

CHAPTER 4: MIXED STRATEGY EQUILIBRIA

§§4.1 & 4.12 Expected Payoffs and Preferences for Lotteries

We next consider probabilistic (or stochastic) choices of actions. We start by giving a few definitions.

Definition. Consider a finite set of possible outcomes $\mathbf{S} = \{s_1, \dots, s_n\}$. A *probability distribution* over $\mathbf{S}$ is a $\mathbf{p} = (p_1, \dots, p_n)$ with $p_i = \Pr(s_i) \geq 0$ and $\sum p_i = 1$. A *random variable* is a function $X : \mathbf{S} \rightarrow \mathbb{R}$. Given a probability distribution $\mathbf{p}$, the *expected value* of a function $X : \mathbf{S} \rightarrow \mathbb{R}$ is given by

$$E(X) = \sum_{s_i \in \mathbf{S}} p_i X(s_i).$$

Example. Consider a single die with outcomes $\mathbf{S} = \{1, \dots, 6\}$ and all the probabilities $p_i = 1/6$. If the random variable is the number facing upward on the die, $X(i) = i$, then

$$E(X) = \frac{1}{6}(1) + \frac{1}{6}(2) + \dots + \frac{1}{6}(6) = 3.5.$$

If we use the random variable of the square of the number on the face, $Y(i) = i^2 = X(i)^2$, then

$$E(Y) = \frac{1}{6}(1) + \frac{1}{6}(4) + \dots + \frac{1}{6}(36) = \frac{91}{6} = 15\frac{1}{6}. \quad \blacksquare$$

We next consider payoffs from lotteries on a finite number of outcomes.

Definition. Let $\mathbf{S} = \{x_i : 1 \leq i \leq n\}$ be a finite set of outcomes. A *lottery* L on these outcomes is a probability distribution on these outcomes:

$$L = \left\{ (x_i, p_i) : \sum_{i=1}^n p_i = 1, \quad p_i \geq 0 \text{ for } 1 \leq i \leq n \right\}.$$

Let $\mathcal{L}(\mathbf{S}) = \{L\}$ be the set of all lotteries on $\mathbf{S}$.

A *preference* on the set of all lotteries $\mathcal{L}(\mathbf{S})$ is a partial ordering $\succeq$ that satisfies the following completeness assumption: For every pair of lotteries L_1 and L_2 in $\mathcal{L}(\mathbf{S})$, either $L_1 \succeq L_2$ or $L_2 \succeq L_1$.

If $u : \mathbf{S} \rightarrow \mathbb{R}$ is a payoff function on individual outcomes, then there is induced a *Bernoulli expected payoff function* U on the set of all lotteries given by

$$U(L) = \sum_{x_i \in \mathbf{S}} p_i u(x_i).$$

In turn, this Bernoulli expected payoff function U induces a preference on the set of all lotteries as follows: $L_1 \succeq L_2$ if and only if $U(L_1) \geq U(L_2)$.

Example. In this example we consider three different types of rescaling of the monetary returns: $g_1(w) = aw + b$ with $a > 0$, $g_2(w) = \sqrt{w}$, and $g_3(w) = w^2$. The different types of rescaling induce different preferences on lotteries.

Consider three lotteries. In each case, a coin is flipped and it is either head or tail. Each possibility has probability $1/2$. In lottery L_A , the monetary return is the same, $u_A(H) = u_A(T) = 5$. In lottery L_B , the monetary returns are $u_B(H) = 9$ and $u_B(T) = 1$. In lottery L_C , the monetary returns are $u_C(H) = 10$ and $u_C(T) = 0$. Each of these lotteries has the same expected monetary return of 5:

$$\begin{aligned} E(u_A) &= \frac{1}{2}(5) + \frac{1}{2}(5) = 5, \\ E(u_B) &= \frac{1}{2}(9) + \frac{1}{2}(1) = 5, \quad \text{and} \\ E(u_C) &= \frac{1}{2}(10) + \frac{1}{2}(0) = 5. \end{aligned}$$

The first rescaling $g_1(w) = aw + b$ is called *risk neutral* because it preserves the equality of the payoffs:

$$\begin{aligned} E(au_A + b) &= \frac{1}{2}(a5 + b) + \frac{1}{2}(a5 + b) = a5 + b, \\ E(au_B + b) &= \frac{1}{2}(a9 + b) + \frac{1}{2}(a1 + b) = a5 + b, \quad \text{and} \\ E(au_C + b) &= \frac{1}{2}(a10 + b) + \frac{1}{2}(a0 + b) = a5 + b. \end{aligned}$$

The book states in Lemma 148.1 that this type of linear rescaling is the only type that preserves the preferences on lotteries.

The second rescaling is $g_2(w) = \sqrt{w}$.

$$\begin{aligned} E(\sqrt{u_A}) &= \frac{1}{2}(\sqrt{5}) + \frac{1}{2}(\sqrt{5}) = \sqrt{5} \approx 2.24, \\ E(\sqrt{u_B}) &= \frac{1}{2}(\sqrt{9}) + \frac{1}{2}(\sqrt{1}) = 2, \quad \text{and} \\ E(\sqrt{u_C}) &= \frac{1}{2}(\sqrt{10}) + \frac{1}{2}(\sqrt{0}) = \frac{1}{2}(\sqrt{10}) \approx 1.58. \end{aligned}$$

Therefore, L_A is preferred to L_B , and L_B is preferred to L_C . A person using this type of payoff function to induce preference on lotteries prefers the certainty of the payoff of 5 of L_A to the possibility of a higher payoff of a lottery like L_C . Such types of rescaling are *risk averse*.

The third rescaling is $g_3(w) = w^2$.

$$\begin{aligned} E(u_A^2) &= \frac{1}{2}(5^2) + \frac{1}{2}(5^2) = 25, \\ E(u_B^2) &= \frac{1}{2}(9^2) + \frac{1}{2}(1^2) = 41 > 25, \quad \text{and} \\ E(u_C^2) &= \frac{1}{2}(10^2) + \frac{1}{2}(0^2) = 50 > 41. \end{aligned}$$

Therefore, L_C is preferred to L_B , and L_B is preferred to L_A . A person using this type of payoff function to induce preference on lotteries prefers the possibility of a higher payoff of L_C to the certainty of the payoff of 5 of L_A . Such types of rescaling are *risk seeking*. ■

Example. In this example, we consider a pair of lotteries with the same expected payoff. The first lottery L_b are bonds that pay 70% return in five years; this return is certain. The second lottery L_s are stocks: they have a $p_1 = 1/2$ chance of increasing 100% and $p_2 = 1/2$ chance of increasing 40%. Assume the investment is \$100. We give three different assessment valuations on the two lotteries.

(1) *Risk neutral*: $u_1(w) = w$.

For this function, $U_1(L_b) = 170$ and $U_1(L_s) = \frac{1}{2}u_1(200) + \frac{1}{2}u_1(140) = \frac{1}{2}[200 + 140] = 170$. This risk neutral valuation evaluates them equivalent in preference.

(2) *Risk averse*: $u_2(w) = \sqrt{w}$.

For this function, $U_2(L_b) = \sqrt{170} = 13.038$ and $U_2(L_s) = \frac{1}{2}[\sqrt{200} + \sqrt{140}] = 12.987$. This risk averse valuation evaluates L_b above L_s . A person using this type of utility function prefers the certainty of the return of bonds over the possibility of higher return for stocks.

(3) *Risk seeking*: $u_3(w) = w^2$.

For this function, $U_3(L_b) = 170^2 = 28,900$ and $U_3(L_s) = \frac{1}{2}[200^2 + 140^2] = 29,800$. This risk seeking valuation evaluates L_s above L_b . A person using this type of utility function prefers possibility of higher return for stocks over the certainty of the return of bonds. ■

Types of functions to rescale the payoffs in the three categories:

Risk neutral: $u_1(w) = aw + b$. This function is an affine function of w .

Risk averse: $u_2(w) = w^a$ for $0 < a < 1$, or $\ln(w)$. These functions are increasing with negative second derivative: they are concave functions. They tend to have a maximum in the interior of the interval of definition.

