Chapter 1: Linear Programming

Math 368

May 22, 2013

For $f : \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}$, $\qquad f(\mathscr{D}) = \{ f(\mathbf{x}) : \mathbf{x} \in \mathscr{D} \}$

is **set of attainable values** of f on $\mathscr{D}$, or **image** of $\mathscr{D}$ by f.

f has a **maximum on $\mathscr{D}$ at $\mathbf{x}_M \in \mathscr{D}$** provided that

$f(\mathbf{x}_M) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \mathscr{D}$.

$\max\{ f(\mathbf{x}) : \mathbf{x} \in \mathscr{D} \} = f(\mathbf{x}_M)$, $\qquad$ **maximum value of f on $\mathscr{D}$**.

$\mathbf{x}_M$ is called a **maximizer of f on $\mathscr{D}$.**

$\arg\max\{ f(\mathbf{x}) : \mathbf{x} \in \mathscr{D} \} = \{ \mathbf{x} \in \mathscr{D} : f(\mathbf{x}) = f(\mathbf{x}_M) \}$.

f has a **minimum on $\mathscr{D}$ at $\mathbf{x}_m \in \mathscr{D}$** provided that

$f(\mathbf{x}_m) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathscr{D}$.

$\min\{ f(\mathbf{x}) : \mathbf{x} \in \mathscr{D} \} = f(\mathbf{x}_m)$, $\qquad$ **minimum of f on $\mathscr{D}$**

$\arg\min\{ f(\mathbf{x}) : \mathbf{x} \in \mathscr{D} \} = \{ \mathbf{x} \in \mathscr{D} : f(\mathbf{x}) = f(\mathbf{x}_m) \}$

set of minimizers

f has an **extremum** at $\mathbf{x}_0$ p.t. $\mathbf{x}_0$ is either a maximizer or minimizer.

Basic Optimization Problem

No maximizer or minimizer of $f(x) = x^3$ on $(0, 1)$,

$\arg\max\{ x^3 : 0 < x < 1 \} = \emptyset$, & $\arg\min\{ x^3 : 0 < x < 1 \} = \emptyset$,

Optimization Problem:

- Does $f(\mathbf{x})$ attain a maximum (or minimum) for some $\mathbf{x} \in \mathscr{D}$?
- If so, what is the maximum value (or minimum value) on $\mathscr{D}$ and what are the points at which $f(\mathbf{x})$ attains a maximum (or minimum) subject to $\mathbf{x} \in \mathscr{D}$?

Notations

$\mathbf{v} \geq \mathbf{w}$ in $\mathbb{R}^n$ means $v_i \geq w_i$ for $1 \leq i \leq n$

$\mathbf{v} \gg \mathbf{w}$ in $\mathbb{R}^n$ means $v_i > w_i$ for $1 \leq i \leq n$

$$\mathbb{R}^n_+ = \{\, \mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \,\} = \{\, \mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n \,\}$$

$$\mathbb{R}^n_{++} = \{\, \mathbf{x} \in \mathbb{R}^n : \mathbf{x} \gg \mathbf{0} \,\} = \{\, \mathbf{x} \in \mathbb{R}^n : x_i > 0 \text{ for } 1 \leq i \leq n \,\}$$

Up to 100 acres of land can be used to grow wheat and/or corn:

x_1 acres used to grow wheat and

x_2 acres used to grow corn: $\quad x_1 + x_2 \leq 100.$

Cost or capital constraint: $5x_1 + 10x_2 \leq 800.$

Labor constraint: $2x_1 + x_2 \leq 150.$

Profit: $f(x_1, x_2) = 80x_1 + 60x_2.$ Objective function

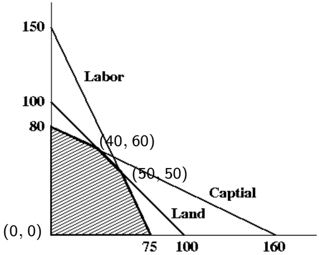
Problem:

$$\begin{array}{ll} \text{Maximize:} & 80x_1 + 60x_2 \quad \text{(profit)}, \\ \text{Subject to:} & x_1 + x_2 \leq 100, \quad \text{(land)} \\ & 5x_1 + 10x_2 \leq 800, \quad \text{(capital)} \\ & 2x_1 + x_2 \leq 150, \quad \text{(labor)} \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

All constraints and objective function are linear: **lin. programming prob.**

Feasible set $\mathscr{F}$ is set of all the points satisfying all constraints

$$x_1 + x_2 \leq 100 \text{ (land)}, \qquad 5x_1 + 10x_2 \leq 800 \text{ (capital)},$$
$$2x_1 + x_2 \leq 150 \text{ (labor)} \qquad x_1 \geq 0,\ x_2 \geq 0.$$



Vertices of the feasible set: $(0,0)$, $(75,0)$, $(50,50)$, $(40,60)$, $(0,80)$.

Other points where two constraints are tight $\left(\frac{140}{3}, \frac{170}{3}\right)$, $(100,0)$, etc

lie outside the feasible set, $\frac{140}{3} + \frac{170}{3} = \frac{310}{3} > 100$, $2(100) > 150$, ...

Example, continued

Since $\nabla f = (80, 60)^{\mathsf{T}} \neq (0, 0)^{\mathsf{T}}$,

maximum must be on boundary of $\mathscr{F}$.

$f(x_1, x_2)$ along an edge is linear combination of values at end points.

If a maximizer were in middle of an edge,

then f would have the same value at two end points of this edge.

Maximizer can be found at one of vertices.

Values at vertices:

$f(0, 0) = 0, \qquad f(75, 0) = 6000, \qquad f(50, 50) = 7000,$

$f(40, 60) = 6800, \qquad f(0, 80) = 4800.$

max value of f is 7000, maximizer $(50, 50)$. **End of Example**

Standard Max Linear Programming Problem (MLP)

Maximize **objective function**: $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = c_1 x_1 + \cdots + c_n x_n,$

Subject to **resource constraints**:

$$\begin{aligned} a_{11} x_1 + \cdots + a_{1n} x_n &\leq b_1 \\ \vdots \qquad\qquad & \vdots \quad\; \vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n &\leq b_m \\ x_j \geq 0 \ \text{ for } \ 1 \leq j &\leq n. \end{aligned}$$

Given data: $\mathbf{c} = (c_1, \dots, c_n)^\mathsf{T}$, $m \times n$ matrix $\mathbf{A} = (a_{ij})$,
$\mathbf{b} = (b_1, \dots, b_m)^\mathsf{T}$ with all $b_i \geq 0$,

Constraints using matrix notation are $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$.

Feasible set: $\mathscr{F} = \{ \mathbf{x} \in \mathbb{R}^n_+ : \mathbf{Ax} \leq \mathbf{b} \}$.

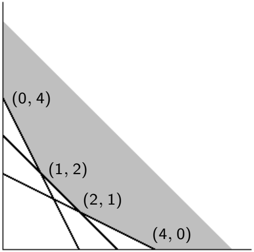
Minimization Example

Minimize: $3x_1 + 2x_2$,

Subject to: $2x_1 + x_2 \geq 4$,

$x_1 + x_2 \geq 3$,

$x_1 + 2x_2 \geq 4$, $\quad x_1 \geq 0$, and $x_2 \geq 0$.



$\mathscr{F}$ is unbounded but $f(\mathbf{x}) \geq 0$ is bded below so $f(\mathbf{x})$ has a minimum.

