

April 29, 2010

CHAPTER 13: EVOLUTIONARY EQUILIBRIUM

Some concepts from biology have been applied to game theory to define a type of equilibrium of a population that is robust against invasion by another type of strategy or state. In evolutionary game theory, there is assumed to be a large population that plays a mixed strategy as a whole: either each player plays a mixed strategy or the mixed strategy gives the probability that a random person plays a given strategy. An individual or invading sub-population plays against a randomly chosen person from the population as a whole in a given symmetric game. Rather than assume rationality of the players, the population is assumed to maintain a state with the maximum fitness, which is measured by the payoff: therefore, the population is assumed to adjust to the maximum payoff.

We assume that a symmetric game is fixed with a finite number of possible actions $\{s_1, \dots, s_n\}$ and payoffs $u(s_i, s_j) = a_{ij}$. Let

$$\Delta = \left\{ \mathbf{p} = (p_1, \dots, p_n)^T : p_i \geq 0 \text{ for all } i, \sum_i p_i = 1 \right\}$$

be the simplex of probability vectors giving mixed strategies. In the context of evolutionary game theory, an element $\mathbf{p} \in \Delta$ is called a *state* (or *strategy*) of the population. A state $\mathbf{p}$ can be thought of as either (i) the mixed strategy played by each individual in the population or (ii) the probability distribution of an individuals within a large population playing each action s_j with probability p_j . We consider the repeated random pairing of an individual against a population in state $\mathbf{p}$, so the individual encounters the states s_j with probability p_j . The *payoff* of an individual who takes action s_i against a population with state $\mathbf{p}$ is assumed to be

$$U(s_i, \mathbf{p}) = \sum_j a_{ij} p_j = (\mathbf{e}^i)^T \mathbf{A} \mathbf{p} = \mathbf{e}^i \cdot \mathbf{A} \mathbf{p},$$

where $\mathbf{e}^i$ is the standard unit vector with a 1 in the i^{th} -coordinate and 0s in the other coordinates. In the same way, the payoff of a state $\mathbf{q}$ playing against a state $\mathbf{p}$ is assumed to be

$$U(\mathbf{q}, \mathbf{p}) = \sum_{ij} q_i a_{ij} p_j = \mathbf{q}^T \mathbf{A} \mathbf{p} = \mathbf{q} \cdot \mathbf{A} \mathbf{p}.$$

The latter quantity $U(\mathbf{q}, \mathbf{p})$ can be thought of as either (i) an individual playing mixed strategy $\mathbf{q}$ against the whole population playing strategy $\mathbf{p}$ or (ii) an invading sub-population which plays the various strategies s_j with probability q_j .

Definition (129.2). A state $\hat{\mathbf{p}}$ is called a *Nash equilibrium* provided that it is a *symmetric Nash equilibrium*, i.e.,

$$U(\hat{\mathbf{p}}, \hat{\mathbf{p}}) \geq U(\mathbf{q}, \hat{\mathbf{p}}) \quad \text{for all } \mathbf{q} \in \Delta,$$

and the state $\mathbf{q} = \hat{\mathbf{p}}$ has the largest payoff in response to a population in the state $\hat{\mathbf{p}}$, or $\hat{\mathbf{p}} \in B(\hat{\mathbf{p}})$.

Definition (cf. 397.1). A state $\hat{\mathbf{p}}$ is an *evolutionary stable state* (or *evolutionary stable strategy*), ESS, provided that for any $\mathbf{q}$ in Δ with $\mathbf{q} \neq \hat{\mathbf{p}}$, there exists an $\epsilon(\mathbf{q}) > 0$ such that

$$U(\hat{\mathbf{p}}, \epsilon \mathbf{q} + (1 - \epsilon) \hat{\mathbf{p}}) > U(\mathbf{q}, \epsilon \mathbf{q} + (1 - \epsilon) \hat{\mathbf{p}})$$

and for all $0 < \epsilon \leq \epsilon(\mathbf{q})$. This means that if a small amount of a new state $\mathbf{q}$ invades to form a new state $\mathbf{q}_\epsilon = \epsilon \mathbf{q} + (1 - \epsilon) \hat{\mathbf{p}}$ then the intruder $\mathbf{q}$ has a lower payoff playing against than new state $\mathbf{q}_\epsilon$ than the original state $\hat{\mathbf{p}}$ has playing against $\mathbf{q}_\epsilon$.

In the definition of an ESS, we can think of $\mathbf{q}$ representing either (i) a mutation my individuals making a random choice of a strategy and ϵ the proportion of the population with this mutation, or (ii) an invading sub-population with state $\mathbf{q}$ with size ϵ in relation to the original population.

