Sequential Rationality

Nash Equilibria for Behavior Strategies

Example 1: Introduce behavior strategies
Consider the game tree given in Figure 1. Player $P_2$ cannot choose between $c$ and $d$ on the information set $R(P_1)$.

\[ E_1(p, q) = pq(6) + p(1 - q)(5) + (1 - p)q(5) + (1 - p)(1 - q)(7) \]
\[ = pq(6 - 5 - 5 + 7) + p(5 - 7) + q(5 - 7) + 7 \]
\[ = 3pq - 2p - 2q + 7. \]

In the same way, the expected payoff of player $P_2$ at the root $R$ is given as follows:

\[ E_2(p, q) = pq(1) + p(1 - q)(4) + (1 - p)q(3) + (1 - p)(1 - q)(2) \]
\[ = pq(1 - 4 - 3 + 2) + p(4 - 2) + q(3 - 2) + 2 \]
\[ = 4pq - 2p + q + 2. \]

Player $P_1$ can change the behavior strategy $p$, and

\[ \frac{\partial E_1(p, q)}{\partial p} = 3q - 2, \]
which equals zero when \( q = \frac{2}{3} \). If \( P_1 \) maximizes \( E_1(p, q) \), then

\[
p = \begin{cases} 
0 & \text{if } q < \frac{2}{3} \\
\text{arbitrary} & \text{if } q = \frac{2}{3} \\
1 & \text{if } q > \frac{2}{3}.
\end{cases}
\]

Player \( P_2 \) can change the behavior strategy \( q \), and

\[
\frac{\partial E_2(p, q)}{\partial q} = -4p + 1,
\]

which equals zero when \( p = \frac{1}{4} \). If \( P_2 \) maximizes \( E_2(p, q) \), then

\[
q = \begin{cases} 
1 & \text{if } p < \frac{1}{4} \\
\text{arbitrary} & \text{if } p = \frac{1}{4} \\
0 & \text{if } p > \frac{1}{4}.
\end{cases}
\]

The only simultaneous solutions of the two criteria occurs for \( p^* = \frac{1}{4} \) and \( q^* = \frac{2}{3} \). See Figure 2 This gives a Nash equilibrium for the whole game in terms of a behavior strategy.

![Figure 2. Possible values of p and q for Example 1.](image)

**Definition** (page 134)

A behavior strategy is a probabilistic choice of edges coming out of a node. If a node \( N_k \) is owned by player \( P_i \) with three edges \( e_1, e_2, \) and \( e_3 \) coming out, then \( P_i \) chooses probabilities \( q_1, q_2, q_3 \geq 0 \) such that \( q_1 + q_2 + q_3 = 1 \). In general, if there are \( m \)-edges \( e_1, \ldots, e_m \) coming out of \( N_k \), then \( P_i \) chooses probabilities \( q_1, \ldots, q_m \geq 0 \) such that \( q_1 + \cdots + q_m = 1 \). On an information set, the player needs to make consistent choices of probabilities at the nodes of the information set. Let \( \sigma_i \) be a choices of probabilities at all the nodes and informations sets owned by player \( P_i \); \( \sigma_i \) is called a behavior strategy for \( P_i \). Let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be the behavior strategies for all the players; \( \sigma \) is called a behavior strategy profile. It is a probabilistic choice of edges coming out of every node owned by a player. (The book uses \( \pi \) or \( \pi_i \) instead of \( \sigma \) or \( \sigma_i \), but we have used \( \pi \) for the payoff so do not use it here.)

A behavior strategy profile \( \sigma \) is said to be completely mixed if every choice at every node is take with a positive probability. This is similar to the notion of an interior mixed strategy.

**Sequential Equilibria and Beliefs**

For the equilibrium in behavior strategy profiles, we want it to be “rational” not only on the whole game but also on parts of the game tree. We can require it to be subgame perfect, but in the game considered above there are no subgames. Therefore, instead, we want to require that it be rational on all the information
sets. Given a behavior strategy profile, we can calculate the expected value at each node, but we don’t know how to weight the nodes without inferring values from the fact that the player knows which information set is being considered. These inferred weights, or probability on the nodes of the information set, is what is called a “belief”. We introduce the ideas through an example.

**Example 5.2: Strategy profile, Beliefs, and payoffs**

Change the payoffs at the end of path $bTL'$ to be $(4, 0)$ rather than $(3, 0)$ and the payoff at the end of the path $bBL'$ to be $(2, 5)$ rather than $(3, 5)$. Also we change the label on node $E$ to be node $Z$, since we use $E$ for the expected value. See Figure 3.

The strategies are labeled as follows.