Vertices: $(4,0)$, $(2,1)$, $(1,2)$, $(0,4)$

Values: $f(4,0) = 12$, $f(2,1) = 8$, $f(1,2) = 7$, $f(0,4) = 8$.

$$\min\{f(\mathbf{x}) : \mathbf{x} \in \mathscr{F}\} = 7 \qquad \arg\min\{f(\mathbf{x}) : \mathbf{x} \in \mathscr{F}\} = \{(1,2)\}.$$

Geometric Method of Solving Linear Prog Problem

1. Determine or draw the feasible set $\mathscr{F}$.

   If $\mathscr{F} = \emptyset$, then problem has no optimal solution,

   problem is called **infeasible**

2. Problem is called **unbounded** and has no solution p.t.

   objective function on $\mathscr{F}$ has

   a. arbitrarily large positive values for a maximization problem, or

   b. arbitrarily large negative values for a minimization problem.

3. A problem is called **bounded** p.t. it is not infeasible nor unbounded;

   an optimal solution exists.

   Determine all the vertices of $\mathscr{F}$ and values at vertices.

   Choose the vertex of $\mathscr{F}$ producing the maximum or minimum value of the objective function.

Rank of a Matrix

Rank of a matrix $\mathbf{A}$ is dimension of column space of $\mathbf{A}$.

i.e., largest number of linearly independent columns of $\mathbf{A}$.

Same as number of pivots (in row reduced echelon form of $\mathbf{A}$).

$\text{rank}(\mathbf{A}) = k$ iff $k \geq 0$ is the largest integer s.t. $\det(\mathbf{A}_k) \neq 0$,

where $\mathbf{A}_k$ is any $k \times k$ submatrix of $\mathbf{A}$

formed by selecting any k columns and any k rows.

$\mathbf{A}_k$ is submatrix of pivot columns and rows.

Sketch: Let $\mathbf{A}'$ be submatrix with k linearly independent columns

that span column space; $\text{rank}(\mathbf{A}') = k$

$\dim(\text{row space of } \mathbf{A}') = k$, so k rows of $\mathbf{A}'$ to get $k \times k$ $\mathbf{A}_k$

with $\text{rank}(\mathbf{A}_k) = k$, so $\det(\mathbf{A}_k) \neq 0$.

Slack Variables

For large number of variables, need a practical algorithm.

Simplex method uses row reduction as solution method.

First step: make all the inequalities of type $x_i \geq 0$.

Inequality of the form $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ for $b_i \geq 0$ is called **resource constraint**.

For resource constraint, introduce **slack variable** s_i by

$$a_{i1}x_1 + \cdots + a_{in}x_n + s_i = b_i \quad \text{with} \quad s_i \geq 0.$$

s_i represents unused resource.

Introduction of a slack variable changes a resource constraint into equality constraint and $s_i \geq 0$.

Introducing Slack Variables into Wheat-Corn Example

$$\begin{aligned} &x_1 + x_2 + s_1 = 100, \\ &5x_1 + 10x_2 + s_2 = 800, \\ &2x_1 + x_2 + s_3 = 150, \qquad s_1,\ s_2,\ s_3 \geq 0. \end{aligned}$$

In matrix form,

$$\text{Maximize:} \quad (80, 60, 0, 0, 0) \cdot (x_1, x_2, s_1, s_2, s_3) = 80\, x_1 + 60\, x_2$$

$$\text{Subj. to:} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 10 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 800 \\ 150 \end{bmatrix} \quad \text{and} \quad \begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0, \\ s_1 &\geq 0, \\ s_2 &\geq 0, \\ s_3 &\geq 0. \end{aligned}$$

Because of 1's and 0's in last 3 columns of matrix, rank is 3.

Initial feasible solution $x_1 = 0 = x_2,\ s_1 = 100,\ s_2 = 800,\ s_3 = 150$

Wheat-Corn Example, continued

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 10 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 800 \\ 150 \end{bmatrix} \quad \text{with} \quad x_i \geq 0, \quad \text{and} \quad s_i \geq 0.$$

Initial feasible sol'n: $\mathbf{p}_0 = (0, 0, 100, 800, 150)^\top$ with $f(\mathbf{p}_0) = 0$.

Rank is 3, so $5 - 3 = 2$ free variables. x_1, x_2.

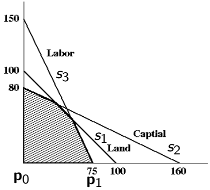
$\mathbf{p}_0$ obtained by setting free variables $x_1 = x_2 = 0$

and solving for dependent variables, which are 3 slack variables.

"Pivot" to make a different pair the free variables equal to zero and

a different triple of positive variables.

Wheat-Corn Example, continued



If leave the vertex $(x_1, x_2) = (0, 0)$, or $\mathbf{p}_0 = (0, 0, 100, 800, 150)$,

making $x_1 > 0$ entering variable while keeping $x_2 = 0$;

first slack variable to become zero is s_3 when $x_1 = 75$.

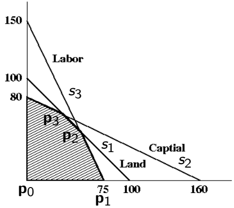
Arrive at the vertex $(x_1, x_2) = (75, 0)$, or $\mathbf{p}_1 = (75, 0, 25, 425, 0)$.

New sol'n has two zero variables and three positive variables.

Move along one edge from $\mathbf{p}_0$ to $\mathbf{p}_1$.

$f(\mathbf{p}_1) = 80(75) = 6000 > 0 = f(\mathbf{p}_0)$.

$\mathbf{p}_1$ is a better feasible sol'n than $\mathbf{p}_0$.



$$\mathbf{p}_1 = (75, 0, 25, 425, 0).$$

Repeat, leaving $\mathbf{p}_1$ by making $x_2 > 0$ entering variable

while keeping $s_3 = 0$. First other variable to become zero is s_1.

Arrive $\mathbf{p}_2 = (50, 50, 0, 50, 0)$

$f(\mathbf{p}_2) = 80(50) + 60(50) = 7000 > 6000 = f(\mathbf{p}_1)$.

$\mathbf{p}_2$ is a better feasible solution than $\mathbf{p}_1$.

Have moved along another edge of the feasible set from

$(x_1, x_2) = (75, 0)$ and arrived at $(x_1, x_2) = (50, 50)$.

Wheat-Corn Example, continued

If leave $\mathbf{p}_2$ by making $s_3 > 0$ entering variable while keeping $s_1 = 0$,

first variable to become zero is s_2, arrive at $\mathbf{p}_3 = (40, 60, 0, 0, 10)$.

$f(\mathbf{p}_3) = 80(40) + 60(60) = 6800 < 7000 = f(\mathbf{p}_2)$.

$\mathbf{p}_3$ is worse feasible solution than $\mathbf{p}_2$.

Let $\mathbf{z} \in \mathscr{F} \smallsetminus \{\mathbf{p_2}\}$. $\mathbf{v} = \mathbf{z} - \mathbf{p}_2$, $\mathbf{v}_j = \mathbf{p}_j - \mathbf{p}_2$.

$f(\mathbf{v}_1) = f(\mathbf{p}_1) - f(\mathbf{p}_2) < 0$, $f(\mathbf{v}_3) = f(\mathbf{p}_3) - f(\mathbf{p}_2) < 0$

$\mathbf{v}_1$ & $\mathbf{v}_3$ are basis of $\mathbb{R}^2$, so $\mathbf{v} = y_1 \mathbf{v}_1 + y_3 \mathbf{v}_3$ with $y_1, y_3 \geq 0$.

$\mathbf{v} \neq \mathbf{0}$ points into $\mathscr{F}$, so (i) $y_1, y_3 \geq 0$ and (ii) $y_1 > 0$ or $y_3 > 0$.

$$f(\mathbf{z}) = f(\mathbf{p}_2) + f(\mathbf{v}) = f(\mathbf{p}_2) + y_1 f(\mathbf{v}_1) + y_3 f(\mathbf{v}_3) < f(\mathbf{p}_2)$$

Since cannot increase f by moving along either edge going out from $\mathbf{p}_2$,

$\mathbf{p}_2$ is an optimal feasible solution.