Comparison	
Rational Game Theorem	Evolutionary Game Theory
Set of player	Population from which the set of players is drawn
Payoff measured in payoff	Fitness of state
Strategy is chosen by a player	Strategy is inherited or endowed by a player
Equilibrium: no player can do better	ESS: no small mutation in the population can survive

The following theorem gives conditions for a Nash equilibrium and an ESS. Osborne takes the assumptions of part (b) of this theorem as the definition (400.3) of ESS.

Theorem.

- a. A state $\hat{\mathbf{p}}$ is a Nash equilibrium if and only if the following two conditions hold:
 - (i) $\mathbf{e}^i \cdot \mathbf{A}\hat{\mathbf{p}} \leq \hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}}$ for all i , and (ii) $\mathbf{e}^i \cdot \mathbf{A}\hat{\mathbf{p}} = \hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}}$ for all i with $\hat{p}_i > 0$.
- b. A state $\hat{\mathbf{p}} \in \Delta$ is an ESS if and only if the following two conditions hold: (i) $\hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} \geq \mathbf{q} \cdot \mathbf{A}\hat{\mathbf{p}}$ for all $\mathbf{q} \in \Delta$ (i.e., $\hat{\mathbf{p}}$ is a Nash equilibrium), and (ii) $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} > \mathbf{q} \cdot \mathbf{A}\mathbf{q}$ whenever $\mathbf{q} \neq \hat{\mathbf{p}}$ and $\hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \mathbf{q} \cdot \mathbf{A}\hat{\mathbf{p}}$.
- c. Let $\hat{\mathbf{p}}$ with $\hat{p}_i > 0$ for all i . The state $\hat{\mathbf{p}}$ is an ESS iff $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} > \mathbf{q} \cdot \mathbf{A}\mathbf{q}$ for all $\mathbf{q} \in \Delta$ with $\mathbf{q} \neq \hat{\mathbf{p}}$.
- d. Any state $\hat{\mathbf{p}}$ is an ESS iff $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} > \mathbf{q} \cdot \mathbf{A}\mathbf{q}$ for all $\mathbf{q} \neq \hat{\mathbf{p}}$ sufficiently close to $\hat{\mathbf{p}}$ in Δ .

Proof. a. If $\hat{\mathbf{p}}$ is a Nash equilibrium, then the inequality (i) holds for all $\mathbf{q}$ and so certainly for for all $\mathbf{e}^i$. Assume that $\hat{p}_i > 0$. Then by (i),

$$\hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \sum_{i \in I} \hat{p}_i \mathbf{e}^i \cdot \mathbf{A}\hat{\mathbf{p}} \leq \sum_{i \in I} \hat{p}_i \hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}}.$$

Since the first and last terms are equal, the inequality is an equality and we must have condition (ii) for all i for which $\hat{p}_i > 0$.

If we assume conditions (i) and (ii), then for any $\mathbf{q} \in \Delta$,

$$\mathbf{q} \cdot \mathbf{A}\hat{\mathbf{p}} = \sum_i q_i \mathbf{e}^i \cdot \mathbf{A}\hat{\mathbf{p}} \leq \sum_i q_i \hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}}.$$

This shows that $\hat{\mathbf{p}}$ is a Nash equilibrium.

b. Assume that $\hat{\mathbf{p}}$ is an ESS and $\mathbf{q} \neq \hat{\mathbf{p}}$. Expanding the condition defining an ESS and collecting terms, gives

$$(*) \quad (1 - \epsilon) (\hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} - \mathbf{q} \cdot \mathbf{A}\hat{\mathbf{p}}) + \epsilon (\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} - \mathbf{q} \cdot \mathbf{A}\mathbf{q}) > 0.$$

Since (*) holds for small $\epsilon > 0$, we get conditions (i) and (ii) of part (b). Conversely, if conditions (i) and (ii) hold, then (*) holds for all $\epsilon > 0$ sufficiently small. (Note that the second term can be negative so the inequality is not necessarily true for all ϵ .)

c. Assume that $\hat{\mathbf{p}}$ is an ESS with $\hat{p}_i > 0$ for all i . Since $\hat{\mathbf{p}}$ is a Nash equilibrium with $\hat{p}_i > 0$ for all i , $\hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \mathbf{e}^i \cdot \mathbf{A}\hat{\mathbf{p}}$ for all i . Take any $\mathbf{q} \neq \hat{\mathbf{p}}$ and sum with weights q_i , $\hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \sum_i q_i \hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \sum_i q_i \mathbf{e}^i \cdot \mathbf{A}\hat{\mathbf{p}} = \mathbf{q} \cdot \mathbf{A}\hat{\mathbf{p}}$. By condition (ii) of ESS, $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} > \mathbf{q} \cdot \mathbf{A}\mathbf{q}$ for this arbitrary $\mathbf{q} \neq \hat{\mathbf{p}}$. This is what we wanted to prove.