Risk seeking: $u_3(w) = w^a$ for $a > 1$. These functions are increasing with positive second derivative: they are convex (concave up) functions. They tend to have a maximum at an endpoint of the interval of definition.

These examples and discussion are meant to illustrate that the values of the payoffs on outcomes is important in inducing the preference on lotteries and mixed strategies. Your homework problem has a rescaling that changes the preferences on a lottery of outcomes. The correct scaling of the monetary value to give a payoff function depends on the individual. In the material we consider, we will usually just give values of the payoffs to use. If you were modeling a situation, you would somehow need to determine a reasonable scaling for the individuals being considered.

We next consider the assumptions for preferences on lotteries that are necessary to insure that they are induced by a Bernoulli expected value of a payoff function.

Definition. Let $\mathbf{S}$ be a finite set of outcomes. A preference on $\mathcal{L}(\mathbf{S})$ on the set of all lotteries on $\mathbf{S}$ that satisfies the following two additional assumptions is called a *von Neumann and Morgenstern preference* or a *vNM preference*:

Continuity: if $L_j \succeq L'$ for $j \geq 1$ and L_j converges to L_∞ , then $L_\infty \succeq L'$.

Independence: If $L_1 \succeq L_2$ and $0 < p \leq 1$, then $p L_1 + (1 - p) L_3 \succeq p L_2 + (1 - p) L_3$.

Theorem. Assume that $\mathcal{L}$ is the set of lotteries on a finite set of outcomes $\mathbf{S}$. Assume that we have a preference $\succeq$ on $\mathcal{L}$ that satisfy the above assumptions, (i) – (ii). Then there is a payoff function u on the outcomes, $u : \mathbf{S} \rightarrow \mathbb{R}$, such that this preference is induced by the Bernoulli expected payoff function $U(L) = \sum_{x_i \in \mathbf{S}} p_i u(x_i)$.

The continuity assumption on the preferences on lotteries is very mild and should be satisfied. The independence assumption is much stronger. The following example gives preferences that are not induced by a payoff function on the individual outcomes, so these preferences on lotteries cannot satisfy the independence assumption.

Example. (See pages 104-5) Consider the four lotteries $L_1 = \{(\$2M, 1)\}$, $L_2 = \{(\$2M, 0.11), (\$0, 0.89)\}$, and $L_3 = \{(\$10M, 0.1), (\$2M, 0.89), (\$0, 0.01)\}$, $L_4 = \{(\$10M, 0.1), (\$0, 0.9)\}$. Assume that $L_1 \succ L_3$ and $L_4 \succ L_2$. We show that this preference cannot be induced by a Bernoulli payoff function derived from a payoff function on the individual outcomes.

Assume there is a utility function u given by a rescaling of the payoffs that induces the preferences. Since $L_1 \succ L_3$,

$$U(L_1) = u(2) > U(L_3) = 0.1 u(10) + 0.89 u(2) + 0.01 u(0),$$

where the amounts are expressed in millions of dollars. Adding $0.89 u(0) - 0.89 u(2)$ to both sides, we get that

$$U(L_2) = 0.11 u(2) + 0.89 u(0) > 0.1 u(10) + 0.9 u(0) = U(L_4).$$

This contradicts the preferences on lotteries L_2 and L_4 and shows that the preferences are not induced by a u on individual outcomes. ■

Example. This example has a set of outcomes that is countably infinite rather than finite. It also indicates the need to rescale the monetary value of the outcomes to get a realistic utility of the outcomes.

In this lottery, a coin is flipped until the first head is obtained. There are a countably infinite number of possible outcomes:

$$H, TH, TTH, \dots, T^{n-1}H, \dots$$

and an infinite sequence of tails, T^∞ . If the coin is fair with $P(H) = P(T) = \frac{1}{2}$ and the outcome of each flip is independent of the previous flips, then the probabilities of the outcomes are

$$P(H) = \frac{1}{2}, P(TH) = P(H)P(T) = \frac{1}{2^2}, P(TTH) = \frac{1}{2^3}, \dots, P(T^{n-1}H) = \frac{1}{2^n}, \dots, P(T^\infty) = 0.$$

Assume that the person playing the lottery is given a monetary return of 2^n dollars if the first head is on the n^{th} flip,

$$X(H) = 2, X(TH) = 2^2, X(TTH) = 2^3, \dots, X(T^{n-1}H) = 2^n, \dots, X(T^\infty) = 0.$$

The expected monetary return is

$$\begin{aligned} E(X) &= \frac{1}{2}(2) + \frac{1}{2^2}(2^2) + \cdots + \frac{1}{2^n}(2^n) + \cdots \\ &= 1 + 1 + \cdots = \infty. \end{aligned}$$

How much would you be willing to pay for this lottery? Since the probability of getting a return greater than 8 is $1/8$, you would probably not be willing to pay \$1,000. This shows that the expected monetary return is not a good measure of the value of this lottery to most or all individuals.

A way to create a measure of the value is to use a function to rescale the monetary return to get a payoff function that can be used to evaluate for a person. Say we take $g(w) = \sqrt{w}$. Consider the random variable $g \circ X = \sqrt{X}$. The expected value of this random variable is

$$\begin{aligned} E(\sqrt{X}) &= \frac{1}{2}(\sqrt{2}) + \frac{1}{2^2}(\sqrt{2^2}) + \cdots + \frac{1}{2^n}(\sqrt{2^n}) + \cdots \\ &= \frac{1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2^2}} + \cdots \right) \\ &= \frac{1}{\sqrt{2}} \left(\frac{1}{1 - \frac{1}{\sqrt{2}}} \right) = \frac{1}{\sqrt{2} - 1} \\ &\approx 2.413 \dots \\ &\approx \sqrt{5.83} \dots \end{aligned}$$

Using this rescaling, we get the worth is \$5.83. This is much more reasonable. Depending on your tolerance for risk, this valuation may be too high or too low for you.

If we use the function $g_2(w) = \ln(w)$ to rescale the monetary return to determine the payoff function, then the worth turns out to be about \$4.00. Which rescaling is the correct one depends on the individual. ■

§§4.2-4.3 Mixed Strategies

We next consider probabilistic choices of actions, called mixed strategies.

Definition. A *mixed strategy* of a player P_i in a strategic game is a probability distribution over the player's possible actions, A_i . If A_i is finite, then the distribution can be given as a function $\alpha_i : A_i \rightarrow [0, 1]$ with $\sum_{a \in A_i} \alpha_i(a) = 1$. Let $\mathcal{A}_i$ be the set of mixed strategies for P_i .

A mixed strategy that equals 1 for one action a_0 and zero for all other actions, $\alpha_i(a_0) = 1$ and $\alpha_i(a) = 0$ for $a \neq a_0$, is called a *pure strategy* or *pure action*.

For a two person game with finite set of actions (bimatrix game), we often let $p_i = \alpha_1(r_i)$ and $q_j = \alpha_2(c_j)$ be the probabilities of the first and second players. Then, $p_i q_j$ is the probability of the pair of actions (r_i, c_j) .

Definition. Consider an n -person strategic game with finite set of actions A_i and payoff functions u_i for $1 \leq i \leq n$. The *Bernoulli payoff functions* for a mixed strategy $(\alpha_1, \dots, \alpha_n)$ are given by

$$U_i(\alpha_1, \dots, \alpha_n) = \sum_{a_1 \in A_1, \dots, a_n \in A_n} \alpha_1(a_1) \cdots \alpha_n(a_n) u_i(a_1, \dots, a_n).$$

This payoff function determines a vNM preferences on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.

Example. In this example we consider two bimatrix games that have the same preferences on pure actions but different preferences on mixed strategies.