Wheat-Corn Example, continued

In this example,

$$\mathbf{p}_0 = (0, 0, 100, 800, 150), \ \ \mathbf{p}_1 = (75, 0, 25, 425, 0),$$
$$\mathbf{p}_2 = (50, 50, 0, 50, 0), \ \ \mathbf{p}_3 = (40, 60, 0, 0, 10),$$
$$\mathbf{p}_4 = (0, 80, 20, 0, 70)$$

are called **basic solutions**

since at most 3 variables are positive,

where 3 is the rank (and number of constraints).

End of Example

Slack Variables added to a Standard Lin Prog Prob, MLP

All resource constraints

Given $\mathbf{b} = (b_1, \ldots, b_m)^\mathsf{T} \in \mathbb{R}^m_+$, $\mathbf{c} = (c_1, \ldots, c_n)^\mathsf{T} \in \mathbb{R}^n$,

$m \times n$ matrix $\mathbf{A} = (a_{ij})$.

Find $\mathbf{x} \in \mathbb{R}^n_+$ and $\mathbf{s} \in \mathbb{R}^m_+$ that

$$
\begin{array}{ll}
\text{maximize:} & f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \\
\text{subject to:} & a_{11}x_1 + \cdots + a_{1n}x_n + s_1 \qquad = b_1 \\
& \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
& a_{m1}x_1 + \cdots + a_{mn}x_n + s_m \qquad = b_m \\
& x_i \geq 0,\ s_j \geq 0 \ \text{ for } \ 1 \leq i \leq n, \quad 1 \leq j \leq m.
\end{array}
$$

Using matrix notation with $\mathbf{I}$ $m \times m$ identity matrix,

$$\text{maximize} \quad f(\mathbf{x}) \quad \text{subject to} \quad [\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b} \quad \text{with} \quad \mathbf{x} \geq 0,\ \mathbf{s} \geq \mathbf{0}.$$

Indicate partition of augmented matrix by extra vertical lines $[\mathbf{A} \,|\, \mathbf{I} \,|\, \mathbf{b}]$

Basic Solutions

Assume $\bar{\mathbf{A}}$ $m \times (n+m)$ matrix with rank m (like $\bar{\mathbf{A}} = [\mathbf{A}, \mathbf{I}]$)

m **dependent variables** are called **basic variables**. (var of pivot col'ns)

n **free variables** are called **non-basic variables**. (var of non-pivot col'ns)

A **basic solution** is a solution $\mathbf{p}$ satisfying $\bar{\mathbf{A}}\mathbf{p} = \mathbf{b}$

such that columns corresponding to $p_i \neq 0$ are linearly indep.

$\leq \text{rank}(\mathbf{A}) = m$.

If $\mathbf{p}$ is also feasible with $\mathbf{p} \geq 0$, then called a **basic feasible solution**.

Obtain by setting n free variables $= 0$, and get basic variables ≥ 0,

allow possibly some basic variables $= 0$

Linear Algebra Solution of Wheat-Corn Problem

Augmented matrix for the original wheat-corn problem is

$$\left[\begin{array}{rr|rrr|r} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 1 & 1 & 1 & 0 & 0 & 100 \\ 5 & 10 & 0 & 1 & 0 & 800 \\ 2 & 1 & 0 & 0 & 1 & 150 \end{array}\right]$$

with free variables x_1 and x_2 and basic variables s_1, s_2, and s_3.

If make $x_1 > 0$ while keeping $x_2 = 0$, x_1 becomes a new basic variable (for new pivot coln) called **entering variable**.

(i) s_1 will become zero when $x_1 = \frac{100}{1} = 100$,

(ii) s_2 will become zero when $x_1 = \frac{800}{5} = 160$, and

(iii) s_3 will become zero when $x_1 = \frac{150}{2} = 75$.

Since s_3 becomes zero for the smallest value of x_1,

s_3 is the **departing variable** and

new pivot is 1st column (for x_1) and 3rd row (old pivot for s_3)

First Pivot

Row reducing to make a pivot in first column third row,

$$\left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 1 & 1 & 1 & 0 & 0 & 100 \\ 5 & 10 & 0 & 1 & 0 & 800 \\ \mathbf{2} & 1 & 0 & 0 & 1 & 150 \end{array}\right] \sim \left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 0 & .5 & 1 & 0 & -.5 & 25 \\ 0 & 7.5 & 0 & 1 & -2.5 & 425 \\ \mathbf{1} & .5 & 0 & 0 & .5 & 75 \end{array}\right]$$

Setting free variables $x_2 = s_3 = 0$,

new basic solution $\mathbf{p}_1 = (75, 0, 25, 425, 0)^{\mathsf{T}}$.

Entries in the right (augmented) column give

values of the new basic variables that are > 0.

Including Objective Function in Matrix

Objective function (or variable) is

$$f = 80x_1 + 60x_2, \quad \text{or} \quad -80x_1 - 60x_2 + f = 0.$$

Adding a row for this equation and a column for variable f

keeps track of the value of f during row reduction.

x_1	x_2	s_1	s_2	s_3	f	
1	1	1	0	0	0	100
5	10	0	1	0	0	800
2	1	0	0	1	0	150
-80	-60	0	0	0	1	0

This matrix including objective function row is called **tableau**.

Entries in column for f are 1 in objective function row and 0 elsewhere.

In objection function row of this tableau, entries for x_i are negative.

First Pivot, continued

Row reducing tableau by making

first column and third row a new pivot,

$$\left[\begin{array}{rr|rrr|r|r} x_1 & x_2 & s_1 & s_2 & s_3 & f & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ 5 & 10 & 0 & 1 & 0 & 0 & 800 \\ \mathbf{2} & 1 & 0 & 0 & 1 & 0 & 150 \\ \hline -80 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{rr|rrr|r|r} x_1 & x_2 & s_1 & s_2 & s_3 & f & \\ \hline 0 & .5 & 1 & 0 & -.5 & 0 & 25 \\ 0 & 7.5 & 0 & 1 & -2.5 & 0 & 425 \\ \mathbf{1} & .5 & 0 & 0 & .5 & 0 & 75 \\ \hline 0 & -20 & 0 & 0 & 40 & 1 & 6000 \end{array}\right]$$

For $x_2 = s_3 = 0$, $x_1 = 75 > 0$, $s_1 = 25 > 0$, $s_2 = 425 > 0$,

Bottom right entry of 6000 is new value of f

Second Pivot

x_1	x_2	s_1	s_2	s_3	f	
0	.5	1	0	-.5	0	25
0	7.5	0	1	-2.5	0	425
1	.5	0	0	.5	0	75
0	-20	0	0	40	1	6000

x_2 & s_3 free (non-basic) variables

If pivot back to make $s_3 > 0$, the value of f becomes smaller,

so select x_2 as the next entering variable, keeping $s_3 = 0$.

(i) s_1 becomes zero when $x_2 = \frac{25}{.5} = 50$, and

(ii) s_2 becomes zero when $x_2 = \frac{425}{7.5} = 56.67$.

(iii) x_1 becomes zero when $x_2 = \frac{75}{.5} = 150$,

Since the smallest positive value of x_1 comes from s_1,

s_1 is the departing variable and pivot on 1st row 2nd column.

Second Pivot, contin.

Pivot on 1st row 2nd column,

x_1	x_2	s_1	s_2	s_3	f	
0	**.5**	1	0	-.5	0	25
0	7.5	0	1	-2.5	0	425
1	.5	0	0	.5	0	75
0	-20	0	0	40	1	6000

$\sim$

x_1	x_2	s_1	s_2	s_3	f	
0	**1**	2	0	-1	0	50
0	0	-15	1	5	0	50
1	0	-1	0	1	0	50
0	0	40	0	20	1	7000

Entries in column for f don't change.