For the converse, for $\mathbf{q} \neq \hat{\mathbf{p}}$, set $\mathbf{q}_\epsilon = (1 - \epsilon)\hat{\mathbf{p}} + \epsilon\mathbf{q}$. So by assumption, $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q}_\epsilon > \mathbf{q}_\epsilon \cdot \mathbf{A}\mathbf{q}_\epsilon$ for all $\epsilon > 0$. Subtracting $(1 - \epsilon)\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q}_\epsilon$ from both sides of the inequality yields

$$\epsilon \hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q}_\epsilon > \epsilon \mathbf{q} \cdot \mathbf{A}\mathbf{q}_\epsilon.$$

Dividing the last inequality by ϵ gives the condition for an ESS.

d. Since part (c) covered the case with all the $\hat{p}_i > 0$, we assume that $\hat{\mathbf{p}}$ is an ESS with some $\hat{p}_i = 0$. Let

$$I(\hat{\mathbf{p}}) = \{i : \hat{p}_i > 0\},$$

$$\Delta_{I(\hat{\mathbf{p}})} = \{\mathbf{q} \in \Delta : q_i > 0 \text{ if } i \in I(\hat{\mathbf{p}}), q_j = 0 \text{ if } j \notin I(\hat{\mathbf{p}})\}, \quad \text{and}$$

$$\Delta_{-I(\hat{\mathbf{p}})} = \{\mathbf{q} \in \Delta : q_i = 0 \text{ for some } i \in I(\hat{\mathbf{p}})\}.$$

The set $\Delta_{I(\hat{\mathbf{p}})}$ is a “vertex”, “edge”, or “face” of Δ where certain q_j are zero.

The set $\Delta_{-I(\hat{\mathbf{p}})}$ is the union of all the faces opposite to $\Delta_{I(\hat{\mathbf{p}})}$. In fact, let $\mathbf{r}$ be any state not equal to $\hat{\mathbf{p}}$. Since $\sum_i r_i = \sum_i \hat{p}_i = 1$, $\sum_i (r_i - \hat{p}_i) = 0$ and some of the $r_i - \hat{p}_i$ must be negative. Let j be the index

and $t > 0$ be such that $0 = \hat{p}_j + t(r_j - \hat{p}_j)$ and all the $\hat{p}_i + t(r_i - \hat{p}_i) \geq 0$. Let $\mathbf{q} = \hat{\mathbf{p}} + t(\mathbf{r} - \hat{\mathbf{p}})$. Note that $\mathbf{q} \in \Delta_{-I(\hat{\mathbf{p}})}$. Since the line segment from $\hat{\mathbf{p}}$ to $\mathbf{q}$ is in Δ , $t \geq 1$. Thus,

$$\mathbf{r} = \frac{1}{t} \mathbf{q} + \left(1 - \frac{1}{t}\right) \hat{\mathbf{p}},$$

with $0 < 1/t \leq 1$. This shows all states in Δ can be written in this form.

For $\mathbf{q} \in \Delta_{-I(\hat{\mathbf{p}})}$, $\mathbf{q} \neq \hat{\mathbf{p}}$. Let $\mathbf{q}_\epsilon = \epsilon \mathbf{q} + (1 - \epsilon) \hat{\mathbf{p}}$. By the definition of ESS, $\mathbf{q} \cdot \mathbf{A} \mathbf{q}_\epsilon < \hat{\mathbf{p}} \cdot \mathbf{A} \mathbf{q}_\epsilon$ for $\epsilon(\mathbf{q}) \geq \epsilon > 0$. Multiplying this inequality by ϵ and adding $(1 - \epsilon) \hat{\mathbf{p}} \cdot \mathbf{A} \mathbf{q}_\epsilon$ to both sides gives $\mathbf{q}_\epsilon \cdot \mathbf{A} \mathbf{q}_\epsilon < \hat{\mathbf{p}} \cdot \mathbf{A} \mathbf{q}_\epsilon$. The $\epsilon(\mathbf{q})$ can all be taken greater than the same $\epsilon_0 > 0$ for all $\mathbf{q} \in \Delta_{-I(\hat{\mathbf{p}})}$ (where we have “compactness”). The union of the line segments from $\hat{\mathbf{p}}$ to all $\mathbf{q} \in \Delta_{-I(\hat{\mathbf{p}})}$ fill up all of Δ , so we get the inequality for all $\mathbf{q}_\epsilon \neq \hat{\mathbf{p}}$ in a neighborhood of $\hat{\mathbf{p}}$.

