\right] \\ &= \frac{16}{15} - \frac{8}{15} - \frac{1}{10} = \frac{13}{30} \approx 0.433. \\ u_2(H, e_H^*, 0.8) &= \frac{1}{2}\left(\frac{4}{5}\right)^2 + \frac{1}{10} = 0.42, \\ u_2(L, e_L^*, 0.8) &= \frac{1}{3}\left(\frac{4}{5}\right)^2 + \frac{1}{10} \approx 0.3133. \end{aligned}$$

■

Examples of Sequential Equilibria

Example 12. (Based on an example of Rosenthal given in [4].)
Consider the game given in Figure 12. The idea behind the game is that there is a chance event in which there is a 5% chance that the second person will always be accomodating and a 95% chance that the second person can choose to be generous or selfish. The first person always has the option of being either generous or selfish. Each player loses \$1 each time she is generous, but gains \$5 each time the other player is generous. If P_1 is selfish the first time, neither player gains or loses, and the game is over. Also, if both players are generous twice then the game is over. Because the second player is always generous in the lower branch, we do not indicate the choices of player P_2. At each stage P_2 knows exactly where she is, but P_1 does not know which subtree she is on. The letters designating the behavioral strategies are indicated in Figure 12.

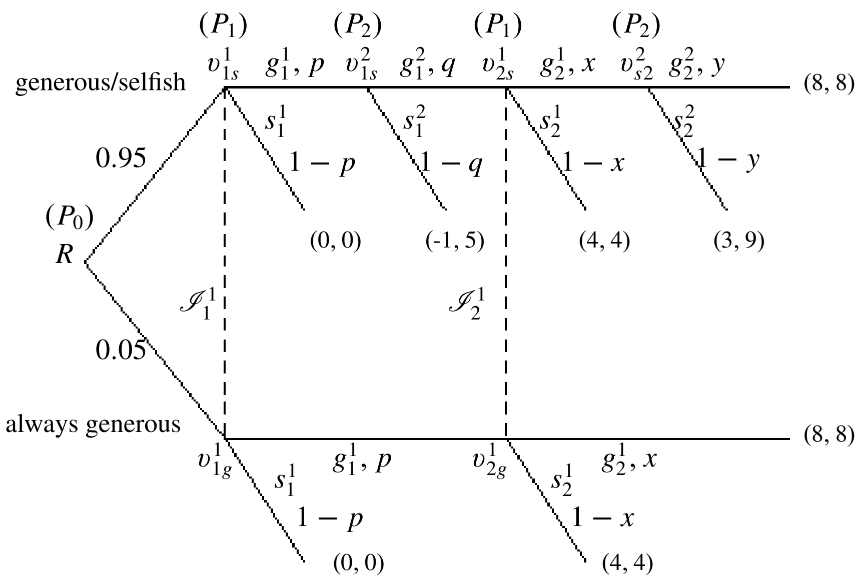

FIGURE 12. Game tree for Example 12

At vertex v_{s2}^2, the payoffs of P_2 are $u_2(g_2^2) = 8$ and $u_2(s_2^2) = 9$, so $y = \beta(g_2^2) = 0$.
For the information set $\mathscr{I}_2^1 = \{v_{2s}^1, v_{2g}^1\}$, the consistent belief $\mu = \mu(v_{2,s}^1)$ of being at vertex $v_{2,s}^1$ satisfies

$$\mu = \mu(v_{2,s}^1) = \begin{cases} \dfrac{0.95pq}{0.95pq + 0.05p} = \dfrac{19q}{19q+1} & \text{for } p \neq 0 \\ \text{arbitrary} & \text{for } p = 0. \end{cases}$$

The consistent belief of being at vertex v^1_{2g} is then $1 - \mu$. The payoffs for s^1_2 and g^1_2 on $\mathscr{I}^1_2$ are

$$E_1(s^1_2|\mathscr{I}^1_2, \mu) = 4 \qquad \text{and}$$
$$E_1(g^1_2|\mathscr{I}^1_2, \mu) = \mu 3 + (1-\mu)8 = 8 - 5\mu.$$

These are equal for $4 = 8 - 5\mu$ or $\mu = 4/5$. The optimal choice of x is the following:

$$x = \beta(g^1_2) = \begin{cases} 1 & \text{if } \mu < 4/5, \quad \text{that is } q < 4/19, \\ \text{arbitrary} & \text{if } \mu = 4/5, \quad \text{that is } q = 4/19, \quad \text{or } p = 0, \\ 0 & \text{if } \mu > 4/5, \quad \text{that is } q > 4/19. \end{cases}$$