$f = 7000$ objective function

Third Pivot

Why does the objective function decrease when moving along the edge

making $s_3 > 0$ an entering variable, keeping $s_1 = 0$?

(i) x_1 becomes zero when $s_3 = \frac{50}{1} = 50$,

(ii) x_2 becomes zero when $s_3 = \frac{50}{-1} = -50$, and

(iii) s_2 becomes zero when $s_3 = \frac{50}{5} = 10$.

Smallest **positive** value of s_3 comes from s_2,

and pivot on the 2nd row 5th column.

x_1	x_2	s_1	s_2	s_3	f	
0	1	2	0	-1	0	50
0	0	-15	1	**5**	0	50
1	0	-1	0	1	0	50
0	0	40	0	20	1	7000

$\sim$

x_1	x_2	s_1	s_2	s_3	f	
0	1	-1	0	0	0	60
0	0	-3	.2	**1**	0	10
1	0	2	-2	0	0	40
0	0	100	-4	0	1	6800

Value of objective function decreases since the entry is **already positive** before pivot in column for s_3 of the **objective function row**

Tableau

Drop the column for the variable f from augmented matrix

since it does not play a role in the row reduction.

(entry in last row stays $= 1$ and others stay $= 0$)

Augmented matrix with objective function row

but without column for objective function variable

is called the **tableau**.

Steps in the Simplex Method for Stand MLP

1. Set up the tableau so that all $b_i \geq 0$. An initial feasible basic solution is determined by setting $x_i = 0$ and solving for s_i.
2. Choose as entering variable any free variable with a negative entry in objection function row. (often most negative)
3. From column selected in previous step, select row for which ratio of entry in augmented column divided by entry in column selected is smallest value ≥ 0; departing variable is basic variable for this row.

   Row reduce the matrix using selected new pivot position.
4. Objective function has no upper bound and no optimal solution when one column has only nonpositive coefficients above a negative coefficient in objective function row.
5. Solution is optimal when all entries in objective function row are nonnegative.
6. If optimal tableau has zero entry in objective row for nonbasic variable and all basic variables are positive, then nonunique solution.

General Constraints: Requirement Constraints

All $b_i \geq 0$ (by multiplying inequalities by -1 if necessary.)

Requirement constraint is given by

$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$ (occur especially for a min problem.)

Require to have at least a minimum amount to quantity.

Can have a surplus of quantity,

so subtract off a **surplus variable** to get equality

$a_{i1}x_1 + \cdots + a_{in}x_n - s_i = b_i$ with $s_i \geq 0$.

To solve equation initially, also add an **artificial variable** $r_i \geq 0$,

$a_{i1}x_1 + \cdots + a_{in}x_n - s_i + r_i = b_i$.

(Walker uses a_i, but we use r_i to distinguish from matrix a_{ij}.)

Initial sol'n sets the artificial variable $r_i = b_i$ while $s_i = 0 = x_j \ \forall j$.

Equality Constraints

For an **equality constraint** $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$,

add one **artificial variable**

$$a_{i1}x_1 + \cdots + a_{in}x_n + r_i = b_i,$$

with an initial solution $r_i = b_i \geq 0$ while the $x_j = 0$.

For general constraints (with either requirement or equality constraint)

initial solution has all $x_i = 0$ and all surplus variables zero,

while slack variables and artificial variables ≥ 0.

This sol'n is not feasible if any artificial variables is positive

for a requirement constraint or an equality constraint

Minimization Example

Assume two foods are consumed in amounts x_1 and x_2

with costs per unit of 15 and 7 respectively, and

yield $(5, 3, 5)$ and $(2, 2, 1)$ units of three vitamins respectively.

Problem is to minimize cost $15\,x_1 + 7\,x_2$ or

$$\begin{array}{ll} \text{Maximize:} & -15\,x_1 - 7\,x_2 \\ \text{Subject to:} & 5\,x_1 + 2\,x_2 \geq 60 \\ & 3\,x_1 + 2\,x_2 \geq 40, \text{ and} \\ & 5\,x_1 + 1\,x_2 \geq 35. \end{array}$$

$\mathbf{x} = \mathbf{0}$ not a feasible sol'n,

initial sol'n involves the artificial variables

Minimization Example, continued

The tableau is

x_1	x_2	s_1	s_2	s_3	r_1	r_2	r_3	
5	2	-1	0	0	1	0	0	60
3	2	0	-1	0	0	1	0	40
5	1	0	0	-1	0	0	1	35
15	7	0	0	0	0	0	0	0

To eliminate artificial variables, preliminary steps

to force all artificial variables to be zero.

Use **artificial objective function** that is

negative sum of equations that contain artificial variables,

$$-13x_1 - 5x_2 + s_1 + s_2 + s_3 + (-r_1 - r_2 - r_3) = -135.$$

$$-13x_1 - 5x_2 + s_1 + s_2 + s_3 + R = -135$$

$R = -r_1 - r_2 - r_3 \leq 0$ new variable, with max of 0.

Minimization Example, continued

Tableau with the artificial objective function included
(but not a column for the variable R) is

$$\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & r_1 & r_2 & r_3 & \\
\hline
5 & 2 & -1 & 0 & 0 & 1 & 0 & 0 & 60 \\
3 & 2 & 0 & -1 & 0 & 0 & 1 & 0 & 40 \\
5 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 35 \\
\hline
15 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-13 & -5 & 1 & 1 & 1 & 0 & 0 & 0 & -135
\end{array}\right]$$

If artificial objective function can be made equal zero,
then this gives an initial feasible basic solution with $r_i = 0$
and only original and slack variables positive.

Then, artificial variables can be dropped and proceed as before.

Minimization Example, continued

$-13 < -5$

x_1	x_2	s_1	s_2	s_3	r_1	r_2	r_3	
5	2	-1	0	0	1	0	0	60
3	2	0	-1	0	0	1	0	40
5	1	0	0	-1	0	0	1	35
15	7	0	0	0	0	0	0	0
-13	-5	1	1	1	0	0	0	-135

$\frac{35}{5} = 7 < \frac{40}{3} \approx 13.3$, $\frac{60}{5} = 12$: Pivoting on a_{31} (making into a pivot)

$\sim$

x_1	x_2	s_1	s_2	s_3	r_1	r_2	r_3	
0	1	-1	0	1	1	0	-1	25
0	$\frac{7}{5}$	0	-1	$\frac{3}{5}$	0	1	$-\frac{3}{5}$	19
1	$\frac{1}{5}$	0	0	$-\frac{1}{5}$	0	0	$\frac{1}{5}$	7
0	4	0	0	3	0	0	-3	-105
0	$-\frac{12}{5}$	1	1	$-\frac{8}{5}$	0	0	$\frac{13}{5}$	-44

Pivoting on a_{22} $\quad (19 \times \frac{5}{7} \approx 13.57 < 25 < 7 \times 5)$

$$\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & r_1 & r_2 & r_3 & \\
\hline
0 & 1 & -1 & 0 & 1 & 1 & 0 & -1 & 25 \\
0 & \frac{7}{5} & 0 & -1 & \frac{3}{5} & 0 & 1 & -\frac{3}{5} & 19 \\
1 & \frac{1}{5} & 0 & 0 & -\frac{1}{5} & 0 & 0 & \frac{1}{5} & 7 \\
\hline
0 & 4 & 0 & 0 & 3 & 0 & 0 & -3 & -105 \\
0 & -\frac{12}{5} & 1 & 1 & -\frac{8}{5} & 0 & 0 & \frac{13}{5} & -44
\end{array}\right]$$

$$\sim \left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & r_1 & r_2 & r_3 & \\
\hline
0 & 0 & -1 & \frac{5}{7} & \frac{4}{7} & 1 & -\frac{5}{7} & -\frac{4}{7} & \frac{80}{7} \\
0 & 1 & 0 & -\frac{5}{7} & \frac{3}{7} & 0 & \frac{5}{7} & -\frac{3}{7} & \frac{95}{7} \\
1 & 0 & 0 & \frac{1}{7} & -\frac{2}{7} & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{30}{7} \\
\hline
0 & 0 & 0 & \frac{20}{7} & \frac{9}{7} & 0 & -\frac{20}{7} & -\frac{9}{7} & -\frac{1115}{7} \\
0 & 0 & 1 & -\frac{5}{7} & -\frac{4}{7} & 0 & \frac{12}{7} & \frac{11}{7} & -\frac{80}{7}
\end{array}\right]$$