1. Let $x = \sigma(a)$ be the weight of the behavior strategy on the edge $a$ and $1 - x = \sigma(b)$ be the weight on the edge $b$. These choices are made by $P_1$ at the node $R$.
2. Let $q = \sigma(T)$ be the weight of the behavior strategy on the edge $T$ and $1 - q = \sigma(B)$ be the weight of the behavior strategy on the edge $B$. These choices are made on the information set $I_1 = \{X, Y\}$ by $P_2$. The same weights must be used at both nodes $X$ and $Y$.
3. Let $p = \sigma(L)$ be the weight of the behavior strategy on the edge $L$ and $1 - p = \sigma(R)$ be the weight of the behavior strategy on the edge $R$. These choices are made by $P_1$ on the information set $I_2 = \{Z, F\}$.
4. Let $r = \sigma(L')$ be the weight of the behavior strategy on the edge $L'$ and $1 - r = \sigma(R')$ be the weight of the behavior strategy on the edge $R'$. These choices are made by $P_1$ on the information set $I_3 = \{G, H\}$.

To calculate the payoffs at each node, we use the behavior behavior strategy to determine the probabilities of taking different paths to terminal nodes and add them up. At node $X$, the probabilities of reaching the various terminal nodes is $\sigma(X, D_1) = \sigma(XZ)\sigma(ZD_1) = qp$, $\sigma(X, D_2) = \sigma(XZ)\sigma(ZD_2) = q(1 - p)$,
\( \sigma(X, D_3) = (1 - q)p \), and \( \sigma(X, D_4) = (1 - q)(1 - p) \). The payoff vectors are as follows:

\[
\begin{align*}
E(Z, \sigma) &= p(4, 2) + (1 - p)(0, 3) = (4p, 3 - p) \\
E(F, \sigma) &= p(1, 7) + (1 - p)(2, 6) = (2 - p, 6 + p) \\
E(G, \sigma) &= r(4, 0) + (1 - r)(2, 4) = (2 + 2r, 4 - 4r) \\
E(H, \sigma) &= r(2, 5) + (1 - r)(4, 3) = (4 - 2r, 3 + 2r) \\
E(X, \sigma) &= q(4p, 3 - p) + (1 - q)(2 - p, 6 + p) \\
&= (2 - 2q - p + 5pq, 6 - 3q + p - 2pq) \\
E(Y, \sigma) &= q(2 + 2r, 4 - 4r) + (1 - q)(4 - 2r, 3 + 2r) \\
&= (4 - 2r - 2q + 4qr, 3 + q + 2r - 6qr) \\
E(R, \sigma) &= x(2 - 2q - p + 3pq, 6 - 3q + p - 2pq) \\
&+ (1 - x)(4 - 2r - 2q + 4qr, 3 + q + 2r - 6qr).
\end{align*}
\]

For a node \( N \), we write \( \sigma(N) \) to mean \( \sigma(RN) \). Thus, \( \sigma(Z) = xq \) and \( \sigma(F) = x(1 - q) \). If \( x > 0 \) and \( P_1 \) knows he is in the information set \( I_2 = \{ Z, F \} \), then \( P_1 \) should rationally expect that the probability of being at node \( Z \) is

\[
\frac{\sigma(Z)}{\sigma(Z) + \sigma(F)} = \frac{xq}{xq + x(1 - q)} = q.
\]

We call this expectation by \( P_1 \) a belief, and denote it by \( \mu_1(Z) \). Similarly, \( \mu_1(F) = 1 - q \).

If \( x = \sigma(a) = 0 \), then the probability of reaching \( I_2 \) is zero, \( \sigma(Z) + \sigma(F) = xq + x(1 - q) = 0 \), and the strategy profile does not immediately induce a system of beliefs on \( I_2 \). However, if we modify the strategy profile slightly and let \( \sigma^n(a) = \frac{1}{n} \) and \( \sigma^n(b) = 1 - \frac{1}{n} \), then the induced belief is

\[
\mu^n(Z) = \frac{\frac{1}{n}\sigma(T)}{\frac{1}{n}\sigma(T) + \frac{1}{n}\sigma(B)} = \frac{\sigma(T)}{\sigma(T) + \sigma(B)} = \frac{q}{q + (1 - q)} = q,
\]

which is the same for all \( n \), so \( \mu_1(Z) = q \). Similarly, \( \mu_1(F) = 1 - q \). Therefore, in this example, the beliefs on \( X \) and \( F \) must be \( q \) and \( 1 - q \) respectively.