The two bimatrix games are

$$\begin{array}{cc} Q_2 & F_2 \\ Q_1 \left(\begin{pmatrix} (2, 2) & (0, 3) \\ (3, 0) & (1, 1) \end{pmatrix} \right) & \text{and} \quad \begin{array}{cc} Q_2 & F_2 \\ Q_1 \left(\begin{pmatrix} (3, 3) & (0, 4) \\ (4, 0) & (1, 1) \end{pmatrix} \right) \end{array} \end{array}$$

(Q stands for quiet and F for fink.) Assume the first player uses probabilistic choices of actions with $p = \alpha_1(Q_1)$ for the probability of Q_1 and $1 - p = \alpha_1(F_1)$ for the probability of F_1 . The second player

also uses probabilistic choices of actions with $q = \alpha_2(Q_2)$ for the probability of Q_2 and $1 - q = \alpha_2(F_2)$ for the probability of F_2 . The probability of combinations of actions are the product of the probability of each action: e.g., the probability of (Q_1, Q_2) is pq . The Bernoulli expected payoff functions are

$$\begin{aligned} U_1(p, q) &= pq u_1(Q_1, Q_2) + p(1 - q) u_1(Q_1, F_2) + (1 - p)q u_1(F_1, Q_2) + (1 - p)(1 - q) u_1(F_1, F_2) \\ &= 2pq + 0(p)(1 - q) + 3(1 - p)q + (1 - p)(1 - q) \\ &= 1 - p + 2q \quad \text{and} \\ U_2(p, q) &= pq u_2(Q_1, Q_2) + p(1 - q) u_2(Q_1, F_2) + (1 - p)q u_2(F_1, Q_2) + (1 - p)(1 - q) u_2(F_1, F_2) \\ &= 2pq + 3p(1 - q) + 0(1 - p)q + (1 - p)(1 - q) \\ &= 1 - 4pq + q + 2p. \end{aligned}$$

Using these payoffs, the payoff U_1 for the mixed strategies $(1, 1)$ and $(0, 11/20)$ are

$$\begin{aligned} U_1(1, 1) &= 2 \\ U_1(0, 11/20) &= \frac{11}{20}(3) + \frac{9}{20}(1) = \frac{42}{20} = 2.1. \end{aligned}$$

Therefore, $U_1(0, 11/20) = 2.1 > 2 = U_1(1, 1)$ and $(0, 11/20)$ is preferred to $(1, 1)$. This is a *vNM preference*.

For the second game,

$$\begin{aligned} \tilde{U}_1(1, 1) &= 3 \\ \tilde{U}_1(0, 11/20) &= \frac{11}{20}(4) + \frac{9}{20}(1) = \frac{53}{20} = 2.65. \end{aligned}$$

Therefore, $\tilde{U}_1(1, 1) = 3 > 2.65 = \tilde{U}_1(0, 11/20)$ and $(1, 1)$ is preferred to $(0, 11/20)$.

This example shows that the vNM preferences induced on the mixed strategies depends on the values of the payoffs on the actions and not just the preferences on the actions. ■

Definition. Consider an n -person strategic game such that for $1 \leq i \leq n$, there are a payoff functions u_i , a finite set of actions A_i , and a set of mixed strategies $\mathcal{A}_i$. Let $\mathcal{A}_{-i} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{i-1} \times \mathcal{A}_{i+1} \times \cdots \times \mathcal{A}_n$ be all the mixed strategies other than those of P_i . An element of $\mathcal{A}_{-i}$ is denoted α_{-i} . Denote the maximum payoff of u_i in response to a mixed strategy α_{-i} by the other players by

$$m_i(\alpha_{-i}) = \max_{\alpha_i \in \mathcal{A}_i} U_i(\alpha_i, \alpha_{-i}).$$

The *best response correspondence* for P_i is the set valued correspondence defined by

$$B_i(\alpha_{-i}) = \{\alpha_i : U_i(\alpha_i, \alpha_{-i}) = m_i(\alpha_{-i})\}.$$

Definition. For a n -person strategic game with payoff functions u_i , a mixed strategy profile $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$ is a *mixed strategy Nash equilibrium* provided that, for all $1 \leq i \leq n$,

$$U_i(\alpha_1^*, \dots, \alpha_n^*) \geq U_i(\alpha_i, \alpha_{-i}^*) \quad \text{for all } \alpha_i \in \mathcal{A}_i,$$

i.e., $\alpha_i^* \in B_i(\alpha_{-i}^*)$ for all $1 \leq i \leq n$.

Example. Consider the bimatrix game given by

$$\begin{pmatrix} (2, 0) & (1, 2) \\ (0, 4) & (2, 1) \end{pmatrix}.$$

It is directly checked that this game has no pure strategy Nash equilibria.

The Bernoulli expected payoff function for P_1 is given by

$$\begin{aligned} U_1(\mathbf{p}, \mathbf{q}) &= p_1 [2q_1 + 1q_2] + p_2 [0q_1 + 2q_2] \\ &= p_1 E_1(r_1, \mathbf{q}) + p_2 E_1(r_2, \mathbf{q}) \end{aligned}$$

where

$$\begin{aligned} E_1(r_1, \mathbf{q}) &= 2q_1 + 1q_2 \quad \text{and} \\ E_1(r_2, \mathbf{q}) &= 0q_1 + 2q_2 \end{aligned}$$

are the expected payoff of the choices of the first and second rows.

Since $U_1(\mathbf{p}, \mathbf{q})$ is an average of $E_1(r_1, \mathbf{q})$ and $E_1(r_2, \mathbf{q})$, its maximum is the maximum of the two values. If the values are equal, then any $\mathbf{p}$ gives a best response, otherwise a single value works.

The values $2q_1 + q_2 = E_1(r_1, \mathbf{q}) = E_1(r_2, \mathbf{q}) = 2q_2$ when

$$\begin{aligned} 2q_1 + q_2 &= 2q_2 \\ 2q_1 &= q_2 = 1 - q_1 \\ 3q_1 &= 1 \\ q_1 &= 1/3. \end{aligned}$$

If $q_1 < 1/3$, then $E_1(r_1, \mathbf{q}) < E_1(r_2, \mathbf{q})$, and if $q_1 > 1/3$, then $E_1(r_1, \mathbf{q}) > E_1(r_2, \mathbf{q})$. Thus, the best response correspondence is

$$B_1(\mathbf{q}) = \begin{cases} \{0\} & \text{if } q_1 < 1/3 \\ [0, 1] & \text{if } q_1 = 1/3 \\ \{1\} & \text{if } q_1 > 1/3. \end{cases}$$

See Figure 1.

In the same way, the Bernoulli expected payoff function for P_2 is given by

$$\begin{aligned} U_2(\mathbf{p}, \mathbf{q}) &= q_1 [4p_2] + q_2 [2p_1 + p_2] \\ &= q_1 E_2(\mathbf{p}, c_1) + q_2 E_2(\mathbf{p}, c_2) \end{aligned}$$

where

$$\begin{aligned} E_2(\mathbf{p}, c_1) &= 4p_2 \quad \text{and} \\ E_2(\mathbf{p}, c_2) &= 2p_1 + p_2 \end{aligned}$$

are the expected payoff of the choices of the first and second columns. The values $4p_2 = E_2(\mathbf{p}, c_1) = E_2(\mathbf{p}, c_2) = 2p_1 + p_2$ when

$$\begin{aligned} 4p_2 &= 2p_1 + p_2 \\ 2p_1 &= 3p_2 = 3 - 3p_1 \\ 5p_1 &= 3 \\ p_1 &= 3/5. \end{aligned}$$

If $p_1 < 3/5$, then $E_2(\mathbf{p}, c_1) > E_2(\mathbf{p}, c_2)$, and if $p_1 > 3/5$, then $E_2(\mathbf{p}, c_1) < E_2(\mathbf{p}, c_2)$. Thus, the best response correspondence is

$$B_2(\mathbf{p}) = \begin{cases} \{1\} & \text{if } p_1 < 3/5 \\ [0, 1] & \text{if } p_1 = 3/5 \\ \{0\} & \text{if } p_1 > 3/5. \end{cases}$$

See Figure 1.

The Nash equilibrium is at the intersection of the graphs of the two best response functions, so $(3/5, 1/3)$.