The proof of the converse is like the converse for part (c), but is only for small $\epsilon > 0$. $\square$

Remark. From part (c) of the theorem, it follows that if an interior Nash equilibrium is an ESS, then there cannot be any other Nash equilibrium. In other words, if there is an interior Nash equilibrium and another Nash equilibrium, then the interior Nash equilibrium is not an ESS.

Examples

Example. Consider the game with matrix $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$. For a state $\mathbf{p} = (p, 1 - p)^T$, the expected payoffs of $\mathbf{e}^1$ and $\mathbf{e}^2$ are $E(\mathbf{e}^1, \mathbf{p}) = 2 - 3p$ and $E(\mathbf{e}^2, \mathbf{p}) = 1 - p$. These are equal for $1 = 2p$, or $p = 1/2$. For $p < 1/2$, $\mathbf{e}^1$ is preferred and $\mathbf{e}^2$ is preferred for $p > 1/2$. The best response in terms of the value q of $(q, 1 - q)$ is

$$B(p, 1 - p) = \begin{cases} \{1\} & \text{if } p < 1/2 \\ [0, 1] & \text{if } p = 1/2 \\ \{0\} & \text{if } p > 1/2. \end{cases}$$

Therefore, the only Nash equilibrium is $p = 1/2$. Let $\hat{\mathbf{p}} = (1/2, 1/2)^T$.

Because $\hat{\mathbf{p}}$ is an interior Nash equilibrium, $\hat{\mathbf{p}} \cdot \mathbf{A} \hat{\mathbf{p}} = \mathbf{q} \cdot \mathbf{A} \hat{\mathbf{p}}$ for all $\mathbf{q}$. Letting $\mathbf{q} = (q, 1 - q)^T$,

$$\begin{aligned} \mathbf{q} \cdot \mathbf{A} \mathbf{q} &= (q, 1 - q)^T \cdot (2 - 3q, 1 - q)^T \\ &= 2q - 3q^2 + 1 - 2q + q^2 = 1 - 2q^2 \quad \text{and} \\ \hat{\mathbf{p}} \cdot \mathbf{A} \mathbf{q} &= (1/2, 1/2)^T \cdot (2 - 3q, 1 - q)^T \\ &= 3/2 - 2q. \end{aligned}$$

Is

$$\begin{aligned} 1 - 2q^2 &\stackrel{?}{<} 3/2 - 2q, \quad \text{or} \\ 0 &\stackrel{?}{<} 2q^2 - 2q + \frac{1}{2} = 2 \left(q^2 - q + \frac{1}{4} \right) = 2 \left(q - \frac{1}{2} \right)^2? \end{aligned}$$

This last inequality is true for all $q \neq 1/2$. Therefore, $\hat{\mathbf{p}}$ is an ESS. $\blacksquare$

Example (Example 404.2). The Dove-Hawk game can be modeled with the matrix $\mathbf{A} = \begin{pmatrix} 1 - c & 2 \\ 0 & 1 \end{pmatrix}$ for $c > 0$, where the first choice is an aggressive bird or hawk and the second choice is passive bird or dove. In comparison to the book, we have taken $v = 2$ and replaced c by $2c$ (to eliminate fractions). The expected payoffs for a state $\mathbf{p} = (p, 1 - p)^T$ are $E(\mathbf{e}^1, \mathbf{p}) = (1 - c)p + 2(1 - p) = 2 - (1 + c)p$ and $E(\mathbf{e}^2, \mathbf{p}) = 1 - p$.

If $0 < c < 1$, then $E(\mathbf{e}^1, \mathbf{p}) - E(\mathbf{e}^2, \mathbf{p}) = 1 - cp > 0$ for all p , $\mathbf{e}^1$ is always preferred, and the only Nash equilibrium is $p = 1$. Since $\mathbf{e}^1$ is a strict Nash equilibrium, it an ESS. This equilibrium corresponds to only hawks or aggressive birds surviving.