Pivoting on a_{14} yields an initial feasible basic solution:

x_1	x_2	s_1	s_2	s_3	r_1	r_2	r_3	
0	0	-1	$\frac{5}{7}$	$\frac{4}{7}$	1	$-\frac{5}{7}$	$-\frac{4}{7}$	$\frac{80}{7}$
0	1	0	$-\frac{5}{7}$	$\frac{3}{7}$	0	$\frac{5}{7}$	$-\frac{3}{7}$	$\frac{95}{7}$
1	0	0	$\frac{1}{7}$	$-\frac{2}{7}$	0	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{30}{7}$
0	0	0	$\frac{20}{7}$	$\frac{9}{7}$	0	$-\frac{20}{7}$	$-\frac{9}{7}$	$-\frac{1115}{7}$
0	0	1	$-\frac{5}{7}$	$-\frac{4}{7}$	0	$\frac{12}{7}$	$\frac{11}{7}$	$-\frac{80}{7}$

$\sim$

x_1	x_2	s_1	s_2	s_3	r_1	r_2	r_3	
0	0	$-\frac{7}{5}$	1	$\frac{4}{5}$	$\frac{7}{5}$	-1	$-\frac{4}{5}$	16
0	1	-1	0	1	1	0	-1	25
1	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{2}{5}$	2
0	0	4	0	-1	-4	0	1	-205
0	0	0	0	0	1	1	1	0

Minimization Example, continued

After these two steps, $(2, 25, 0, 16, 0)$ is an initial feasible basic solution and artificial variables can be dropped.

Pivoting on a_{15} yields final solution:

$$
\left[\begin{array}{cc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & \\
\hline
0 & 0 & -\frac{7}{5} & 1 & \mathbf{\frac{4}{5}} & 16 \\
0 & 1 & -1 & 0 & 1 & 25 \\
1 & 0 & \frac{1}{5} & 0 & -\frac{2}{5} & 2 \\
\hline
0 & 0 & 4 & 0 & \mathbf{-1} & -205
\end{array}\right]
\sim
\left[\begin{array}{cc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & \\
\hline
0 & 0 & -\frac{7}{4} & \frac{5}{4} & \mathbf{1} & 20 \\
0 & 1 & \frac{3}{4} & -\frac{5}{4} & 0 & 5 \\
1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 10 \\
\hline
0 & 0 & \frac{9}{4} & \frac{5}{4} & 0 & -185
\end{array}\right].
$$

All entries in the obj fn are positive, so maximal solution: $(10, 5, 0, 0, 20)$ with an value of -185.

For original problem, minimal sol'n has value of 185.

End of Example

1. Make all $b_i \geq 0$ of constraints by multiplying by -1 if necessary.
2. Add a slack variable for each resource inequality, add a surplus variable and an artificial variable for each requirement constraint, and add an artificial variable for each equality constraint.
3. If either requirement constraints or equality constraints are present, then form **artificial objective function** by taking negative sum of all equations that contain artificial variables,
   dropping terms involving artificial variables.

   Set up tableau matrix. (The row for artificial objective function has zeroes in the columns of the artificial variables.)

   An initial solution of equations including artificial variables is determined by setting all original variables $x_j = 0$,
   all slack variables $s_i = b_i$, all the surplus variables $s_i = 0$,
   and all artificial variables $r_i = b_i$.

Steps for General Constraints, continued

4. Apply simplex algorithm using artificial objective function.

   a. If it is not possible to make artificial objective function equal to zero, then there is no feasible solution.

   b. If the artificial variables can be made equal to zero, then drop artificial variables and artificial objective function from tableau and continue using initial feasible basic solution constructed.

5. Apply simplex algorithm to actual objective function.
   Solution is optimal when all entries in objective function row are nonnegative.

Example with Equality Constraint

Consider the problem of

Maximize: $3x_1 + 4x_2$

Subject to: $-2x_1 + x_2 \leq 6$, and

$2x_1 + 2x_2 \geq 24$,

$x_1 = 8$,

$x_1 \geq 0, \quad x_2 \geq 0$.

With slack, surplus, and artificial variables added the problem becomes

Maximize: $3x_1 + 4x_2$

Subject to: $-2x_1 + x_2 + s_1 = 6$

$2x_1 + 2x_2 - s_2 + r_2 = 24$

$x_1 + r_3 = 8$.

Artificial obj fn is negative sum of the 2nd and 3rd rows

$$-3x_1 - 2x_2 + s_2 + R = -32 \quad \text{where } R = -r_2 - r_3$$

Example, continued

The tableau with variables is

x_1	x_2	s_1	s_2	r_2	r_3	
-2	1	1	0	0	0	6
2	2	0	-1	1	0	24
1	0	0	0	0	1	8
-3	-4	0	0	0	0	0
-3	-2	0	1	0	0	-32

Pivoting on a_{31} and then a_{22},

$\sim$

x_1	x_2	s_1	s_2	r_2	r_3	
0	1	1	0	0	2	22
0	**2**	0	-1	1	-2	8
1	0	0	0	0	1	8
0	-4	0	0	0	3	24
0	**-2**	0	1	0	3	-8

$\sim$

x_1	x_2	s_1	s_2	r_2	r_3	
0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	3	18
0	**1**	0	$-\frac{1}{2}$	$\frac{1}{2}$	-1	4
1	0	0	0	0	1	8
0	0	0	-2	2	-1	40
0	0	0	0	1	1	0

Attained a feasible solution of $(x_1, x_2, s_1, s_2) = (8, 4, 18, 0)$.

Can now drop artificial objective function and artificial variables.

Pivoting on $a_{1,4}$

$$\left[\begin{array}{cc|cc|c} x_1 & x_2 & s_1 & s_2 & \\ \hline 0 & 0 & 1 & \frac{1}{2} & 18 \\ 0 & 1 & 0 & -\frac{1}{2} & 4 \\ 1 & 0 & 0 & 0 & 8 \\ \hline 0 & 0 & 0 & -2 & 40 \end{array}\right] \sim \left[\begin{array}{cc|cc|c} x_1 & x_2 & s_1 & s_2 & \\ \hline 0 & 0 & 2 & 1 & 36 \\ 0 & 1 & 1 & 0 & 22 \\ 1 & 0 & 0 & 0 & 8 \\ \hline 0 & 0 & 4 & 0 & 112 \end{array}\right]$$

Optimal solution of $f = 112$ for $(x_1, x_2, s_1, s_2) = (8, 22, 0, 36)$.

End of Example

Duality: Introductory Example

Return to wheat-corn problem (MLP).

$$\begin{aligned} \text{Maximize:} \quad & z = 80x_1 + 60x_2 \\ \text{subject to:} \quad & x_1 + x_2 \leq 100, \ \text{(land)} \\ & 5x_1 + 10x_2 \leq 800, \ \text{(capital)} \\ & 2x_1 + x_2 \leq 150, \ \text{(labor)}, \qquad x_1, x_2 \geq 0 \end{aligned}$$

Assume excess resources can be rented out or shortfalls rented for prices of y_1, y_2, and y_3. (**Shadow prices** of inputs.)