The payoffs of both players in response to this fixed mixed strategy by the other player are given by

$$\begin{aligned} U_1(\mathbf{p}, 1/3) &= p_1 \left(\frac{2}{3} + \frac{2}{3} \right) + p_2 \left(\frac{4}{3} \right) = \frac{4}{3} (p_1 + p_2) = \frac{4}{3} \\ U_2(3/5, \mathbf{q}) &= q_1 \left(\frac{8}{5} \right) + q_2 \left(\frac{8}{5} \right) = \frac{8}{5} (p_1 + p_2) = \frac{8}{5}; \end{aligned}$$

each of these payoffs are independent of the choice of their own mixed strategy so they satisfy the conditions for a Nash equilibrium, as they must. ■

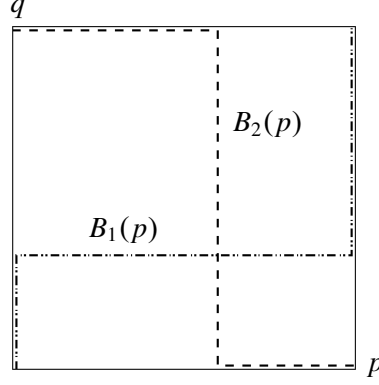


FIGURE 1. Best responses

For a general bimatrix game with mixed strategy profile (α_1, α_2) , we let $p_i = \alpha_1(r_i)$, $q_j = \alpha_2(c_j)$, $\mathbf{p} = (p_1, \dots, p_m)$ where $m = \#(A_1)$, and $\mathbf{q} = (q_1, \dots, q_n)$ where $n = \#(A_2)$. Then,

$$\begin{aligned}
 E_1(r_i, \mathbf{q}) &= \sum_{j=1}^n q_j u_1(r_i, c_j), \\
 U_1(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^m \sum_{j=1}^n p_i q_j u_1(r_i, c_j) \\
 &= \sum_{i=1}^m p_i E_1(r_i, \mathbf{q}) \\
 E_2(\mathbf{q}, c_j) &= \sum_{i=1}^m p_i u_2(r_i, c_j), \\
 U_2(\mathbf{p}, \mathbf{q}) &= \sum_{j=1}^n \sum_{i=1}^m q_j p_i u_2(r_i, c_j) \\
 &= \sum_{j=1}^n q_j E_2(\mathbf{p}, c_j).
 \end{aligned}$$

The set of all mixed strategies for each player is a simplex, which we denote by

$$\begin{aligned}
 \mathcal{A}_1 = \Delta_m &= \left\{ (p_1, \dots, p_m) : p_i \geq 0 \text{ \& } \sum_{i=1}^m p_i = 1 \right\} \quad \text{and} \\
 \mathcal{A}_2 = \Delta_n &= \left\{ (q_1, \dots, q_n) : q_j \geq 0 \text{ \& } \sum_{j=1}^n q_j = 1 \right\}
 \end{aligned}$$

For $\Delta = \Delta_m$ or Δ_n , let $\mathcal{P}_c(\Delta)$ be the set of nonempty closed subsets of Δ . Then the best responses $B_1(\mathbf{q})$ and $B_2(\mathbf{p})$ are subsets of Δ_m and Δ_n respectively, so elements of $\mathcal{P}_c(\Delta_m)$ and $\mathcal{P}_c(\Delta_n)$ respectively.

Theorem (cf. Proposition 116.2). Assume $m = \#(A_1) < \infty$ and $n = \#(A_2) < \infty$. Let $m_1(\mathbf{q}) = \max_{\mathbf{p} \in \mathcal{A}_1} U_1(\mathbf{p}, \mathbf{q})$ and $m_2(\mathbf{p}) = \max_{\mathbf{q} \in \mathcal{A}_2} U_2(\mathbf{p}, \mathbf{q})$. Then the following conditions hold.

- a. (i) $m_1(\mathbf{q}) = \max_{1 \leq i \leq m} E_1(r_i, \mathbf{q})$.
- (ii) Let $A_1^{\max}(\mathbf{q}) = \{r_i \in A_1 : E_1(r_i, \mathbf{q}) = m_1(\mathbf{q})\}$. Then,

$$B_1(\mathbf{q}) = \{ \mathbf{p} \in \mathcal{A}_1 : p_i = 0 \text{ if } r_i \notin A_1^{\max}(\mathbf{q}) \}.$$

Therefore, $B_1(\mathbf{q})$ is a convex combination of the pure actions with the maximum payoff in response to $\mathbf{q}$; $B_1(\mathbf{q})$ is nonempty, closed, and convex.

- b. (i) $m_2(\mathbf{p}) = \max_{1 \leq j \leq n} E_2(\mathbf{p}, c_j)$.
(ii) Let $A_2^{\max}(\mathbf{p}) = \{c_j \in A_2 : E_2(\mathbf{p}, c_j) = m_2(\mathbf{p})\}$. Then,

$$B_2(\mathbf{p}) = \{\mathbf{q} \in \mathcal{A}_2 : q_j = 0 \text{ if } c_j \notin A_2^{\max}(\mathbf{p})\}.$$

Therefore, $B_2(\mathbf{p})$ is a convex combination of the pure actions with the maximum payoff in response to $\mathbf{p}$; $B_2(\mathbf{p})$ is nonempty, closed, and convex.

Theorem (Proposition 119.1). Assume $m = \#(A_1) < \infty$ and $n = \#(A_2) < \infty$. Then, there exists a mixed strategy (or pure strategy) Nash equilibrium.

Proof. The set valued correspondence

$$(\mathbf{p}, \mathbf{q}) \mapsto B_1(\mathbf{q}) \times B_2(\mathbf{p}) \in \mathcal{P}_c(\Delta_m) \times \mathcal{P}_c(\Delta_n)$$

has an image in $\mathcal{P}_c(\Delta_m) \times \mathcal{P}_c(\Delta_n)$ that is closed and bounded, and so is what is called an upper semi-continuous correspondence. Also, each $B_1(\mathbf{q}) \times B_2(\mathbf{p})$ is convex. By the Kakutani fixed point theorem, there exists a point $(\mathbf{p}^*, \mathbf{q}^*)$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in B_1(\mathbf{q}^*) \times B_2(\mathbf{p}^*)$, i.e., a Nash equilibrium. $\square$

§§4.3, 4.10 Examples of Mixed Strategy Nash Equilibria for Bimatrix Games

Example (Both Mixed and Pure Equilibria). Consider the bimatrix game

$$\begin{pmatrix} (1, 3) & (1, 0) \\ (0, 1) & (3, 2) \end{pmatrix}.$$

Let (p_1, p_2) and (q_1, q_2) be the mixed strategies of the players. The expected payoffs of player one for rows one and two are

$$\begin{aligned} E_1(r_1, \mathbf{q}) &= q_1 + q_2 = 1 \\ E_1(r_2, \mathbf{q}) &= q_1(0) + q_2(3) = 3q_2. \end{aligned}$$

The payoffs are equal for $1 = 3q_2$, $q_2 = 1/3$ and $q_1 = 2/3$. For $q_1 < 2/3$ and $q_2 > 1/3$, $E_1(r_1, \mathbf{q}) < E_1(r_2, \mathbf{q})$; for $q_1 > 2/3$ and $q_2 < 1/3$, $E_1(r_1, \mathbf{q}) > E_1(r_2, \mathbf{q})$. Therefore, the best response by player 1 (giving the values of p_1) is the following :

$$B_1(\mathbf{q}) = \begin{cases} \{0\} & \text{for } q_1 < 2/3 \\ [0, 1] & \text{for } q_1 = 2/3 \\ \{1\} & \text{for } q_1 > 2/3. \end{cases}$$

The expected payoffs of player two for columns one and two are

$$\begin{aligned} E_2(\mathbf{p}, c_1) &= p_1(3) + p_2(1) = (p_1 + p_2) + 2p_1 = 1 + 2p_1, \\ E_2(\mathbf{p}, c_2) &= p_1(0) + p_2(2) = 2 - 2p_1. \end{aligned}$$

The payoffs are equal for $1 + 2p_1 = 2 - 2p_1$, $p_1 = 1/4$. For $p_1 < 1/4$, $E_2(\mathbf{p}, c_1) < E_2(\mathbf{p}, c_2)$; for $p_1 > 1/4$, $E_2(\mathbf{p}, c_1) > E_2(\mathbf{p}, c_2)$. Therefore, the maximum of U_2 occurs at the following values of q_1 :

$$B_2(\mathbf{p}) = \begin{cases} \{0\} & \text{for } p_1 < 1/4 \\ [0, 1] & \text{for } p_1 = 1/4 \\ \{1\} & \text{for } p_1 > 1/4. \end{cases}$$