Now consider the payoff for player P_2 at v^2_{1s}:

$$E_2(s^2_1|x) = 5 \qquad \text{and}$$
$$E_2(g^2_1|x) = x9 + (1-x)4 = 4 + 5x.$$

These are equal for $5 = 4 + 5x$ or $x = 1/5$. The choice of q which maximizes the payoff is

$$q = \beta(g^2_1) = \begin{cases} 0 & \text{if } x < 1/5 \\ \text{arbitrary} & \text{if } x = 1/5 \\ 1 & \text{if } x > 1/5. \end{cases}$$

Combining the choices at v^2_{1s} with those at $\mathscr{I}^1_2$, we have the following cases.

$x < 1/5$**:** $\Rightarrow q = 0 \Rightarrow x = 1$. This is a contradiction.
$x = 1/5$**:** $\Rightarrow q = 4/19$ and $\mu = 4/5 \Rightarrow x = 1/5$. This is a compatible choice.
$x > 1/5$**:** $\Rightarrow q = 1 \Rightarrow x = 0$. This is a contradiction.

Thus, the only compatible choices are $x = 1/5$, $q = 4/19$, and $\mu = 4/5$.

Next, we calculate the payoff vectors for this choice of a behavioral strategy profile.

$$E(v^1_{2g}) = \frac{1}{5}(8,8) + \frac{4}{5}(4,4) = \left(\frac{24}{5}, \frac{24}{5}\right)$$
$$E(v^1_{2,s}) = \frac{1}{5}(3,9) + \frac{4}{5}(4,4) = \left(\frac{3+16}{5}, \frac{9+16}{5}\right) = \left(\frac{19}{5}, 5\right)$$
$$E(v^2_{1s}) = \frac{4}{19}\left(\frac{19}{5}, 5\right) + \frac{15}{19}(-1,5) = \left(\frac{4\cdot 19 - 15\cdot 5}{5\cdot 19}, 5\right) = \left(\frac{1}{5\cdot 19}, 5\right).$$

Turning to the expectation of P_1 on $\mathscr{I}^1_1$,

$$E_1(s^1_1) = 0 \qquad \text{and}$$
$$E_1(g^1_1) = 0.95\left(\frac{1}{5\cdot 19}\right) + 0.05\left(\frac{24}{5}\right)$$
$$= \frac{1}{20}\left(\frac{1}{5} + \frac{24}{5}\right) = \frac{1}{4}.$$

Since $E_1(g^1_1) = 1/4 > 0 = E_1(s^1_1)$, g^1_1 is the better choice or $p = 1$. Thus, we have shown that the only sequential equilibrium is

$$p = \beta(g^1_1) = 1,\ x = \beta(g^1_2) = 1/5,\ q = \beta(g^2_1) = 4/19,\ y = \beta(g^2_2) = 0, \quad \text{and} \quad \mu(v^1_{2,s}) = 4/5.$$

The payoff for for P_1 for the sequential equilibrium is $1/4$. For P_2, it is

$$E_2(R) = 0.95(5) + 0.05\left(\frac{24}{5}\right) = 4.75 + 0.24$$
$$= 4.99.$$

In this game, P_1 is generous the first time to take advantage of the fact that P_2 may be a generous type. ■

Example 13. Based on Example 5.10 in [1]. Consider the extensive game given in Figure 13.