Profit $P = 80x_1 + 60x_2 + (100 - x_1 - x_2)y_1$
$$+(800 - 5x_1 - 10x_2)y_2 + (150 - 2x_1 - x_2)y_3$$

If profit by renting outside land, then competitors raise price of land force farmer $100 - x_1 - x_2 \geq 0$.

If $100 - x_1 - x_2 > 0$, then market sets $y_1 = 0$.

$(100 - x_1 - x_2)y_1 = 0$ at optimal

Similar other constraints, Farmer's perspective yields original MLP

Duality: Introductory Example, contin.

$P = 80x_1 + 60x_2 + (100 - x_1 - x_2)y_1 + (800 - 5x_1 - 10x_2)y_2 + (150 - 2x_1 - x_2)y_3$

If a resource is slack, $100 - x_1 - x_2 > 0$, then market sets $y_1 = 0$.

Market is minimizing P as function of (y_1, y_2, y_3)

Rearranging profit function from markets perspective yields

$P = (80 - y_1 - 5y_2 - 2y_3)x_1 + (60 - y_1 - 10y_2 - y_3)x_2 + 100y_1 + 800y_2 + 150y_3$

Coefficients of x_i represents net profit after costs of unit of i^{th}-good

If net profit > 0, the competitors grow wheat/corn. Force ≤ 0,

$$80 - y_1 - 5y_2 - 2y_3 \leq 0 \qquad \text{or} \qquad 80 \leq y_1 + 5y_2 + 2y_3$$
$$60 - y_1 - 10y_2 - y_3 \leq 0 \qquad \text{or} \qquad 60 \leq y_1 + 10y_2 + y_3$$

If a resource is slack, $100 - x_1 - x_2 > 0$, then market sets $y_1 = 0$.

$$0 = (100 - x_1 - x_2)y_1 \qquad 0 = (800 - 5x_1 - 10x_2)y_2$$
$$0 = (150 - 2x_1 - x_2)y_3$$

Market is minimizing $P = 100y_1 + 800y_2 + 150y_3$

Duality: Introductory Example, contin.

Market's perspective results in dual minimization problem:

$$\begin{aligned} \text{Minimize:} \quad & w = 100y_1 + 800y_2 + 150y_3 \\ \text{Subject to:} \quad & y_1 + 5y_2 + 2y_3 \geq 80, \ \ \text{(wheat)} \\ & y_1 + 10y_2 + y_3 \geq 60, \ \ \text{(corn)} \\ & y_1,\ y_2,\ y_3 \geq 0. \end{aligned}$$

$$\text{MLP:} \quad \begin{aligned} \text{Maximize:} \quad & z = 80x_1 + 60x_2 \\ \text{Subject to:} \quad & x_1 + x_2 \leq 100, \\ & 5x_1 + 10x_2 \leq 800, \\ & 2x_1 + x_2 \leq 150, \qquad x_1, x_2 \geq 0 \end{aligned}$$

1. Coefficient matrices of x_i and y_i are transposes of each other.

2. Coefficients for objective function of MLP
become constants for inequalities of dual mLP.

3. Constants for inequalities of MLP
become coefficients for objective function of dual mLP.

Duality: Introductory Example, contin.

For the wheat-corn MLP problem, final tableau

$$\left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 1 & 1 & 1 & 0 & 0 & 100 \\ 5 & 10 & 0 & 1 & 0 & 800 \\ 2 & 1 & 0 & 0 & 1 & 150 \\ \hline -80 & -60 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 0 & 1 & 2 & 0 & -1 & 50 \\ 0 & 0 & -15 & 1 & 5 & 50 \\ 1 & 0 & -1 & 0 & 1 & 50 \\ \hline 0 & 0 & 40 & 0 & 20 & 7000 \end{array}\right]$$

Optimal sol'n MLP is

$x_1 = 50$ and $x_2 = 50$ with a payoff of 7000.

Optimal sol'n dual mLP is (by theorem given later)

$y_1 = 40$, $y_2 = 0$, and $y_3 = 20$ with the same payoff,

where 40, 0, and 20 are entries in bottom row of final tableau in columns associated with slack variables.

End of Example

Example, Bicycle Manufacturing

A bicycle manufacturer manufactures x_1 3-speeds and x_2 5-speeds.

Maximize profits given by $z = 12x_1 + 15x_2$.

Constraints are

$$\begin{array}{ll} 20\,x_1 + 30\,x_2 \leq 2400 & \text{finishing time in minutes,} \\ 15\,x_1 + 40\,x_2 \leq 3000 & \text{assembly time in minutes,} \\ x_1 + x_2 \leq 100 & \text{frames used for assembly,} \\ x_1 \geq 0 \quad x_2 \geq 0. & \end{array}$$

Dual problem is

$$\begin{array}{ll} \text{Minimize:} & w = 2400\,y_1 + 3000\,y_2 + 100\,y_3, \\ \text{Subject to:} & 20\,y_1 + 15\,y_2 + y_3 \geq 12, \\ & 30\,y_1 + 40\,y_2 + y_3 \geq 15, \\ & y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0. \end{array}$$

Bicycle Manufacturing, contin.

MLP

x_1	x_2	s_1	s_2	s_3	
20	30	1	0	0	2400
15	40	0	1	0	3000
1	1	0	0	1	100
-12	-15	0	0	0	0

$\sim$

x_1	x_2	s_1	s_2	s_3	
0	**10**	1	0	-20	400
0	25	0	1	-15	1500
1	1	0	0	1	100
0	-3	0	0	12	1200

$\sim$

x_1	x_2	s_1	s_2	s_3	
0	**1**	$\frac{1}{10}$	0	-2	40
0	0	$-\frac{5}{2}$	1	35	500
1	0	$-\frac{1}{10}$	0	3	60
0	0	$\frac{3}{10}$	0	6	1320

Optimal sol'n has $x_1 = 60$ 3-speeds, $x_2 = 40$ 5-speeds,

with a profit of $1320.

Bicycle Manufacturing, continued

$$\left[\begin{array}{cc|ccc|r} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 0 & 1 & \frac{1}{10} & 0 & -2 & 40 \\ 0 & 0 & -\frac{5}{2} & 1 & 35 & 500 \\ 1 & 0 & -\frac{1}{10} & 0 & 3 & 60 \\ \hline 0 & 0 & \frac{3}{10} & 0 & 6 & 1320 \end{array}\right]$$

y_i are **marginal values** of corresponding constraint.

Dual problem has a solution of

$y_1 = \frac{3}{10}$ profit per finishing minute,

$y_2 = 0$ profit per assembly minute

$y_3 = 6$ profit per frame.

Additional units of the exhausted resources, finishing time and frames, contribute to the profit but not assembly time.