Plotting the best response functions in the (p_1, q_1) -space, we get that they intersection at $(p_1, q_1) = (0, 0)$, $(1/4, 2/3)$, and $(1, 1)$. Thus, these three pairs are the Nash equilibria (for mixed and pure strategies). $\blacksquare$

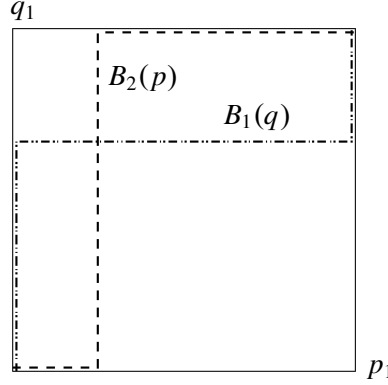


FIGURE 2. Best responses

Example (Finding all the Mixed and Pure Equilibria). Consider the bimatrix game

$$\begin{pmatrix} (1, 1) & (0, 0) & (0, 0) \\ (0, 0) & (0, 2) & (3, 0) \\ (0, 0) & (2, 0) & (0, 3) \end{pmatrix}.$$

Let (p_1, p_2, p_3) be the mixed strategy for the first player and (q_1, q_2, q_3) be the mixed strategy for the second player.

The expected payoff functions are

$$\begin{aligned} E_1(r_1, \mathbf{q}) &= q_1, & E_1(r_2, \mathbf{q}) &= 3q_3, & E_1(r_3, \mathbf{q}) &= 2q_2, \\ E_2(\mathbf{p}, c_1) &= p_1, & E_2(\mathbf{p}, c_2) &= 2p_2, & E_2(\mathbf{p}, c_3) &= 3p_3, \end{aligned}$$

and

$$U_1 = p_1 q_1 + 3p_2 q_3 + 2p_3 q_2 \quad \text{and} \quad U_2 = p_1 q_1 + 2p_2 q_2 + 3p_3 q_3.$$

a. Interior mixed strategy Nash equilibrium.

At an interior mixed strategy Nash equilibrium, all the expected payoffs must be equal:

$$\begin{aligned} q_1 &= 3q_3 = 2q_2 \quad \text{and} \\ p_1 &= 2p_2 = 3p_3. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &= q_1 + q_2 + q_3 = q_1 \left(1 + \frac{1}{2} + \frac{1}{3}\right) = q_1 \left(\frac{6+3+2}{6}\right) \\ q_1 &= \frac{6}{11}, \quad q_2 = \frac{3}{11}, \quad q_3 = \frac{2}{11}, \\ 1 &= p_1 + p_2 + p_3 = p_1 \left(1 + \frac{1}{2} + \frac{1}{3}\right) = p_1 \left(\frac{6+3+2}{6}\right) \\ p_1 &= \frac{6}{11}, \quad p_2 = \frac{3}{11}, \quad p_3 = \frac{2}{11}. \end{aligned}$$

The expected payoffs for these choices are

$$\begin{aligned} U_1 &= \frac{6}{11} \cdot \frac{6}{11} + 3 \cdot \frac{3}{11} \cdot \frac{2}{11} + 2 \cdot \frac{2}{11} \cdot \frac{3}{11} = \frac{66}{11 \cdot 11} = \frac{6}{11}, \quad \text{and} \\ U_2 &= \frac{6}{11} \cdot \frac{6}{11} + 2 \cdot \frac{3}{11} \cdot \frac{3}{11} + 3 \cdot \frac{2}{11} \cdot \frac{2}{11} = \frac{66}{11 \cdot 11} = \frac{6}{11}. \end{aligned}$$

b. Mixed strategy Nash equilibrium with $p_1 = 0$ or $q_1 = 0$. If $p_1 = 0$, then the expected payoff for the columns are, $E_2((0, p_2, p_3), c_1) = 0$, $E_2((0, p_2, p_3), c_2) = 2p_2$, $E_2((0, p_2, p_3), c_3) = 3p_3$. The maximum cannot involve c_1 , so $q_1 = 0$. (If $p_1 = 0$, then the second and the third columns weakly dominate the first

column.) A similar argument shows that if $q_1 = 0$ then $p_1 = 0$. (If $q_1 = 0$, then the second and the third rows weakly dominate the first row.) Therefore, if either is zero, then both are zero.

For both $p_1 = 0$ and $q_1 = 0$, and the others are nonzero, then

$$\begin{aligned} 3q_3 &= 2q_2 > q_1 = 0 & \text{and} \\ 2p_2 &= 3p_3 > p_1 = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &= q_2 + q_3 = q_2 \left(1 + \frac{2}{3}\right) = q_2 \left(\frac{5}{3}\right), \\ q_2 &= \frac{3}{5} \quad \text{and} \quad q_3 = \frac{2}{5}, \\ 1 &= p_2 + p_3 = p_2 \left(1 + \frac{2}{3}\right) = p_2 \left(\frac{5}{3}\right), \\ p_2 &= \frac{3}{5} \quad \text{and} \quad p_3 = \frac{2}{5}. \end{aligned}$$

Therefore, $(p_1, p_2, p_3) = (0, \frac{3}{5}, \frac{2}{5})$ and $(q_1, q_2, q_3) = (0, \frac{3}{5}, \frac{2}{5})$ is a Nash equilibrium. This mixed strategy has a payoff of $U_1 = \frac{6}{5}$ and $U_2 = \frac{6}{5}$.

c. Mixed strategy Nash equilibrium with $p_2 = 0$, $p_3 = 0$, $q_2 = 0$, or $q_3 = 0$. The following argument using weak dominance to show if one is zero then all are zero.

If $p_2 = 0$, then $E_2((p_1, 0, p_3), c_2) = 0$, the maximum cannot involve c_2 , and $q_2 = 0$.

If $q_2 = 0$, then $E_1(r_3, (q_1, 0, q_3)) = 0$, the maximum cannot involve r_3 , and $p_3 = 0$.

If $p_3 = 0$, then $E_2((p_1, p_2, 0), c_3) = 0$, the maximum cannot involve c_3 , and $q_3 = 0$.

If $q_3 = 0$, then $E_1(r_2, (q_1, q_2, 0)) = 0$, the maximum cannot involve r_2 , and $p_2 = 0$.

Therefore, if any of these quantities are zero, then $(p_1, p_2, p_3) = (1, 0, 0)$ and $(q_1, q_2, q_3) = (1, 0, 0)$.

Therefore, this is a pure strategy Nash equilibrium. It has a payoff of $U_1 = 1$ and $U_2 = 1$.

Summarizing: There are three Nash equilibria.

- a. In the interior, there is a mixed strategy Nash equilibrium

$$(p_1, p_2, p_3) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11}) \quad \text{and} \quad (q_1, q_2, q_3) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11}),$$

with payoffs $U_1 = \frac{6}{11}$ and $U_2 = \frac{6}{11}$.

- b. On a “face” with one variable equal to zero, there is the mixed strategy Nash equilibrium

$$(p_1, p_2, p_3) = (0, \frac{3}{5}, \frac{2}{5}) \quad \text{and} \quad (q_1, q_2, q_3) = (0, \frac{3}{5}, \frac{2}{5}),$$

with payoffs $U_1 = \frac{6}{5}$ and $U_2 = \frac{6}{5}$.

- c. Finally, there is a pure strategy Nash equilibrium

$$(p_1, p_2, p_3) = (1, 0, 0) \quad \text{and} \quad (q_1, q_2, q_3) = (1, 0, 0),$$

with payoffs $U_1 = 1$ and $U_2 = 1$.

Security level analysis of the three Nash equilibria: We only analyze player 1, but the analysis for player 2 is similar.