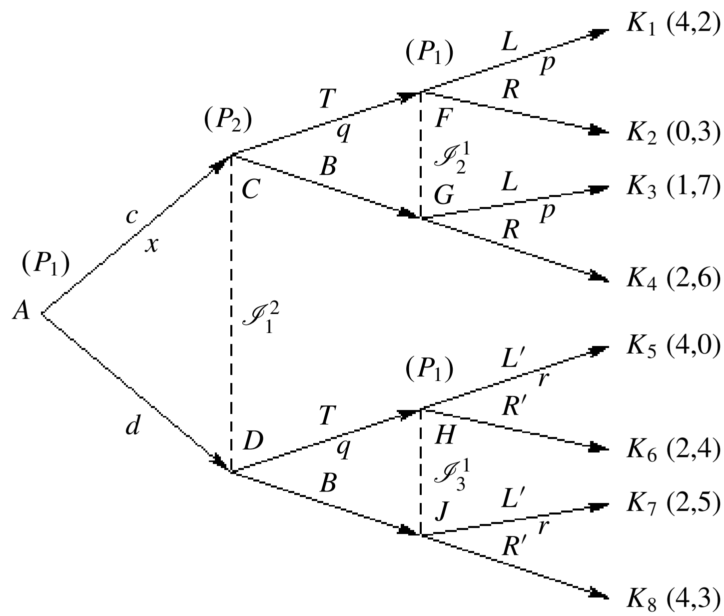

FIGURE 13. Game tree for Example 13

The strategies are labeled as follows.

1. At vertex A owned by P_1, let the behavioral strategy be $x = \beta_1(c)$ and $1 - x = \beta_1(d)$.
2. On the information set $\mathscr{I}_1^2 = \{C, D\}$ owned by P_2, let the behavioral strategy be $q = \beta_2(T)$ and $1 - q = \beta_2(B)$.
3. On the information set $\mathscr{I}_2^1 = \{F, G\}$ owned by P_1, let the behavioral strategy be $p = \beta_1(L)$ and $1 - p = \beta_1(R)$.
4. On the information set $\mathscr{I}_3^1 = \{H, J\}$ owned by P_1, let the behavioral strategy be $r = \beta_1(L')$ and $1 - r = \beta_1(R')$.

We start with the information set $\mathscr{I}_2^1$. The probabilities $\Pr^\beta(F) = \beta(R, F) = xq$ and $\Pr^\beta(G) = \beta(R, G) = x(1 - q)$. The consistent belief system is

$$\mu_1^\beta(F) = \begin{cases} \dfrac{\Pr^\beta(F)}{\Pr^\beta(F) + \Pr^\beta(G)} = \dfrac{xq}{xq + x(1-q)} = q & \text{if } x \neq 0, \\ \text{arbitrary} & \text{if } x = 0. \end{cases}$$

Similarly, $\mu_1^\beta(G) = 1 - q$, If $x = \beta(c) = 0$, then the probability of reaching $\mathscr{I}_2$ is zero, $\Pr^\beta(F) + \Pr^\beta(G) = xq + x(1 - q) = 0$, and the strategy profile does not induce a system of beliefs on $\mathscr{I}_2$. The payoffs for P_1 are

$$E_1(L|\mathscr{I}_2^1) = q(4) + (1 - q)(1) = 1 + 3q \qquad \text{and}$$
$$E_1(R|\mathscr{I}_2^1) = q(0) + (1 - q)(2) = 2 - 2q.$$

These are equal for $1 + 3q = 2 - 2q$, $5q = 1$, or $q = 1/5$. Therefore, the payoff for P_1 is is maximized on $\mathscr{I}_2^1$ for

$$p = \begin{cases} 0 & \text{if } q < 1/5, \\ \text{arbitrary} & \text{if } q = 1/5 \text{ or } x = 0, \\ 1 & \text{if } q > 1/5. \end{cases}$$

Similarly, a weakly consistent system of beliefs on $\mathscr{I}_3^1 = \{H, J\}$, is $\mu_1^\beta(H) = q$ and $\mu_1^\beta(J) = 1 - q$ if $x \neq 1$. The payoffs for P_1 are

$$E_1(L'|\mathscr{I}_3^1) = q(4) + (1-q)(2) = 2 + 2q \qquad \text{and}$$
$$E_1(R'|\mathscr{I}_3^1) = q(2) + (1-q)(4) = 4 - 2q.$$

These are equal for $2 + 2q = 4 - 2q$, $4q = 2$, or $q = 1/2$. Therefore, the payoff for P_1 is is maximized on $\mathscr{I}_3$ for

$$r = \begin{cases} 0 & \text{if } q < 1/2 \\ \text{arbitrary} & \text{if } q = 1/2 \quad \text{or} \quad x = 1, \\ 1 & \text{if } q > 1/2. \end{cases}$$

On $\mathscr{I}_1^2 = \{C, D\}$, a consistent system of beliefs is $\mu_2^\beta(C) = x$, and $\mu_2^\beta(D) = 1 - x$. The payoffs for P_2 are

$$\begin{aligned} E_2(T|\mathscr{I}_1^2) &= x[p2 + (1-p)(3)] + (1-x)[r(0) + (1-r)4] \\ &= 4 - 4r + x(-1 - p + 4r) \qquad \text{and} \\ E_2(B|\mathscr{I}_1^2) &= x[p7 + (1-p)6] + (1-x)[r(5) + (1-r)3] \\ &= 3 + 2r + x(3 + p - 2r). \end{aligned}$$