Rules for Forming Dual LP

Maximization Problem, MLP	**Minimization Problem, mLP**
i^{th} constraint $\sum_j a_{ij}x_j \leq b_i$	i^{th} variable $0 \leq y_i$
i^{th} constraint $\sum_j a_{ij}x_j \geq b_i$	i^{th} variable $0 \geq y_i$
i^{th} constraint $\sum_j a_{ij}x_j = b_i$	i^{th} variable y_i unrestricted
j^{th} variable $0 \leq x_j$	j^{th} constraint $\sum_i a_{ij}y_i \geq c_j$
j^{th} variable $0 \geq x_j$	j^{th} constraint $\sum_i a_{ij}y_i \leq c_j$
j^{th} variable x_j unrestricted	j^{th} constraint $\sum_i a_{ij}y_i = c_j$

Standard conditions for MLP, $\sum_j a_{ij}x_j \leq b_i$ or $0 \leq x_j$, corresp to standard conditions for mLP, $0 \leq y_i$ or $\sum_i a_{ij}y_i \geq c_j$

Nonstandard conditions, $\geq b_i$ or $0 \geq x_j$, corresp to nonstand conditions

Equality constraints correspond to unrestricted variables

These rules follow from proof of Duality Theorem (given subsequently)

Example

$$\begin{aligned}
\text{Minimize :} \quad & 8y_1 + 10y_2 + 4y_3 \\
\text{Subject to :} \quad & 4y_1 + 2y_2 - 3y_3 \geq 20 \\
& 2y_1 + 3y_2 + 5y_3 \leq 150 \\
& 6y_1 + 2y_2 + 4y_3 = 40 \\
& y_1 \text{ unrestricted, } y_2 \geq 0,\ y_3 \geq 0
\end{aligned}$$

By table, dual maximization problem is

$$\begin{aligned}
\text{Maximize :} \quad & 20x_1 + 150x_2 + 40x_3 \\
\text{Subject to :} \quad & 4x_1 + 2x_2 + 6x_3 = 8 \\
& 2x_1 + 3x_2 + 2x_3 \leq 10 \\
& -3x_1 + 5x_2 + 4x_3 \leq 4 \\
& x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted}
\end{aligned}$$

Example, continued

$$\begin{aligned} \text{Maximize :} \quad & 20x_1 + 150x_2 + 40x_3 \\ \text{Subject to :} \quad & 4x_1 + 2x_2 + 6x_3 = 8 \\ & 2x_1 + 3x_2 + 2x_3 \leq 10 \\ & \text{-}3x_1 + 5x_2 + 4x_3 \leq 4 \\ & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted} \end{aligned}$$

By making change of variables $x_2 = \text{-}v_2$ and $x_3 = v_3 - w_3$,

all restrictions on variables are ≥ 0:

$$\begin{aligned} \text{Maximize :} \quad & 20x_1 - 150v_2 + 40v_3 - 40w_3 \\ \text{Subject to :} \quad & 4x_1 - 2v_2 + 6v_3 - 6w_3 = 8 \\ & 2x_1 - 3v_2 + 2v_3 - 2w_3 \leq 10 \\ & \text{-}3x_1 - 5v_2 + 4x_3 - 4w_3 \leq 4 \\ & x_1 \geq 0,\ v_2 \geq 0,\ v_3 \geq 0,\ w_3 \geq 0 \end{aligned}$$

Tableau for maximization problem with variables x_1, v_2, v_3, w_3, with artificial variable r_1, and with slack variables s_2 and s_3 is

x_1	v_2	v_3	w_3	s_2	s_3	r_1	
4	-2	6	-6	0	0	1	8
2	-3	2	-2	1	0	0	10
-3	-5	4	-4	0	1	0	4
-20	150	-40	40	0	0	0	0
-4	2	-6	6	0	0	0	-8

$\sim$

x_1	v_2	v_3	w_3	s_2	s_3	r_1	
1	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	0	0	$\frac{1}{4}$	2
0	-2	-1	1	1	0	$-\frac{1}{2}$	6
0	$-\frac{13}{2}$	$\frac{17}{2}$	$-\frac{17}{2}$	0	1	$\frac{3}{4}$	10
0	140	-10	10	0	0	5	40
0	0	0	0	0	0	1	0

.

Drop artificial objective function but keep artificial variable
to determine value of its dual variable.

x_1	v_2	v_3	w_3	s_2	s_3	r_1	
1	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	0	0	$\frac{1}{4}$	2
0	-2	-1	1	1	0	$-\frac{1}{2}$	6
0	$-\frac{13}{2}$	$\mathbf{\frac{17}{2}}$	$-\frac{17}{2}$	0	1	$\frac{3}{4}$	10
0	140	$\mathbf{-10}$	10	0	0	5	40

$\sim$

x_1	v_2	v_3	w_3	s_2	s_3	r_1	
1	$\frac{11}{17}$	0	0	0	$-\frac{3}{17}$	$\frac{2}{17}$	$\frac{4}{17}$
0	$-\frac{47}{17}$	0	0	1	$\frac{2}{17}$	$-\frac{7}{17}$	$\frac{122}{17}$
0	$-\frac{13}{17}$	**1**	-1	0	$\frac{2}{17}$	$\frac{3}{34}$	$\frac{20}{17}$
0	$\frac{2250}{17}$	0	0	0	$\frac{20}{17}$	$\frac{100}{17}$	$\frac{880}{17}$

Example, continued

$$\left[\begin{array}{cccc|cc|c|c} x_1 & v_2 & v_3 & w_3 & s_2 & s_3 & r_1 & \\ \hline 1 & \frac{11}{17} & 0 & 0 & 0 & -\frac{3}{17} & \frac{2}{17} & \frac{4}{17} \\ 0 & -\frac{47}{17} & 0 & 0 & 1 & \frac{2}{17} & -\frac{7}{17} & \frac{122}{17} \\ 0 & -\frac{13}{17} & 1 & -1 & 0 & \frac{2}{17} & \frac{3}{34} & \frac{20}{17} \\ \hline 0 & \frac{2250}{17} & 0 & 0 & 0 & \frac{20}{17} & \frac{100}{17} & \frac{880}{17} \end{array}\right]$$

Optimal solution of MLP is

$$x_1 = \frac{4}{17}, \qquad x_2 = -v_2 = 0,$$

$$x_3 = v_3 - w_3 = \frac{20}{17} - 0 = \frac{20}{17},$$

$$s_2 = \frac{122}{17}, \quad \text{and} \quad s_3 = r_1 = 0$$

with a maximal value $20\left(\frac{4}{17}\right) + 150(0) + 40\left(\frac{20}{17}\right) = \frac{880}{17}$.

Optimal solution for original mLP can be also read off final tableau,

$$\left[\begin{array}{cccc|cc|c|c}
x_1 & v_2 & v_3 & w_3 & s_2 & s_3 & r_1 & \\
\hline
1 & \frac{11}{17} & 0 & 0 & 0 & -\frac{3}{17} & \frac{2}{17} & \frac{4}{17} \\
0 & -\frac{47}{17} & 0 & 0 & 1 & \frac{2}{17} & -\frac{7}{17} & \frac{122}{17} \\
0 & -\frac{13}{17} & 1 & -1 & 0 & \frac{2}{17} & \frac{3}{34} & \frac{20}{17} \\
\hline
0 & \frac{2250}{17} & 0 & 0 & \mathbf{0} & \mathbf{\frac{20}{17}} & \mathbf{\frac{100}{17}} & \frac{880}{17}
\end{array}\right]$$

$$y_1 = \frac{100}{17}, \qquad y_2 = 0, \qquad y_3 = \frac{20}{17},$$

and minimal value

$$8\left(\frac{100}{17}\right) + 10\,(0) + 4\left(\frac{20}{17}\right) = \frac{880}{17}.$$

Example, continued

Alternatively method: First write minimization problem in

with variables ≥ 0 by setting $y_1 = u_1 - z_1$,

$$
\begin{aligned}
\text{Minimize :} \quad & 8u_1 - 8z_1 + 10y_2 + 4y_3 \\
\text{Subject to :} \quad & 4u_1 - 4z_1 + 2y_2 - 3y_3 \geq 20 \\
& 2u_1 - 2z_1 + 3y_2 + 5y_3 \leq 150 \\
& 6u_1 - 6z_1 + 2y_2 + 4y_3 = 40 \\
& u_1 \geq 0,\ z_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0
\end{aligned}
$$

Dual MLP will now have a different tableau than before

but the same solution.

Remark on Tableau with Surplus/Equal

After artificial objective function is zero

drop this row but keep artificial variables

to determine values of dual variables.

Proceed to make all the entries of row for objective function

≥ 0 in columns of x_i, slack and surplus variable,

but allow negative values in artificial variable columns.