- (a) $(p_1, p_2, p_3) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11})$: This implies that

$$U_1 = \frac{6}{11} + 3 \cdot \frac{3}{11}(1 - q_1 - q_2) + 2 \cdot \frac{2}{11}q_2 = \frac{9}{11} - \frac{3}{11}q_1 - \frac{5}{11}q_2,$$

for $0 \leq q_1 + q_2 \leq 1$. This is minimized at $q_1 = 0$ and $q_2 = 1$, with a value of $U_1 = \frac{4}{11}$. Thus, with this choice, the first player is guaranteed to receiving at least $\frac{4}{11}$.

- (b) $(p_1, p_2, p_3) = (0, \frac{3}{5}, \frac{2}{5})$: This implies that

$$U_1 = 5 \cdot \frac{3}{5}(1 - q_1 - q_2) + 2 \cdot \frac{2}{5}q_2 = \frac{9}{5} - \frac{9}{5}q_1 - q_2,$$

which is minimized at $q_1 = 1$ and $q_2 = 0$ with a value of $U_1 = 0$.

- (c) $(p_1, p_2, p_3) = (1, 0, 0)$: This implies that $U_1 = q_1$, which has a minimum of $U_1 = 0$ at $q_1 = 0$.

Therefore, although the Nash equilibrium $(p_1, p_2, p_3) = (6/11, 3/11, 2/11)$ has a lower expected payoff than the other, it has the largest guaranteed payoff. Thus, one justification for playing the mixed strategy would be that it has a larger guaranteed payoff.

Reasons for playing a mixed strategy

- i. In some bimatrix games there is no pure strategy Nash equilibrium but only a mixed strategy Nash equilibrium.
- ii. If there are both a mixed strategy Nash equilibrium and a pure strategy Nash equilibrium, the guaranteed payoff from the mixed strategy Nash equilibrium can be higher than the pure strategy Nash equilibrium.

■

Example (A Plane of Mixed Strategy Equilibria). Consider the bimatrix game

$$\begin{pmatrix} (1, 0) & (0, 1) & (1, 0) & (0, 1) \\ (0, 1) & (1, 0) & (0, 1) & (1, 0) \\ (0, 1) & (1, 0) & (1, 0) & (0, 1) \\ (1, 0) & (0, 1) & (0, 1) & (1, 0) \end{pmatrix}.$$

The best response in each column has a 1 is first payoff; the best response in each row has a 1 is second payoff; there are no pure strategy Nash equilibria.

Let (p_1, p_2, p_3, p_4) be the mixed strategy for the first player and (q_1, q_2, q_3, q_4) be the mixed strategy for the second player. The expected payoffs for player one and player two for the various rows and columns are

$$\begin{aligned} E_1(r_1, \mathbf{q}) &= q_1 + q_3, & E_2(\mathbf{p}, c_1) &= p_2 + p_3 \\ E_1(r_2, \mathbf{q}) &= q_2 + q_4, & E_2(\mathbf{p}, c_2) &= p_1 + p_4 \\ E_1(r_3, \mathbf{q}) &= q_2 + q_3, & E_2(\mathbf{p}, c_3) &= p_2 + p_4 \\ E_1(r_4, \mathbf{q}) &= q_1 + q_4, & E_2(\mathbf{p}, c_4) &= p_1 + p_3. \end{aligned}$$

For an interior mixed strategy Nash equilibria, all the expected payoffs for player one must be equal:

$$q_1 + q_3 = q_2 + q_4 = q_2 + q_3 = q_1 + q_4.$$

From the first and third expression, we get that $q_1 = q_2$; from the second and third, $q_3 = q_4$. Using these in the third and fourth expression, $q_1 + q_4 = q_2 + q_3 = 1 - (q_1 + q_4)$, $2(q_1 + q_4) = 1$, and $q_1 + q_4 = 1/2$. Since $q_1 + q_4 = 1/2$ and both are positive, each can be at most $1/2$. If we set $q_1 = 0.5t$ for $0 \leq t \leq 1$, then $q_4 = 0.5(1 - t)$, and $(q_1, q_2, q_3, q_4) = t(0.5, 0.5, 0, 0) + (1 - t)(0, 0, 0.5, 0.5)$.

Similarly, setting the expected payoffs for player two equal:

$$p_2 + p_3 = p_1 + p_4 = p_2 + p_4 = p_1 + p_3.$$

If we set $p_1 = 0.5s$ for $0 \leq s \leq 1$, then $(p_1, p_2, p_3, p_4) = s(0.5, 0.5, 0, 0) + (1 - s)(0, 0, 0.5, 0.5)$.

Thus, for any choice of $0 \leq s \leq 1$ and $0 \leq t \leq 1$, the mixed strategy

$$\begin{aligned} (p_1, p_2, p_3, p_4) &= s(0.5, 0.5, 0, 0) + (1 - s)(0, 0, 0.5, 0.5) \quad \text{and} \\ (q_1, q_2, q_3, q_4) &= t(0.5, 0.5, 0, 0) + (1 - t)(0, 0, 0.5, 0.5) \end{aligned}$$

is a mixed strategy Nash equilibrium. This example has a plane, so a continuum, of Nash equilibria. ■

Continuous probability

In the next topic on private value auctions, we need to use a continuous probability distribution that is given by a density function. A *density function* is a function $f(t) \geq 0$ for all t with

$$\int_{-\infty}^{\infty} f(t) dt = 1.$$

Such a density function induces a *probability distribution* by

$$\Pr(t \leq v) = \int_{-\infty}^v f(t) dt.$$

Then,

$$\begin{aligned} \Pr(t = v) &= \int_v^v f(t) dt = 0 \quad \text{and} \\ \Pr(t < v) &= \int_{-\infty}^v f(t) dt = \Pr(t \leq v). \end{aligned}$$

The *normal distribution* is given by

$$\Pr(t \leq v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^v e^{-t^2/2} dt.$$

A *uniform density function* and *uniform distribution* on the closed interval $[v_L, v_H]$ are given by

$$\begin{aligned} f(t) &= \begin{cases} 0 & \text{if } t \notin [v_L, v_H] \\ \frac{1}{v_H - v_L} & \text{if } t \in [v_L, v_H] \end{cases} \quad \text{and} \\ \Pr(t \leq v) &= \int_{-\infty}^v f(t) dt = \begin{cases} 0 & \text{if } t \leq v_L \\ \frac{v - v_L}{v_H - v_L} & \text{if } v_L \leq v \leq v_H \\ 1 & \text{if } v_H \leq v. \end{cases} \end{aligned}$$

First Price Individual Private Value Auction

There are n players with valuations $(v_1, \dots, v_n)$ with $v_i \geq 0$, where each player knows their own valuation but not the exact valuations of the other players. It is assumed that there is a fixed interval $[v_L, v_H]$, with a low and a high valuation, such that each player believes the valuation of each of the other players is given by a uniform distribution over $[v_L, v_H]$. We consider a first price auction where the payoff for a unique highest bidder is $v_i - b_i$. If the bidders tie, they split the payoff as indicated below. We will consider a 2-person auction and then an auction with three or more players.

Two Person Auction.

Given i and $j \neq i$, let

$$u_i(b_1, b_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(v_i - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j. \end{cases}$$

We claim that the strategy profile with

$$b_i^*(v_i) = \frac{1}{2}(v_i + v_L) \leq \frac{1}{2}(v_L + v_H) \quad \text{for } i = 1, 2$$

is a Nash equilibrium with the maximum expected payoff over the distribution of the other players v_j .