If $c \geq 1$, then $E(\mathbf{e}^1, \mathbf{p}) = E(\mathbf{e}^2, \mathbf{p})$ for $2 - (1 + c)p = 1 - p$, $1 = cp$, or $\hat{p}_c = 1/c$. The best response in terms of the value q of $(q, 1 - q)$ is

$$B(p, 1 - p) = \begin{cases} \{1\} & \text{if } p < 1/c \\ [0, 1] & \text{if } p = 1/c \\ \{0\} & \text{if } p > 1/c. \end{cases}$$

The only Nash equilibrium where $p \in B(p)$ is $\hat{\mathbf{p}}_c = (1/c, (c - 1)/c)$.

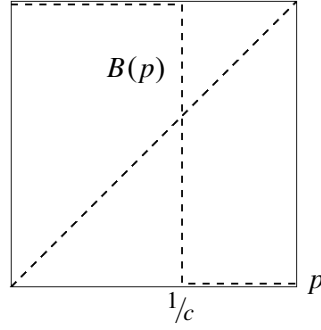


FIGURE 1. Best responses

For $c = 1$, it can be checked directly that $\hat{\mathbf{p}}_1 = \mathbf{e}^1$ is an ESS.

For $c > 1$, the equilibrium $\hat{\mathbf{p}} = \hat{\mathbf{p}}_c$ is interior, so to be a ESS, we need $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} > \mathbf{q} \cdot \mathbf{A}\mathbf{q}$ for all $\mathbf{q} \neq \hat{\mathbf{p}}$. Letting $\mathbf{q} = (q, 1 - q)$,

$$\begin{aligned} \hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} &= \left(\frac{1-c}{c}, \frac{1+c}{c} \right) \mathbf{q} = -2q + \frac{1+c}{c} \quad \text{and} \\ \mathbf{q} \cdot \mathbf{A}\mathbf{q} &= (1-c)q^2 + 2q(1-q) + (1-q)^2 = -cq^2 + 1. \end{aligned}$$

Therefore, we need

$$\begin{aligned} -2q + \frac{c+1}{c} &\stackrel{?}{>} -cq^2 + 1, \\ cq^2 - 2q + \frac{1}{c} &\stackrel{?}{>} 0, \quad \text{or} \\ \frac{1}{c} (cq - 1)^2 &\stackrel{?}{>} 0. \end{aligned}$$

This last inequality is true for all $q \neq 1/c = \hat{p}_c$, so $\hat{\mathbf{p}}_c$ is an ESS for $c > 1$. This equilibrium corresponds to both aggressive and passive birds surviving. ■

Example. Consider the game with matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. For a state $\mathbf{p}$, the expected payoffs are

$E(\mathbf{e}^1, \mathbf{p}) = p_2 + p_3$, $E(\mathbf{e}^2, \mathbf{p}) = p_1 + p_3$, and $E(\mathbf{e}^3, \mathbf{p}) = p_1 + p_2$. A straight forward consideration shows that these expected payoffs are equal only for $p_1 = p_2 = p_3 = 1/3$. Let $\hat{\mathbf{p}} = (1/3, 1/3, 1/3)^T$.

Because $\hat{\mathbf{p}}$ is an interior Nash equilibrium, $\hat{\mathbf{p}} \cdot \mathbf{A}\hat{\mathbf{p}} = \mathbf{q} \cdot \mathbf{A}\hat{\mathbf{p}}$ for all $\mathbf{q}$. We need to show that $\mathbf{q} \cdot \mathbf{A}\mathbf{q} < \hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q}$ for $\mathbf{q} \neq \hat{\mathbf{p}}$. The payoffs are $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{q} = (2/3, 2/3, 2/3)\mathbf{q} = 2/3$ and $\mathbf{q} \cdot \mathbf{A}\mathbf{q} = 2q_1q_2 + 2q_1q_3 + 2q_2q_3$. Substituting $q_1 = x_1 + 1/3$, $q_2 = x_2 + 1/3$, and $q_3 = 1/3 - x_1 - x_2$, $\mathbf{q} \cdot \mathbf{A}\mathbf{q} = -2x_1^2 - 2x_2^2 - 2x_1x_2 + 2/3$. The quadratic terms have the matrix $\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$, which has determinant $3 > 0$ and negative upper left entry. By the test for negative definite, $\mathbf{q} \cdot \mathbf{A}\mathbf{q} < 2/3$ for $(x_1, x_2) \neq (0, 0)$, i.e., for $(q_1, q_2, q_3) \neq \hat{\mathbf{p}}$. This shows that $\hat{\mathbf{p}}$ is an ESS.