Similarly, a consistent system of beliefs on \( I_3 = \{ G, H \} \), is \( \mu_1(G) = q \) and \( \mu_1(H) = 1 - q \). Finally, for \( P_2 \) on \( I_1 = \{ X, Y \} \), a consistent system of beliefs is \( \mu_2(X) = x \), and \( \mu_2(Y) = 1 - x \). We fixed this consistent system of beliefs \( \mu^a \).

In the homework, there is an example where the belief system on an information set is arbitrary.

Thus, in any case, on the information set \( I_2 \), the payoff for \( P_1 \) is

\[
E_1(I_2, \sigma, \mu^a) = q E_1(Z, \sigma) + (1 - q) E_1(F, \sigma) = E_1(X, \sigma) = 2 - 2q - p + 5pq.
\]

Similarly, on the information set \( I_3 \), the payoff for \( P_1 \) is

\[
E_1(I_3, \sigma, \mu^a) = E_1(Y, \sigma) = 4 - 2r - 2q + 4qr.
\]

Also, on the information set \( I_1 \), the payoff for \( P_2 \) is

\[
E_2(I_1, \sigma, \mu^a) = E_2(R, \sigma) = x(6 - 3q + p - 2pq) + (1 - x)(3 + q + 2r - 6qr).
\]

We give the definition, state the theorem, and calculate some simpler examples before we show how to pick the values to maximize the payoff on all the information sets.
We write \( \sigma(N_1, N_k) \) for the choice of the behavior strategy on the edge \((N_1, N_k)\). If \( N_1, \ldots, N_k \) is a path in the game tree, then we write

\[
\sigma(N_1, N_k) = \sigma(N_1, N_2) \sigma(N_2, N_3) \cdots \sigma(N_{k-1}, N_k).
\]

This quantity is the probability of taking this path from \( N_1 \) to \( N_k \) with the given behavior strategy given that the path starts at \( N_1 \). If \( R \) is the root, then we write \( \sigma(N_k) \) for \( \sigma(R, N_k) \).

We need to calculate the expected payoffs for different players given a behavior strategy \( \sigma \). Let \( X_1, \ldots, X_k \) be the terminal nodes. Let \( u_j(X_i) \) be the payoff for player \( P_j \) at \( X_i \). At any node \( N \), the expected payoff for \( P_j \) is given by

\[
E_j(N, \sigma) = \sum_{i=1}^{k} \sigma(N, X_i) u_j(X_i).
\]

It considers only payoffs at terminal nodes which can be reached by path from the node \( N \), and it takes the sum of the payoffs at these terminal nodes weighted with the probability of getting from the node \( N \) to the terminal node determined by the behavior strategy profile. We combine these to form the payoff vector at \( N \),

\[
E(N, \sigma) = (E_1(N, \sigma), \ldots, E_n(N, \sigma)).
\]

If we start at the root \( R \), then we say \( E_j(\sigma) \equiv E_j(R, \sigma) \) is the payoff for player \( P_j \) for the behavior strategy profile.

**Definition** (page 134)

A *system of beliefs* \( \mu_i \) for \( P_i \) is a function which assigns a probability distribution to the nodes in each information set of player \( P_i \):

1. If \( N_i \) are the nodes owned by \( P_i \), then \( \mu_i(N) \geq 0 \) for each node \( N \) in \( N_i \).
2. If \( I \) is an information set of \( P_i \), then \( \sum_{N \in I} \mu_i(N) = 1 \).

If \( N \) is in \( I \), then \( \mu_i(N) \) is the belief by \( P_i \) that we are at \( N \) given that \( P_i \) knows we are in \( I \). We write \( \mu = (\mu_1, \ldots, \mu_n) \) for the beliefs of all the players. For a node \( N \) owned by \( P_i \), we write \( \mu(N) \) to mean \( \mu_i(N) \), with the understanding that we use the beliefs of player \( P_i \) on a node owned by \( P_i \).

If \( I \) is an information set, with \( \sigma(I) = \sum_{N \in I} \sigma(N) > 0 \), then Bayes formula gives that

\[
\sigma(N_j|I) = \frac{\sigma(I|N_j) \sigma(N_j)}{\sigma(I)} = \frac{\sigma(N_j)}{\sigma(I)},
\]

for \( N_j \) in \( I \). The belief \( \mu \) is *consistent* with a strategy profile \( \sigma \) if

\[
\mu(N_j) = \frac{\sigma(N_j)}{\sum_{N \in I} \sigma(N)}.
\]

We write \( \mu^\sigma \) or \( \mu^\sigma \) for the consistent belief system for \( \sigma \).