Fix i and $j \neq i$.

$$\begin{aligned} (Eu_i)(b_i, b_j^*) &= \Pr(b_i > b_j^*)u_i(b_i, b_j^*) + \Pr(b_i = b_j^*)u_i(b_i, b_j^*) + \Pr(b_i < b_j^*)(0) \\ &= \Pr(b_i > b_j^*)(v_i - b_i) = (v_i - b_i) \Pr(\{v_j : b_i > b_j^*(v_j)\}). \end{aligned}$$

The bid $b_i > b_j^*(v_j) = \frac{1}{2}(v_j + v_L)$ when $2b_i - v_L > v_j$, so

$$\Pr(\{v_j : b_i > b_j^*(v_j)\}) = \begin{cases} 0 & \text{if } b_i \leq v_L \\ \frac{(2b_i - v_L) - v_L}{v_H - v_L} = \frac{2(b_i - v_L)}{v_H - v_L} & \text{if } v_L \leq b_i \leq \frac{v_H + v_L}{2} \\ 1 & \text{if } \frac{v_H + v_L}{2} \leq b_i, \end{cases}$$

$$(Eu_i)(b_i, b_j^*) = \begin{cases} 0 & \text{if } b_i \leq v_L \\ \frac{2(b_i - v_L)(v_i - b_i)}{v_H - v_L} & \text{if } v_L \leq b_i \leq \frac{v_H + v_L}{2} \\ v_i - b_i & \text{if } \frac{v_H + v_L}{2} \leq b_i. \end{cases}$$

The critical point satisfies

$$\begin{aligned} \frac{\partial}{\partial b_i}(Eu_i)(b_i, b_j^*) &= \frac{2}{v_H - v_L} [v_i - b_i - b_i + v_L] = 0 \\ 2b_i &= v_i + v_L \\ b_i &= \frac{1}{2}(v_i + v_L) \leq \frac{1}{2}(v_H + v_L). \end{aligned}$$

This last equation is the same formula as was given for $b_i^*(v_i)$. The second derivative

$$\frac{\partial^2}{\partial b_i^2}(Eu_i)(b_i, b_j^*) = \frac{-4}{v_H - v_L} < 0,$$

so the critical point is a maximum. This argument works for $i = 1, 2$.

Since $b_i^* - v_L = \frac{1}{2}(v_i - v_L)$ and $v_i - b_i^* = \frac{1}{2}(v_i - v_L)$, the expected payoff is

$$(Eu_i)(b_i^*, b_j^*) = \frac{2 \cdot \frac{1}{2}(v_i - v_L) \frac{1}{2}(v_i - v_L)}{v_H - v_L} = \frac{(v_i - v_L)^2}{2(v_H - v_L)},$$

which depends of u_H .

Three or More Person Auction.

For a bid profile $\mathbf{b} = (b_1, \dots, b_n)$, we let

$$M(\mathbf{b}_{-i}) = \max\{b_j : j \neq i\}.$$

The payoff is

$$u_i(b_1, \dots, b_n) = \begin{cases} v_i - b_i & \text{if } b_i > M(\mathbf{b}_{-i}) \\ \frac{1}{r}(v_i - b_i) & \text{if } b_i = M(\mathbf{b}_{-i}) \text{ with } r \text{ finalists} \\ 0 & \text{if } b_i < M(\mathbf{b}_{-i}). \end{cases}$$

We claim that the strategy profile with each player using following strategy is a Nash equilibrium:

$$b_i^*(v_i) = \frac{1}{n}[(n-1)v_i + v_L] \quad \text{for } 1 \leq i \leq n.$$

We need to calculate $\Pr(b_i > M(\mathbf{b}_{-i}^*)) = \Pr(b_i > b_j^* \text{ for all } j \neq i)$. Fix $1 \leq i \leq n$, and let $j \neq i$ with $1 \leq j \leq n$. The bid $b_i > b_j^* = \frac{1}{n}[(n-1)v_j + v_L]$ iff $nb_i - v_L > (n-1)v_j$ iff $\frac{1}{n-1}[nb_i - v_L] > v_j$. Note that $\frac{1}{n-1}[nb_i - v_L] - v_L = \frac{n}{n-1}[b_i - v_L]$. So

$$\Pr(b_i > b_j^*) = \Pr(\{v_j : v_j < \frac{1}{n-1}[nb_i - v_L]\}) = \frac{n}{n-1} \left[\frac{b_i - v_L}{v_H - v_L} \right].$$

The probabilities on the values of the v_j are independent events, so

$$\Pr(b_i > M(\mathbf{b}_{-i}^*)) = \Pr(b_i > b_j^* \text{ for all } j \neq i) = \left(\frac{n}{n-1}\right)^{n-1} \left[\frac{b_i - v_L}{v_H - v_L}\right]^{n-1}.$$

Therefore,

$$\begin{aligned} (Eu_i)(b_i, b_{-i}^*) &= (v_i - b_i) \Pr(\{v_{-i} : b_i > M(\mathbf{b}_{-i}^*)\}) \\ &= \left(\frac{n}{(n-1)(v_H - v_L)}\right)^{n-1} (b_i - v_L)^{n-1} (v_i - b_i). \end{aligned}$$

The critical point satisfies

$$\begin{aligned} 0 &= \frac{\partial}{\partial b_i} (Eu_i)(b_i, b_{-i}^*) = \left(\frac{n}{(n-1)(v_H - v_L)}\right)^{n-1} [(n-1)(b_i - v_L)^{n-2} (v_i - b_i) - (b_i - v_L)^{n-1}], \\ (n-1)(v_i - b_i) &= (b_i - v_L), \\ (n-1)v_i + v_L &= nb_i, \\ b_i &= \frac{1}{n}[(n-1)v_i + v_L]. \end{aligned}$$

This is the same formula as was given for $b_i^*(v_i)$. The second derivative

$$\begin{aligned} \left(\frac{(n-1)(v_H - v_L)}{n}\right)^{n-1} \frac{\partial^2}{\partial b_i^2} (Eu_i)(b_i^*, b_{-i}^*) &= \\ &= (n-1)(n-2)(b_i^* - v_L)^{n-3} (v_i - b_i^*) - 2(n-1)(b_i^* - v_L)^{n-2} \\ &= (n-1)(b_i^* - v_L)^{n-3} [(n-2)(v_i - b_i^*) - 2(b_i^* - v_L)] \\ &= (n-1)(b_i^* - v_L)^{n-3} \left[\frac{(n-2)}{n}(v_i - v_L) - \frac{2(n-1)}{n}(v_i - v_L)\right] \\ &= -(n-1)(b_i^* - v_L)^{n-3} \frac{n+1}{n} (v_i - v_L) < 0, \end{aligned}$$

so the critical point is a maximum.

§§4.11 All-Pay Auction: Continuum of Actions

We consider the case with 2 bidders on a good of value K . The bids are $0 \leq a_i \leq K$. Each person pays what they bid even when they lose. The payoff is

$$u_i(a_1, a_2) = \begin{cases} -a_i & \text{if } a_i < a_j \\ \frac{1}{2}K - a_i & \text{if } a_i = a_j \\ K - a_i & \text{if } a_i > a_j. \end{cases}$$

There is no pure strategy Nash equilibrium as is seen by the following arguments.

- (i) $a_i > a_j$: P_i can increase payoff by decreasing a_i slightly, so this is not a NE.
- (ii) $a_i = a_j < K$: P_i can increase payoff by increasing a_i slightly, so this is not a NE.
- (iii) $a_i = a_j \geq K$: P_i can increase payoff by changing to $a_i = 0$, so this is not a NE.

A mixed strategy is a probability distribution on $[0, K]$. We look for a mixed Nash equilibrium among the uniform probability distributions on intervals $[x_i, y_i] \subset [0, K]$. Such a uniform distribution is given by

$$F_i(v) = \Pr(a_i < v) = \frac{v - x_i}{y_i - x_i} \quad \text{for } x_i \leq v \leq y_i.$$

By Proposition 142.2 in the book, a pair (F_1^*, F_2^*) is a mixed Nash equilibrium provided

- (i) $F_i^*(S) = 0$ if $(Eu_i)(a_i, F_j^*) < (Eu_i)(F_i^*, F_j^*)$ for all $a_i \in S$, and
- (ii) $(Eu_i)(a_i, F_j^*) \leq (Eu_i)(F_i^*, F_j^*)$ for all a_i .

The expected payoffs are as follows:

- (a) If $a_i < x_j$, then $a_i \leq a_j^*$ with probability one, so the expected payoff is $-a_i < 0$.
- (b) If $a_i > y_j$, then $a_i > a_j^*$ with probability one, so the expected payoff is $K - a_i$.
- (c) Finally, if $x_j \leq a_i \leq y_j$, then (i) $a_j < a_i$ with probability $F_j^*(a_i)$ and payoff $K - a_i$ for P_i , and (ii) $a_j > a_i$ with probability $1 - F_j^*(a_i)$ and payoff $-a_i$ for P_i . Therefore, if $x_j \leq a_i \leq y_j$, then the expected payoff is

$$(K - a_i) F_j^*(a_i) + (-a_i) (1 - F_j^*(a_i)) = K F_j^*(a_i) - a_i.$$

Because of the criterion in Proposition 142.2, the payoff must be constant on the interval $x_i \leq a_i \leq y_i$, and less off this interval.