Since $\hat{\mathbf{p}}$ is a completely mixed ESS, there cannot be any other Nash equilibrium. ■

Example. Consider the game given by the 3×3 matrix $\mathbf{A} = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$. The expected payoffs against

a state of $\mathbf{p}$ are $E(\mathbf{e}^1, \mathbf{p}) = 6p_2 - 4p_3$, $E(\mathbf{e}^2, \mathbf{p}) = -3p_1 + 5p_3$, and $E(\mathbf{e}^3, \mathbf{p}) = -p_1 + 3p_2$.

When $\mathbf{p} = \mathbf{e}^1$, $E(\mathbf{e}^1, \mathbf{e}^1) = 0$, which is greater than $E(\mathbf{e}^2, \mathbf{e}^1) = -3$ and $E(\mathbf{e}^3, \mathbf{e}^1) = -1$, so $\mathbf{e}^1 = (1, 0, 0)$ is one Nash equilibrium. Since $\mathbf{q} \cdot \mathbf{A}\mathbf{e}^1 = -3q_2 - q_3 < 0 = \mathbf{e}^1 \cdot \mathbf{A}\mathbf{e}^1$ for all $\mathbf{q} \neq \mathbf{e}^1$, $\mathbf{e}^1$ is a strict Nash equilibrium. Since there are no states $\mathbf{q} \neq \mathbf{e}^1$, with $\mathbf{q} \cdot \mathbf{A}\mathbf{e}^1 = \mathbf{e}^1 \cdot \mathbf{A}\mathbf{e}^1$, $\mathbf{e}^1$ is automatically an ESS.

For an interior equilibrium, the expected payoffs must all be equal. A direct calculation shows that $p_1 = p_2 = p_3 = 1/3$, and $\hat{\mathbf{p}} = (1/3, 1/3, 1/3)$ is an interior Nash equilibrium. Since there is another Nash equilibrium, $\hat{\mathbf{p}}$ cannot be an ESS. Alternatively, since $\hat{\mathbf{p}} \cdot \mathbf{A}\mathbf{e}^1 = -4/3 < 0 = \mathbf{e}^1 \cdot \mathbf{A}\mathbf{e}^1$, $\hat{\mathbf{p}}$ is not an ESS.

This example has both an interior Nash equilibrium and one on the boundary; the one on the boundary is an ESS and the one in the interior is not an ESS. ■

Example (Two ESS). Consider the game with

$$\begin{bmatrix} 5 & 7 & 2 \\ 8 & 6 & 5 \\ 1 & 8 & 4 \end{bmatrix}.$$

There are three Nash equilibria, one completely mixed equilibrium and two mixed between two states.

The expected values in playing against $\mathbf{p} = (p_1, p_2, p_3)$ are

$$E(\mathbf{e}^1, \mathbf{p}) = 5p_1 + 7p_2 + 2p_3$$

$$E(\mathbf{e}^2, \mathbf{p}) = 8p_1 + 6p_2 + 5p_3$$

$$E(\mathbf{e}^3, \mathbf{p}) = 1p_1 + 8p_2 + 4p_3.$$

If all three are equal, then we have the equations

$$5p_1 + 7p_2 + 2p_3 = 8p_1 + 6p_2 + 5p_3 = 1p_1 + 8p_2 + 4p_3 \quad \text{and}$$

$$p_1 + p_2 + p_3 = 1.$$

This leads to the equations

$$1 = p_1 + p_2 + p_3$$

$$0 = 3p_1 - p_2 + 3p_3$$

$$0 = -7p_1 + 2p_2 - p_3.$$

Solving the augmented system by row reduction is as follows:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & 0 \\ -7 & 2 & -1 & 0 \end{array} \right] &\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & 0 & -3 \\ 0 & 9 & 6 & 7 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 6 & \frac{1}{4} \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{24} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{24} \end{array} \right] \end{aligned}$$

We get the completely mixed Nash equilibrium of $(\frac{5}{24}, \frac{3}{4}, \frac{1}{24})$.

If $E(\mathbf{e}^1, \mathbf{p}) = E(\mathbf{e}^2, \mathbf{p}) > E(\mathbf{e}^3, \mathbf{p})$, then $p_3 = 0$, $p_1 = p$, and $p_2 = 1 - p$. We get the equation

$$5p + 7(1 - p) = 8p + 6(1 - p)$$

$$1 = 4p$$

$$p = \frac{1}{4}.$$

Then $E(\mathbf{e}^1, \mathbf{p}) = E(\mathbf{e}^2, \mathbf{p}) = 5 \cdot \frac{1}{4} + 7 \cdot \frac{3}{4} = \frac{26}{4}$ and $E(\mathbf{e}^3, \mathbf{p}) = \frac{1}{4} + 8 \cdot \frac{3}{4} = \frac{25}{4} < \frac{26}{4}$. Thus, $(\frac{1}{4}, \frac{3}{4}, 0)$ is a Nash equilibrium.