When \( \sigma \) is not a completely mixed belief strategy, then it is possible for \( \sum_{N \in I} \sigma(N) = 0 \) for an information set \( I \). In this case, we can put a consistent behavior systems on \( I \) by taking any sequence of completely mixed behavior strategies \( \sigma^n \) which converge to \( \sigma \) and then let \( \mu^n \) be the limit of \( \mu^\sigma^n \). This gives a system of beliefs on branches that are not accessible by the behavior strategy profile. Different choices of the \( \sigma^n \) can lead to different limit belief systems \( \mu^\sigma \). In Example 5 below, we show that the system of beliefs can be arbitrary on an information set \( I \) for which \( \sigma(I) = 0 \). However, in Example 5.2 discussed above, the system of beliefs on \( I_2 \) and \( I_3 \) are determined to be \( \mu(Z) = \mu(G) = q \) and \( \mu(F) = \mu(H) = 1 - q \) even when \( x = 0 \).

In order to maximize the payoff on an information set, we need to calculate the expected payoffs for an information set \( I \) in terms of a behavior strategy \( \sigma \) and a belief system \( \mu \). Let \( X_1, \ldots, X_k \) be the terminal
Let $u_j(X_i)$ be the payoff for player $P_j$ at $X_i$. The expected payoff for $P_j$ on the information set $I$ is given by

$$E_j(I, \sigma, \mu) = \sum_{N \in I} \mu(N) \sum_{i=1}^{k} \sigma(N, X_i) u_j(X_i).$$

Thus, we need to weight the payoffs at all the nodes of the information set by means of the belief $\mu$.

**Definition** (page 144)

A sequential (Nash) equilibrium for an $n$-person sequential game with imperfect information is a pair $(\sigma^*, \mu^*)$ where $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n)$ is a behavior strategy profile and $\mu^* = (\mu^*_1, \ldots, \mu^*_n)$ is a system of beliefs consistent with $\sigma^*$ such that, on any information set $I$ (which could be a single node) owned by a player $P_j$, the expected payoff

$$E_j(I, \sigma^*, \mu^*) = \max_{\sigma_j} E_j(I, \sigma^*_1, \ldots, \sigma^*_j, \ldots, \sigma^*_n, \mu^*),$$

where the maximum is taken by changing their behavior strategy $\sigma_j$ of $P_j$ while keeping fixed the behavior strategies $\sigma_i$ for $i \neq j$ of the other players and the system of beliefs $\mu^*$.

**Theorem 5.9 (Kreps-Wilson)** (page 145)

Every sequential game with imperfect information and perfect recall has a sequential Nash equilibrium.

We have defined various types of Nash equilibrium.

**Strategic form games:**
1. Pure strategies.

**Sequential games:**
1. Basic definition of a Nash equilibrium for a sequential game.
2. Subgame perfect (Nash) equilibrium.
3. Nash equilibrium in behavior strategy profiles.
4. Sequential (Nash) equilibrium which is rational on all the information sets. For this equilibrium, we need a system of beliefs in addition to a behavior strategy profile.

**Calculating Sequential Equilibria**

**Example 3: Sequential equilibrium** (Based on an example of Rosenthal given in the book by Myerson)

Consider the game given in Figure 4. The idea behind the game is that there is a chance event in which there is a 5% chance that the second person will always be accommodating and a 95% chance that the second person can choose to be generous or selfish. The first person always the option of being either generous or selfish. Each player loses $1 each time she is generous, but gains $5 each time the other player is generous. If a player is selfish, neither player gains or loses, and the game is over. Also, if both players are generous twice then the game is over. Because the second player is always generous in the lower “branch”, we do not indicate the choices of player $P_2$. At each stage $P_2$ knows exactly where she is, but $P_1$ does not know which branch she is on. The letters designating the behavior strategies are indicated in the figure.

At node $V_0$, the payoff of $P_2$ for $y = 1$ is 8 while the payoff for $y = 0$ is 9, so $y = 0$ is the best choice.

Considering the information set $I_2 = \{V_4, V_5\}$, to determine the compatible belief $\mu = \mu(V_4)$ of being at node $V_4$, we use the Bayes formula,

$$\mu = \mu(V_4) = \frac{0.95pq}{0.95pq + 0.05p} = \frac{19q}{19q + 1}.$$

The compatible belief of being at node $V_5$ is then $1 - \mu$. The payoff for player $P_1$ on $I_2$ is

$$E_1(I_2, \sigma, \mu) = \mu[x(3) + (1 - \mu)(4)] + (1 - \mu) \left[x(8) + (1 - x)(4)\right]$$.
and

\[
\frac{\partial E_1(I_2, \sigma, \mu)}{\partial x} = \mu(3 - 4) + (1 - \mu)(8 - 4) = 4 - 5\mu.
\]