These are equal for $4-4r+x(-1-p+4r) = 3+2r+x(3+p-2r)$, $0 = 1-6r+x(-4-2p+6r) \equiv \Delta$. Therefore, the payoff for P_1 is is maximized on $\mathscr{I}_2$ for

$$q = \begin{cases} 0 & \text{if } \Delta < 0 \\ \text{arbitrary} & \text{if } \Delta = 0 \\ 1 & \text{if } \Delta > 0. \end{cases}$$

Finally, at A,

$$\begin{aligned} E_1(c) &= q\,[p(4) + (1-p)(0)] + (1-q)\,[p(1) + (1-p)(2)] \\ &= 4pq + (1-q)[2-p] = 2 - 2q - p + 5pq \\ E_1(d) &= q\,[r(4) + (1-r)(2)] + (1-q)\,[r(2) + (1-r)(4)] \\ &= q[2+2r] + (1-q)[4-2r] = 4 - 2q - 2r + 4rq. \end{aligned}$$

These are equal for $0 = p(5q-1) + r(2-4q) - 2 \equiv \Theta$. Therefore, the payoff for P_1 is maximized at A for

$$x = \begin{cases} 0 & \text{if } \Theta < 0 \\ \text{arbitrary} & \text{if } \Theta = 0 \\ 1 & \text{if } \Theta > 0. \end{cases}$$

We combine the calculations.

1. $q < 1/5 \Rightarrow p = 0\ \&\ r = 0 \Rightarrow \Theta = -2 \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1$. This is a contradiction.
2. $q = 1/5 \Rightarrow p$ arbitrary & $r = 0 \Rightarrow \Theta = -2 \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1$. This is a contradiction.
3. $1/5 < q < 1/2 \Rightarrow p = 1\ \&\ r = 0 \Rightarrow \Theta = -2 + (5q-1) < -1/2 \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1$. This is a contradiction.
4. $q = 1/2 \Rightarrow p = 1$ & r arbitrary $\Rightarrow \Theta < -1/2 \Rightarrow x = 0 \Rightarrow \Delta = 1 - 6r$. To allow $q = 1/2$, we need $\Delta = 0$, so $r = 1/6$. This gives a compatible solution.
5. $1/2 < q < 1 \Rightarrow p = 1\ \&\ r = 1 \Rightarrow \Theta = q - 1 < 0 \Rightarrow x = 0. \Rightarrow \Delta = 1 - 6 - 0 = -5 < 0$ and $q = 0$. This is a contradiction.
6. $q = 1 \Rightarrow p = 1\ \&\ r = 1 \Rightarrow \Theta = 0 \Rightarrow x$ arbitrary. $\Rightarrow \Delta = 1 - 6 - x(4+2-6) = -5 < 0$ and $q = 0$. This is a contradiction.

Therefore, the only solution is $x = \beta_1(c) = 0$, $q = \beta_2(T) = 1/2$, $p = \beta_1(L) = 1$, and $r = \beta_1(L') = 1/6$. The consistent system of beliefs is $\mu_2(C) = 0$, $\mu_2(D) = 1$, $\mu_1(F) = 1/2$, $\mu_1(G) = 1/2$, $\mu_1(H) = 1/2$, and $\mu_1(J) = 1/2$. ■

Example 14. (cf. Exercise page 210 in Myerson [4].) Consider the game given in Figure 14. Chance owns the root and has a $2/3$ probability of selecting the edge to vertex A and $1/3$ probability of selecting the edge to vertex B.