Similarly, if $E(\mathbf{e}^2, \mathbf{p}) = E(\mathbf{e}^3, \mathbf{p}) > E(\mathbf{e}^1, \mathbf{p})$, then $p_1 = 0$, $p_2 = p$, and $p_3 = 1 - p$. We get the equation

$$6p + 5(1 - p) = 8p + 4(1 - p)$$

$$1 = 3p$$

$$p = \frac{1}{3}.$$

Then $E(\mathbf{e}^2, \mathbf{p}) = E(\mathbf{e}^3, \mathbf{p}) = 6 \cdot \frac{1}{3} + 5 \cdot \frac{2}{3} = \frac{16}{3}$ and $E(\mathbf{e}^1, \mathbf{p}) = 7 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{11}{3} < \frac{16}{3}$. Thus, $(0, \frac{1}{3}, \frac{2}{3})$ is a Nash equilibrium.

It can be checked that there are not any other Nash equilibria, pure or with $E(\mathbf{e}^1, \mathbf{p}) = E(\mathbf{e}^3, \mathbf{p}) > E(\mathbf{e}^2, \mathbf{p})$.

Since there are other Nash equilibria, the completely mixed Nash equilibrium cannot be an ESS.

Consider the Nash equilibrium $\hat{\mathbf{p}} = (\frac{1}{4}, \frac{3}{4}, 0)$. If $U(\hat{\mathbf{p}}, \hat{\mathbf{p}}) = U(\mathbf{q}, \hat{\mathbf{p}})$, then $\mathbf{q} = (q, 1 - q, 0)$. We get the following payoffs,

$$U(\hat{\mathbf{p}}, \mathbf{q}) = \frac{1}{4}(5 + 3 \cdot 8, 7 + 3 \cdot 6) \begin{bmatrix} q \\ 1 - q \end{bmatrix} = \frac{1}{4}(29 - 25)q + \frac{25}{4} = q + \frac{25}{4},$$

$$\begin{aligned} U(\mathbf{q}, \mathbf{q}) &= 5q^2 + (8 + 7)q(1 - q) + 6(1 - q)^2 = 5q^2 + 15q - 15q^2 + 6 - 12q + 6q^2 \\ &= -4q^2 + 3q + 6. \end{aligned}$$

To be an ESS, for $q \neq \frac{1}{4}$, we need

$$\begin{aligned} q + \frac{25}{4} &\stackrel{?}{>} -4q^2 + 3q + 6 \\ 4q^2 - 2q + \frac{1}{4} &\stackrel{?}{>} 0 \\ 4\left(q^2 - \frac{1}{2}q + \frac{1}{16}\right) &\stackrel{?}{>} 0 \\ 4\left(q - \frac{1}{4}\right)^2 &\stackrel{?}{>} 0. \end{aligned}$$

This is indeed true, so this is an ESS.

The case for the Nash equilibrium $(0, \frac{1}{3}, \frac{2}{3})$ is similar. The only $\mathbf{p}$ that give the same payoff with $U(\hat{\mathbf{p}}, \hat{\mathbf{p}}) = U(\mathbf{q}, \hat{\mathbf{p}})$ have $\mathbf{q} = (0, q, 1 - q)$. We get the following payoffs,

$$\begin{aligned} U(\hat{\mathbf{p}}, \mathbf{q}) &= \frac{1}{3}(6q + 5(1 - q)) + \frac{2}{3}(8q + 4(1 - q)) = \left(\frac{6 - 5 + 16 - 8}{3}\right)q + \frac{5 + 8}{3} \\ &= 3q + \frac{13}{3}, \end{aligned}$$

$$\begin{aligned} U(\mathbf{q}, \mathbf{q}) &= 6q^2 + (8 + 5)q(1 - q) + 4(1 - q)^2 = 6q^2 + 13q - 13q^2 + 4 - 8q + 4q^2 \\ &= -3q^2 + 5q + 4. \end{aligned}$$

To be an ESS, for $q \neq \frac{1}{3}$, we need

$$\begin{aligned} 3q + \frac{13}{3} &\stackrel{?}{>} -3q^2 + 5q + 4 \\ 3q^2 - 2q + \frac{1}{3} &\stackrel{?}{>} 0 \\ 3\left(q^2 - \frac{2}{3}q + \frac{1}{9}\right) &= 3\left(q - \frac{1}{3}\right)^2 \stackrel{?}{>} 0. \end{aligned}$$