This is optimized for

\[
x \begin{cases} = 1 & \text{if } \mu < \frac{4}{5} \text{ or } q < \frac{4}{19} \\ \text{arbitrary} & \text{if } \mu = \frac{4}{5} \text{ or } q = \frac{4}{19} \\ = 0 & \text{if } \mu > \frac{4}{5} \text{ or } q > \frac{4}{19}.
\end{cases}
\]

Now consider the payoff for player \(P_2\) at \(V_3\).

\[
E_2(V_3) = q [x9 + (1 - x)4] + (1 - q)5
\]

\[
\frac{\partial E_2(V_3)}{\partial q} = 9x + 4 - 4x - 5 = 5x - 1.
\]

The choice of \(q\) which maximizes the payoff is

\[
q \begin{cases} = 0 & \text{if } x < \frac{1}{5} \\ \text{arbitrary} & \text{if } x = \frac{1}{5} \\ = 1 & \text{if } x > \frac{1}{5}.
\end{cases}
\]

Combining the choices at \(V_3\) with those at \(I_2\), we have the following cases.

\(x < \frac{1}{5}\): \(\Rightarrow q = 0 \Rightarrow x = 1\). This is a contradiction.

\(x = \frac{1}{5}\): \(\Rightarrow q = \frac{4}{19}\) and \(\mu = \frac{4}{5}\). This is a compatible choice.

\(x > \frac{1}{5}\): \(\Rightarrow q = 1 \Rightarrow x = 0\). This is a contradiction.

Thus, the only compatible choices are \(x = \frac{1}{5}, q = \frac{4}{19},\) and \(\mu = \frac{4}{5}\).

Next, we calculate the payoff vectors for this choice of a behavior strategy profile.

\[
E(V_5) = \frac{1}{5} (8, 8) + \frac{4}{5} (4, 4) = \left(\frac{24}{5}, \frac{24}{5}\right)
\]

\[
E(V_4) = \frac{1}{5} (3, 9) + \frac{4}{5} (4, 4) = \left(\frac{3 + 16}{5}, \frac{9 + 16}{5}\right) = \left(\frac{19}{5}, \frac{5}{5}\right)
\]

\[
E(V_3) = \frac{4}{19} \left(\frac{19}{5}, 5\right) + \frac{15}{19} (-1, 5) = \left(\frac{4 \cdot 19 - 15 \cdot 5}{5 \cdot 19}, \frac{5}{5}\right) = \left(\frac{1}{5}, \frac{1}{5}\right).
\]
Turning to the expectation of $P_1$ on $I_1$, 
\[ E_1(I_1) = 0.95p \left( \frac{1}{5}, \frac{24}{19} \right) + 0.05p \left( \frac{24}{5} \right) \]
\[ = \frac{p}{20} \left( \frac{1}{5} + \frac{24}{5} \right) = \frac{p}{4}, \]
which is maximized for $p = 1$. Thus, we have shown that the only sequential equilibrium is 
\[ p = \sigma_1(g_1) = 1, \quad x = \sigma_1(g_3) = \frac{1}{5}, \quad q = \sigma_2(g_2) = \frac{4}{19}, \quad y = \sigma_2(g_4) = 0, \quad \text{and} \quad \mu(V_4) = \frac{4}{5}. \]
The payoff for $P_1$ for the sequential equilibrium is $1/4$. For $P_2$, it is 
\[ E_2(R) = 0.95(5) + 0.05 \left( \frac{24}{5} \right) = 4.75 + 0.24 = 4.99. \]

In this game, $P_1$ is generous the first time to take advantage of the fact that $P_2$ may be a generous type. ■

**Example 5**: (cf. Myerson exercise page 210)
Consider the game given in Figure 5. Nature owns the root and has a $2/3$ probability of selecting the edge to node $A$ and $1/3$ probability of selecting the edge to node $B$.

\[ R \]
\[ 2/3 \]
\[ (P_1) \]
\[ 1 - w \]
\[ (5, 6) \]
\[ 1/3 \]
\[ (P_1) \]
\[ 1 - y \]
\[ (2, 6) \]
\[ B \]
\[ A \]
\[ w \]
\[ x \]
\[ C \]
\[ (9, 5) \]
\[ 1 - x \]
\[ x \]
\[ (9, 0) \]
\[ D \]
\[ (0, 3) \]
\[ (0, 3) \]

**Figure 5. Game tree for Example 5**

If $(w, y) \neq (0, 0)$, then the system of beliefs of $P_2$ on the information set $I$ is 
\[ \mu(C) = \frac{2w}{3w + \frac{2}{3}y} \]
\[ = \frac{2w}{2w + y} \]
\[ \mu(D) = \frac{y}{2w + y}. \]