Since the payoff $-a_i$ is less than the other payoffs, it cannot be included: If $a_i < x_j$, then $a_i < x_i$, i.e., $x_j \leq x_i$. Reversing the roles, $x_i \leq x_j$ so $x_i = x_j$.

The quantity $K - a_i$ is not a constant, so it cannot be included: If $a_i > y_j$, then $a_i > y_i$ and $y_j \geq y_i$. Reversing the roles gives the other inequality, so $y_i = y_j$. Thus, $x_1 = x_2 < y_1 = y_2$.

Because the payoff must be constant on the interval $x_1 = x_2 \leq a_i \leq y_1 = y_2$, there is a constant c_i such that

$$\begin{aligned} c_i &= K F_j^*(a_i) - a_i \quad \text{for } x_i \leq a_i \leq y_i, \\ &= K \left[\frac{a_i - x_j}{y_j - x_j} \right] - a_i, \\ c_i(y_j - x_j) &= K(a_i - x_j) - a_i(y_j - x_j), \\ c_i(y_j - x_j) + Kx_j &= a_i [K - (y_j - x_j)]. \end{aligned}$$

Since this last equality holds as a_i varies, $K = y_j - x_j$. Also $0 \leq x_j \leq y_j \leq K$, so $x_j = 0$ and $y_j = K$. Thus, F_1 and F_2^* must be uniform distributions on all of $[0, K]$.

§§4.7-4.8 Symmetric Equilibria

Definition (cf. 51.1 & 129.1). A two person strategic game is *symmetric* provided that (i) the set of actions are the same for both players, $\mathbf{A} = \mathbf{A}_1 = \mathbf{A}_2$, and (ii) if the payoff functions are u_1 and u_2 then $u_1(s, s') = u_2(s', s)$ for all actions $s, s' \in \mathbf{A}$. (The second conditions says that if either person plays action s and the other plays action s' then the payoff is the same.)

If there is a finite number of possible actions $\{s_1, \dots, s_k\}$, then the condition to be the same is as follows: The payoff is given by a single $k \times k$ matrix $\mathbf{A}$ where $a_{ij} = u_1(s_i, s_j)$ is the payoff for the first player playing s_i against the second player playing s_j . It is symmetric (in strategies) because the payoff for the second player playing s_j against s_i is a_{ji} , so the payoff of the second player has matrix $\mathbf{A}^T$. The matrix $\mathbf{A}$ is not necessarily symmetric. Osborne gives both $\mathbf{A}$ and $\mathbf{A}^T$, but we give only $\mathbf{A}$.

For mixed strategies $\mathbf{p}$ by P_1 and $\mathbf{q}$ by P_2 , the payoff for P_1 is $\mathbf{p} \cdot \mathbf{A} \mathbf{q} = \mathbf{p}^T \mathbf{A} \mathbf{q}$.

Definition (cf. 52.1 & 129.2). For a two-person symmetric strategic game with a finite number of possible actions, a mixed strategy $\hat{\mathbf{p}}$ is a *symmetric Nash equilibrium* provided that

$$\hat{\mathbf{p}} \cdot \mathbf{A} \hat{\mathbf{p}} \geq \mathbf{q} \cdot \mathbf{A} \hat{\mathbf{p}}$$

for all mixed strategies $\mathbf{q}$, i.e., the mixed strategy $\mathbf{q} = \hat{\mathbf{p}}$ has the largest payoff in response to a population in the state $\hat{\mathbf{p}}$, or $\hat{\mathbf{p}} \in B_1(\hat{\mathbf{p}})$. Note that this implies that

$$\hat{\mathbf{p}}^T \mathbf{A}^T \hat{\mathbf{p}} = (\hat{\mathbf{p}}^T \mathbf{A} \hat{\mathbf{p}})^T \geq (\mathbf{q}^T \mathbf{A} \hat{\mathbf{p}})^T = \hat{\mathbf{p}}^T \mathbf{A}^T \mathbf{q},$$

and $\hat{\mathbf{p}} \in B_2(\hat{\mathbf{p}})$. Thus, this is slightly different than the usual Nash equilibrium where $\hat{\mathbf{p}} \in B_1(\hat{\mathbf{q}})$ and $\hat{\mathbf{q}} \in B_2(\hat{\mathbf{p}})$.

Example. Consider the game with matrix $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$. For a mixed strategy $\mathbf{p} = (p, 1 - p)^T$, the expected payoffs of $\mathbf{e}^1$ and $\mathbf{e}^2$ are $E(\mathbf{e}^1, \mathbf{p}) = -1(p) + 2(1 - p) = 2 - 3p$ and $E(\mathbf{e}^2, \mathbf{p}) = 1 - p$. These are equal for $1 = 2p$, or $p = 1/2$. For $p < 1/2$, $\mathbf{e}^1$ is preferred; $\mathbf{e}^2$ is preferred for $p > 1/2$. The best response in terms of the value q of $(q, 1 - q)$ is

$$B(p, 1 - p) = \begin{cases} \{1\} & \text{if } p < 1/2 \\ [0, 1] & \text{if } p = 1/2 \\ \{0\} & \text{if } p > 1/2. \end{cases}$$

Therefore, the only Nash equilibrium is $\hat{p} = 1/2$ or the complete strategy of $\hat{\mathbf{p}} = (1/2, 1/2)^T$. ■

Example (Section 4.8). We consider the example of reporting a crime. The two actions are C for “call” and D for “don’t call”. We assume there are n potential callers or players. We assume there are parameters $0 < c < v$ such that the payoff functions are given as follows:

$$\begin{aligned} u_1(D, \dots, D) &= 0 \\ u_1(C, *, \dots, *) &= v - c \\ u_1(D, *, C, *, \dots, *) &= v. \end{aligned}$$

In the three cases, (i) no one calls, (ii) P_1 calls, and (iii) someone else calls but not P_1 . The payoffs for the other players are similar. Since the role of each player is interchangeable, this game is a symmetric n -person game.

We consider only symmetric strategies where each player is the same. Let $p = \alpha(C)$ and $1 - p = \alpha(D)$. Then

$$\begin{aligned} E_1(C, p) &= v - c \\ E_1(D, p) &= 0 \cdot \Pr(\text{no one calls}) + v \Pr(\text{at least one person calls}). \end{aligned}$$

The probability that one person does not call is $1 - p$, so the probability that no one calls (among the $n - 1$ other people) is $(1 - p)^{n-1}$. Therefore, the probability that at least one person calls is $1 - (1 - p)^{n-1}$. Thus,

$$E_1(D, p) = v [1 - (1 - p)^{n-1}].$$

To find an interior equilibrium with $0 < p < 1$, the two expected payoffs must be equal:

$$\begin{aligned} v - c &= v - v(1 - p)^{n-1} \\ \frac{c}{v} &= (1 - p)^{n-1} \\ 1 - p &= \left(\frac{c}{v}\right)^{\frac{1}{n-1}} \\ p_n^* &= 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}. \end{aligned}$$

How does the probability change with n ? As n increases, $1/(n - 1)$ decreases to 0, $(c/v)^{\frac{1}{n-1}}$ increases up to 1, so $p_n^* = 1 - (c/v)^{\frac{1}{n-1}}$ decreases to 0. Thus, each person is less likely to call. Moreover, the probability that no one calls is

$$(1 - p_n^*)^n = \left(\frac{c}{v}\right)^{\frac{n}{n-1}} = \left(\frac{c}{v}\right) \cdot \left(\frac{c}{v}\right)^{\frac{1}{n-1}}.$$

Since $n/(n - 1)$ decreases with limit 1, the probability that no one calls increases to c/v , and it is less likely that anyone calls. Note that this is always a positive probability that someone calls. ■