This is indeed true, so this is also an ESS. ■

Example (13.2.6: No ESS). Some games have no ESS. Consider

$$\begin{bmatrix} \gamma & -1 & 1 \\ 1 & \gamma & -1 \\ -1 & 1 & \gamma \end{bmatrix}.$$

with $0 < \gamma < 1$. The expected payoffs against a state $\mathbf{p}$ are

$$\begin{aligned} E(\mathbf{e}^1, \mathbf{p}) &= \gamma p_1 - p_2 + p_3, \\ E(\mathbf{e}^2, \mathbf{p}) &= p_1 + \gamma p_2 - p_3, \quad \text{and} \\ E(\mathbf{e}^3, \mathbf{p}) &= -p_1 + p_2 + \gamma p_3. \end{aligned}$$

(i) No pure strategy is a NE.

(ii) If $p_3 = 0$ and $p_1, p_2 > 0$, then $\gamma p_1 - p_2 = p_1 + \gamma p_2$ iff $0 = (1 - \gamma)p_1 + (1 + \gamma)p_2$, so $p_1 = p_2 = 0$. It is impossible for all the $p_i = 0$.

(iii) The case of $p_1 = 0$ or $p_2 = 0$ is similar to case (ii).

(iv) The expected values are all equal for $p_1 = p_2 = p_3 = 1/3$. A calculation shows that these are the only values with all three expected values equal.

Thus, $\mathbf{p}^* = (1/3, 1/3, 1/3)$ is the only Nash equilibrium.

For any pure strategies has a payoff $U(\mathbf{e}^i, \mathbf{p}^*) = \gamma/3 = U(\mathbf{p}^*, \mathbf{p}^*)$. Since $U(\mathbf{e}^i, \mathbf{e}^i) = \gamma > \gamma/3 = U(\mathbf{e}^i, \mathbf{p}^*)$, $\mathbf{p}^*$ is not an ESS. This shows that the only Nash equilibrium is not an ESS and there are no ESS. ■

Example (13.2.5 Bach versus Stravinsky). Because we want to set the problem up in a symmetric manner, we do not use the possible actions of Bach or Stravinsky, but rather of (i) choosing favorite concert F and (ii) deferring to the preference of your friend D . The payoffs are then assumed to be given by the following matrix:

$$\begin{pmatrix} & F & D \\ F & [(0, 0) & (2, 1)] \\ D & [(1, 2) & (0, 0)] \end{pmatrix}.$$

The payoffs in response to a state of $\mathbf{p} = (p, 1 - p)$ are

$$\begin{aligned} E(\mathbf{e}^1, \mathbf{p}) &= 2(1 - p) \\ E(\mathbf{e}^2, \mathbf{p}) &= p. \end{aligned}$$

They are equal for $2 - 2p = p$, $2 = 3p$, or $p = 2/3$. For $p < 2/3$, $\mathbf{e}^1$ is preferred so $p = 1$ is the best response. For $p > 2/3$, $\mathbf{e}^2$ is preferred so $p = 0$ is the best response. The best response correspondence is

$$B(p, 1 - p) = \begin{cases} \{1\} & \text{if } p < \frac{2}{3} \\ [0, 1] & \text{if } p = \frac{2}{3} \\ \{0\} & \text{if } p > \frac{2}{3}. \end{cases}$$

The only symmetric equilibrium is $\hat{\mathbf{p}} = (2/3, 1/3)$. There is no pure strategy symmetric equilibrium, even though we earlier found non-symmetric ones.

To check for an ESS, we need to compare the payoffs

$$\begin{aligned} U(q, q) &= 2q(1 - q) + 1(1 - q)q = 3q - 3q^2 \quad \text{and} \\ U(\hat{\mathbf{p}}, q) &= 2 \cdot \frac{2}{3}(1 - q) + 1 \cdot \frac{1}{3}q = \frac{4}{3} - q. \end{aligned}$$

Is

$$3q - 3q^2 \stackrel{?}{<} \frac{4}{3} - q, \quad \text{or}$$

$$0 \stackrel{?}{<} 3q^2 - 4q + \frac{4}{3} = 3 \left(q^2 - \frac{4}{3}q + \frac{4}{9} \right) = 3 \left(q - \frac{2}{3} \right)^2 \stackrel{?}{>}$$

This latter quantity is positive for $q \neq \frac{2}{3} = \hat{p}$. Therefore, this is an ESS. ■