We show that if $(w, y) = (0, 0)$, then $\mu(C)$ and $\mu(D) = 1 - \mu(C)$ are arbitrary. For any $0 \leq p \leq 1$, consider $w_n = \frac{3p}{2n}$ and $y_n = \frac{3(1-p)}{n}$. Then, $\sigma_n(I) = \frac{p}{n} + (1 - p)/n = 1/n > 0$. The compatible systems of beliefs is 
\[ \mu_n(C) = \frac{p}{p + (1-p)} = p \quad \text{and} \]
\[ \mu_n(D) = \frac{(1-p)}{p + (1-p)} = 1 - p. \]
For \((w, y) \neq 0\), the expected value for \(P_2\) on \(I\) is
\[
E_2(I) = \left(\frac{2w}{2w+y}\right) [5x + (1-x)3] + \left(\frac{y}{2w+y}\right) [x(0) + (1-x)3] \\
= \frac{2w(3+2x) + 3y(1-x)}{2w+y}.
\]
Taking the derivative with respect to \(x\) (the behavior which \(P_2\) chooses on \(I\)),
\[
\frac{\partial E_2(I)}{\partial x} = \frac{4w - 3y}{2w+y} \begin{cases} 
< 0 & \text{if } w < \frac{3}{4}y \\
= 0 & \text{if } w = \frac{3}{4}y \\
> 0 & \text{if } w > \frac{3}{4}y.
\end{cases}
\]
Therefore, for \(P_2\) to maximize \(E_2(I)\),
\[
x \begin{cases} 
\text{arbitrary} & \text{if } (w, y) = (0, 0) \\
= 0 & \text{if } w < \frac{3}{4}y \\
\text{arbitrary} & \text{if } w = \frac{3}{4}y \\
= 1 & \text{if } w > \frac{3}{4}y.
\end{cases}
\]
Next, turning to \(P_1\) at the node \(A\),
\[
E_1(A) = (1-w)5 + w [x(9) + (1-x)(0)] \\
= 5(1-w) + 9wx,
\]
\[
\frac{\partial E_1(A)}{\partial w} = -5 + 9x \begin{cases} 
< 0 & \text{if } x < \frac{5}{9} \\
= 0 & \text{if } x = \frac{5}{9} \\
> 0 & \text{if } x > \frac{5}{9}.
\end{cases}
\]
Therefore, for \(P_1\) to maximize \(E_1(A)\),
\[
w \begin{cases} 
= 0 & \text{if } x < \frac{5}{9} \\
\text{arbitrary} & \text{if } x = \frac{5}{9} \\
= 1 & \text{if } x > \frac{5}{9}.
\end{cases}
\]
Finally, considering node \(B\) owned by \(P_1\),
\[
E_1(B) = 2(1-y) + 9yx,
\]
\[
\frac{\partial E_1(B)}{\partial y} = -2 + 9x \begin{cases} 
< 0 & \text{if } x < \frac{2}{5} \\
= 0 & \text{if } x = \frac{2}{5} \\
> 0 & \text{if } x > \frac{2}{5}.
\end{cases}
\]
Therefore, for \(P_1\) to maximize \(E_1(A)\),
\[
y \begin{cases} 
= 0 & \text{if } x < \frac{2}{5} \\
\text{arbitrary} & \text{if } x = \frac{2}{5} \\
= 1 & \text{if } x > \frac{2}{5}.
\end{cases}
\]
Combining the restrictions on the behavior strategies, we get the following.
\(x < \frac{2}{9}\): \(\Rightarrow\) \(w = 0 \& y = 0 \Rightarrow x\) is arbitrary.
Therefore, one solution is \(w = 0, y = 0,\) and \(0 \leq x < \frac{2}{9}\).
\(x = \frac{2}{9}\): \(\Rightarrow\) \(w = 0 \& y\) arbitrary \(\Rightarrow x = 0\) or \(y = 0\). We cannot have both \(x = \frac{2}{9}\) and 0.
Therefore, the only solution is \(w = 0, y = 0,\) and \(x = \frac{2}{9}\).
\(\frac{2}{9} < x < \frac{5}{9}\): \(\Rightarrow\) \(w = 0 \& y = 1 \Rightarrow x = 0\). This is a contradiction.
Therefore, one solution is \( w = \frac{3}{4}, y = 1 \), and \( x = \frac{5}{9} \).

Therefore, one solution is \( w = 1, y = 1 \) and \( x = 1 \).