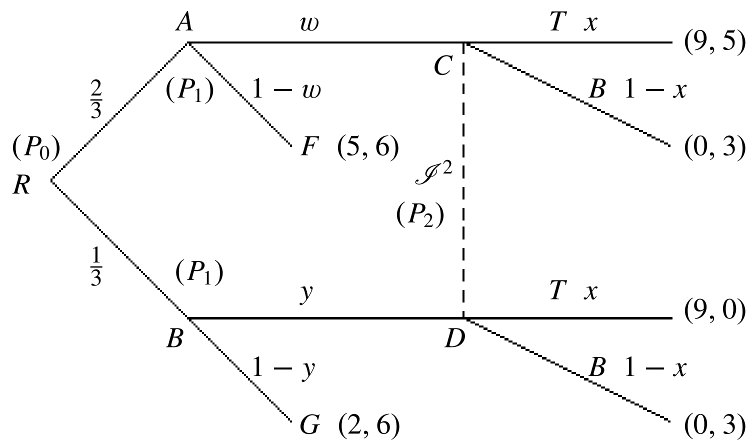

FIGURE 14. Game tree for Example 14

The system of beliefs of P_2 on the information set $\mathscr{I}$ is

$$\mu(C) = \begin{cases} \dfrac{\frac{2}{3}w}{\frac{2}{3}w + \frac{1}{3}y} = \dfrac{2w}{2w+y} & \text{if } (w, y) \neq (0,0), \\ \text{arbitrary} & \text{if } (w, y) = (0,0), \end{cases}$$
$$\mu(D) = 1 - \mu(C).$$

For $(w, y) \neq 0$, on $\mathscr{I}^2$ the expected payoffs for P_2 are

$$E_2(T) = \left(\frac{2w}{2w+y}\right) 5 + 0 \qquad \text{and}$$
$$E_2(B) = 3.$$

The edges T is preferred for $10w > 6w + 3y$ or $4w > 3y$. Therefore, the payoff for P_2 is optimized for

$$x = \beta(T) = \begin{cases} 0 & \text{if } w < \frac{3}{4}y \\ \text{arbitrary} & \text{if } w = \frac{3}{4}y \text{ or } (w, y) = (0,0), \\ 1 & \text{if } w > \frac{3}{4}y. \end{cases}$$

At the vertex A,

$$E_1(AC) = x(9) + (1-x)(0) = 9x, \qquad \text{and}$$
$$E_1(AF) = 5.$$

The edge AC is preferred for $9x > 5$. Therefore, the optimal payoff for P_1 occurs for

$$w = \beta(AC) = \begin{cases} 0 & \text{if } x < \frac{5}{9} \\ \text{arbitrary} & \text{if } x = \frac{5}{9} \\ 1 & \text{if } x > \frac{5}{9}. \end{cases}$$

Finally, considering vertex B owned by P_1,

$$E_1(BD) = x(9) + (1-x)(0) = 9x, \qquad \text{and}$$
$$E_1(BG) = 2.$$

The edge BD is preferred for $9x > 2$. Therefore, the optimal payoff for P_1 occurs for

$$y = \beta(BD) = \begin{cases} 0 & \text{if } x < \frac{2}{9} \\ \text{arbitrary} & \text{if } x = \frac{2}{9} \\ 1 & \text{if } x > \frac{2}{9}. \end{cases}$$

Combining the restrictions on the behavioral strategies, we get the following.

1. $x < 2/9 \Rightarrow\ w = 0\ \&\ y = 0 \Rightarrow\ x$ is arbitrary.
Therefore, one solution is $w = 0$, $y = 0$, and $0 \le x < 2/9$.

2. $x = 2/9\ \Rightarrow\ w = 0\ \&\ y$ arbitrary . Since $x \neq 0$, $y = \frac{4}{3}w = 0$.
One solution is $w = 0$, $y = 0$, and $x = 2/9$.

3. $2/9 < x < 5/9] \Rightarrow\ w = 0\ \&\ y = 1\ \Rightarrow\ x = 0$. This is a contradiction.

4. $x = 5/9\ \Rightarrow\ y = 1\ \&\ w$ arbitrary $\Rightarrow\ w = \frac{3}{4}y = \frac{3}{4}$.
Therefore, one solution is $w = \frac{3}{4}$, $y = 1$, and $x = 5/9$.

5. $x > 5/9\ \Rightarrow\ w = 1\ \&\ y = 1\ \Rightarrow\ x = 1$ (by the condition on $\mathscr{I}$.)
Therefore, one solution is $w = 1$, $y = 1$, and $x = 1$.

Summarizing, we have found three sequential equilibria:

$$\left\{ w = 0,\ y = 0,\ 0 \le x \le 2/9 \right\},$$
$$\left\{ w = 3/4,\ y = 1,\ x = 5/9 \right\},$$
$$\{ w = 1,\ y = 1,\ x = 1 \}.$$

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