Summarizing, we have found three sequential equilibria:

\[
\{ \ w = 0, \ y = 0, \ 0 \leq x \leq \frac{2}{9} \ \} , \\
\{ \ w = \frac{3}{4}, \ y = 1, \ x = \frac{5}{9} \ \} , \\
\{ \ w = 1, \ y = 1, \ x = 1 \ \} .
\]

Example 4: Sequential equilibrium for poker

This is a slight variation of the earlier game of poker considered. There are two players who each get one card that can be either an ace \( A \), a king \( K \), or a queen \( Q \). Between the two players there are nine possible hands each one occurring with probability \( \frac{1}{9} \). We assume the ante is 4 and the bet is 6. To simplify the consideration, we assume that each player will always bid with an ace and always fold with a queen. (If other possibilities are considered, it turns out that this is the correct choice.) Let \( p \) be the behavior strategy of \( P_1 \) bidding with a king and \( q \) be the behavior strategy of \( P_2 \) bidding with a king. In Figure 6, we dot the bids not taken when the player holds a queen. (Some other lines not taken are omitted.)

The information set \( \mathcal{I}_2 \) that \( P_2 \) holds a king is the set of nodes which are circled in the figure. With these possibilities, we need to determine the compatible belief on \( \mathcal{I}_2 \).

\[
\mu(\text{AK}) = \frac{\frac{1}{9}}{\frac{1}{9} + \left(\frac{1}{9}\right) (p) + \left(\frac{1}{9}\right) (0)} = \frac{1}{1 + p} \\
\mu(\text{KK}) = \frac{\left(\frac{1}{9}\right) (p)}{\left(\frac{1}{9}\right) (p + 1)} = \frac{p}{1 + p} \\
\mu(\text{QK}) = \frac{\left(\frac{1}{9}\right) (0)}{\left(\frac{1}{9}\right) (p + 1)} = 0.
\]
Now we can calculate the expected payoff for $P_2$ on $I_2$.

\[
E_2(I_2) = \frac{1}{1+p} [q(-10) + (1-q)(-4)] + \frac{p}{1+p} [q(0) + (1-q)(-4)] + 0
\]

\[
= \frac{-10q - 4(1-q) - 4p(1-q)}{1+p}
\]

and

\[
\frac{\partial E_2(I_2)}{\partial q} = \frac{-10 + 4 + 4p}{1+p} = \frac{4p - 6}{1+p} < 0
\]

for all $0 \leq p \leq 1$, so $q^* = 0$, i.e., $P_2$ always folds with a king.

Now, consider the information set $I_1$ where $P_1$ holds a king. These nodes are surrounded by squares in the figure. All the hands in $I_1$ are equally likely, so $P_1$ should give them a belief of $\frac{1}{3}$ probability each. Then,

\[
E_1(I_1) = \frac{1}{3} [p \cdot 1(-10) + (1-p)(-4)] + \frac{1}{3} [p(q \cdot 0) + p(1-q)4 + (1-p)(-4)]
\]

\[
+ \frac{1}{3} [p(1 \cdot 4) + (1-p)(-4)]
\]

\[
= \frac{1}{3} [-10p - 4 + 4p + 4p - 4pq - 4 + 4p + 4p - 4 + 4p]
\]

\[
= \frac{1}{3} [-10p - 4pq - 12].
\]

Taking the partial derivative with respect to $p$,

\[
\frac{\partial E_1(I_1)}{\partial p} = \frac{1}{3} [10 - 4q].
\]

This partial derivative is negative when evaluated at $q^* = 0$, so $p^* = 1$. Thus, player $P_1$ always bids with a king. The reader can calculate the expected payoff for this game. ■

**Example 5.2 & 5.10: Sequential equilibrium**

We now calculate the values which give an equilibrium which is optimal on every informations set for the modified Example 5.2 presented before. On $\mathcal{I}_3 = \{ G, H \}$, $E_1(\mathcal{I}_3, \sigma, \mu^\sigma) = 4 - 2r - 2q + 4q^r$, so

\[
\frac{\partial E_1(\mathcal{I}_3, \sigma, \mu^\sigma)}{\partial r} = 4q - 2 \begin{cases} < 0 & \text{if } q < \frac{1}{2} \\ = 0 & \text{if } q = \frac{1}{2} \\ > 0 & \text{if } q > \frac{1}{2}. \end{cases}
\]

Therefore, $E_1(\mathcal{I}_3, \sigma, \mu^\sigma)$ is maximized by

\[
\begin{cases} r = 0 & \text{if } q < \frac{1}{2} \\ \text{arbitrary} & \text{if } q = \frac{1}{2} \\ = 1 & \text{if } q > \frac{1}{2}. \end{cases}
\]

On $\mathcal{I}_2 = \{ Z, F \}$, $E_1(\mathcal{I}_2, \sigma, \mu^\sigma) = 2 - 2q - p + 5pq$, so

\[
\frac{\partial E_1(\mathcal{I}_2, \sigma, \mu^\sigma)}{\partial p} = -1 + 5q \begin{cases} < 0 & \text{if } q < \frac{1}{5} \\ = 0 & \text{if } q = \frac{1}{5} \\ > 0 & \text{if } q > \frac{1}{5}. \end{cases}
\]

Therefore, $E_1(\mathcal{I}_2, \sigma, \mu^\sigma)$ is maximized by

\[
\begin{cases} p = 0 & \text{if } q < \frac{1}{5} \\ \text{arbitrary} & \text{if } q = \frac{1}{5} \\ = 1 & \text{if } q > \frac{1}{5}. \end{cases}
\]
On \( \mathcal{I}_1 = \{ X, Y \} \), \( E_2(\mathcal{I}_1, \sigma, \mu^\sigma) = x(6 - 3q + p - 2pq) + (1 - x)(3 + q + 2r - 6qr) \), so
\[
\frac{\partial E_2(\mathcal{I}_1, \sigma, \mu^\sigma)}{\partial q} = x(-3 - 2p) + (1 - x)(1 - 6r) = 1 - 6r - x(4 + 2p - 6r) \equiv \Delta.
\]
Therefore, \( E_2(\mathcal{I}_1, \sigma, \mu^\sigma) \) is maximized by
\[
q = \begin{cases} 
0 & \text{if } \Delta < 0 \\
\text{arbitrary} & \text{if } \Delta = 0 \\
1 & \text{if } \Delta > 0.
\end{cases}
\]
Finally, at \( R \),
\[
E(R, \sigma) = x(2 - 2q - p + 5pq, 6 - 3q + p - 2pq) + (1 - x)(4 - 2r - 2q + 4qr, 3 + q + 2r - 6qr),
\]
so
\[
\frac{\partial E_1(R, \sigma)}{\partial x} = (2 - 2q - p + 5pq) - (4 - 2r - 2q + 4qr) = p(5q - 1) - r(4q - 2) - 2.
\]
We combine the calculations.

1. \( q < \frac{1}{5} \Rightarrow p = 0 \& r = 0 \Rightarrow \frac{\partial E_1(R)}{\partial x} = -2 \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1. \) This is a contradiction.

2. \( q = \frac{1}{5} \Rightarrow p \text{ arbitrary} \& r = 0 \Rightarrow \frac{\partial E_1(R)}{\partial x} = -2 \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1. \) This is a contradiction.

3. \( \frac{1}{5} < q < \frac{1}{2} \Rightarrow p = 1 \& r = 0 \Rightarrow \frac{\partial E_1(R)}{\partial x} = -2 + (5q - 1) < -\frac{1}{2} \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1. \) This is a contradiction.

4. \( q = \frac{1}{2} \Rightarrow p = 1 \& r \text{ arbitrary} \Rightarrow \frac{\partial E_1(R)}{\partial x} < -\frac{1}{2} \Rightarrow x = 0 \Rightarrow \Delta = 1 - 6r. \) To allow \( q = \frac{1}{2} \), we need \( \Delta = 0 \), so \( r = \frac{1}{6}. \) This gives a compatible solution.

5. \( \frac{1}{2} < q < 1 \Rightarrow p = 1 \& r = 1 \Rightarrow \frac{\partial E_1(R)}{\partial x} = q - 1 < 0 \Rightarrow x = 0. \Rightarrow \Delta = 1 - 6 - 0 = -5 < 0 \) and \( q = 0. \) This is a contradiction.

6. \( q = 1 \Rightarrow p = 1 \& r = 1 \Rightarrow \frac{\partial E_1(R)}{\partial x} = 0 \Rightarrow x \text{ arbitrary}. \Rightarrow \Delta = 1 - 6 - x(4 + 2 - 6) = -5 < 0 \) and \( q = 0. \) This is a contradiction.

Therefore, the only solution is \( x = \sigma_1(a) = 0, q = \sigma_2(T) = \frac{1}{2}, p = \sigma_1(L) = 1, \) and \( r = \sigma_1(L') = \frac{1}{6}. \)

The consistent system of beliefs is \( \mu(X) = 0, \mu(Y) = 1, \mu(Z) = \frac{1}{2}, \mu(F) = \frac{1}{2}, \mu(G) = \frac{1}{2}, \) and \( \mu(H) = \frac{1}{2}. \) The book starts with these values and verifies that they give a sequential equilibrium.