Dual variables are entries in row for objective function

in columns of **slack variables** and **artificial variables**.

For pair of surplus and artificial variable columns,

value in artificial variable column is ≤ 0 and

–1 of value in surplus variable column.

Notation for Duality Theorems

MLP: (primal) maximization linear programming problem

Maximize:	$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$
Subject to:	$\sum_j a_{ij} x_j \leq b_i, \ \geq b_i,$ or $= b_i$ for $1 \leq i \leq m$ and
	$x_j \geq 0, \ \leq 0,$ or unrestricted for $1 \leq j \leq n$
Feasible set:	$\mathscr{F}_M$.

mLP: (dual) minimization linear programming problem

Minimize:	$g(\mathbf{y}) = \mathbf{b} \cdot \mathbf{y}$
Subject to:	$\sum_i a_{ij} y_i \geq c_j, \ \leq c_j,$ or $= c_j$ for $1 \leq j \leq n$ and
	$y_i \geq 0, \ \leq 0,$ or unrestricted for $1 \leq i \leq m$
Feasible set:	$\mathscr{F}_m$.

Dual of the minimization problem is the maximization problem.

Weak Duality Theorem

Theorem (Weak Duality Theorem)

Let $\mathbf{x} \in \mathscr{F}_M$ *for MLP and* $\mathbf{y} \in \mathscr{F}_m$ *for mLP, any feasible solutions.*

a. *Then,* $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \leq \mathbf{b} \cdot \mathbf{y} = g(\mathbf{y})$.

Thus, optimal value M *to either problem satisfies*

$\mathbf{c} \cdot \mathbf{x} \leq M \leq \mathbf{b} \cdot \mathbf{y}$ *for any* $\mathbf{x} \in \mathscr{F}_M$ *and* $\mathbf{y} \in \mathscr{F}_m$.

b. $\mathbf{c} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{y}$ *iff* $\mathbf{x}$ *&* $\mathbf{y}$ *satisfy* **complementary slackness**

$$0 = y_j(b_j - a_{j1}x_1 - \cdots - a_{jn}x_n) \qquad \text{for } 1 \leq j \leq m, \qquad \text{and}$$

$$0 = x_i(a_{1i}y_1 + \cdots + a_{mi}y_m - c_i) \qquad \text{for } 1 \leq i \leq n.$$

In matrix notation,

$$0 = \mathbf{y} \cdot (\mathbf{b} - \mathbf{A}\mathbf{x}) \qquad \text{and}$$

$$0 = \mathbf{x} \cdot \left(\mathbf{A}^\top \mathbf{y} - \mathbf{c}\right).$$

Complementary Slackness

For linear programming usually solve by simplex method, row reduction

In nonlinear programming with inequalities often solve

complementary slackness equations,

Karush-Kuhn-Tucker equations

Proof of Weak Duality Theorem

(1) If $\sum_j a_{ij}x_j \leq b_i$ then $y_i \geq 0$ and $y_i(\mathbf{Ax})_i = y_i \sum_j a_{ij}x_j \leq y_i\, b_i$.

If $\sum_j a_{ij}x_j \geq b_i$ then $y_i \leq 0$ and $y_i(\mathbf{Ax})_i = y_i \sum_j a_{ij}x_j \leq y_i\, b_i$.

If $\sum_j a_{ij}x_j = b_i$ then y_i is arb and $y_i(\mathbf{Ax})_i = y_i \sum_j a_{ij}x_j = y_i\, b_i$.

Summing over i

$$\mathbf{y}\cdot\mathbf{Ax} = \sum_i y_i(\mathbf{Ax})_i \leq \sum_i y_i\, b_i = \mathbf{y}\cdot\mathbf{b} \quad \text{or} \quad \mathbf{y}^\top(\mathbf{b} - \mathbf{Ax}) \geq 0.$$

(2) By same type of argument as (1),

$$\mathbf{c}\cdot\mathbf{x} \leq \mathbf{x}\cdot\left(\mathbf{A}^\top\mathbf{y}\right) = \left(\mathbf{A}^\top\mathbf{y}\right)^\top\mathbf{x} = \mathbf{y}^\top(\mathbf{Ax}) = \mathbf{y}\cdot\mathbf{Ax}$$

$$\left(\mathbf{A}^\top\mathbf{y} - \mathbf{c}\right)^\top\mathbf{x} \geq 0.$$

$$\mathbf{c}\cdot\mathbf{x} \leq \mathbf{y}\cdot\mathbf{Ax} \leq \mathbf{y}\cdot\mathbf{b} \qquad \textbf{(a)}$$

(b) $\mathbf{y}\cdot\mathbf{b} - \mathbf{c}\cdot\mathbf{x} = \mathbf{y}^\top(\mathbf{b} - \mathbf{Ax}) + \left(\mathbf{A}^\top\mathbf{y} - \mathbf{c}\right)^\top\mathbf{x} = 0$ iff

$$0 = \left(\mathbf{A}^\top\mathbf{y} - \mathbf{c}\right)\cdot\mathbf{x} \quad \text{and} \quad 0 = \mathbf{y}\cdot(\mathbf{b} - \mathbf{Ax}).$$

QED

Feasibility/Boundedness

Corollary

Assume that MLP and mLP both have feasible solutions.

Then MLP is bounded above and has an optimal solution.

Also, mLP is bounded below and has an optimal solution.

Proof.

If $\mathbf{y}_0 \in \mathscr{F}_m$ and $\mathbf{x} \in \mathscr{F}_M$, then

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \leq \mathbf{b} \cdot \mathbf{y}_0$$

so f is bounded above, and has an optimal solution.

Similarly, if $\mathbf{x}_0 \in \mathscr{F}_M$, and $\mathbf{y} \in \mathscr{F}_m$ then

$$g(\mathbf{y}) = \mathbf{b} \cdot \mathbf{y} \geq \mathbf{c} \cdot \mathbf{x}_0$$

so g is bounded below, and has an optimal solution. □

Necessary Conditions for Optimal Solution

Proposition

If $\bar{\mathbf{x}}$ is an optimal solution for MLP,

then there is a feasible solution $\bar{\mathbf{y}} \in \mathscr{F}_m$ of the dual mLP

that satisfies complementary slackness equations,

1. $\bar{\mathbf{y}} \cdot (\mathbf{b} - \mathbf{A}\bar{\mathbf{x}}) = 0$,
2. $\left(\mathbf{A}^{\top}\bar{\mathbf{y}} - \mathbf{c}\right) \cdot \bar{\mathbf{x}} = 0$.

Similarly, if $\bar{\mathbf{y}} \in \mathscr{F}_m$ is optimal solution of mLP,

then there is a feasible solution $\bar{\mathbf{x}} \in \mathscr{F}_M$ that satisfy 1-2.

Proof is longest of duality arguments.

Prove similar necessary conditions for nonlinear situation.

Proof of Necessary Conditions

Let $\mathbf{E}$ be set of i such that $b_i = \sum_j a_{ij}\bar{x}_j$, i.e. is tight or effective.

Gradient of this constraint is transpose of i^{th}-row of $\mathbf{A}$, $\mathbf{R}_i^\top$,

Let $\mathbf{E}'$ be set of i such that $x_i = 0$ is tight.

$-\mathbf{e}_i = (0, \dots, -1, \dots, 0)^\top$ is negative of gradient

Assume nondegenerate, so gradients of constraints

$\{\mathbf{R}_i^T\}_{i\in\mathbf{E}} \cup \{-\mathbf{e}_i\}_{i\in\mathbf{E}'} = \{\mathbf{w}_i\}_{i\in\mathbf{E}''}$ are linearly independent.

(Otherwise take an appropriate subset in following argument.)

f has a maximum at $\bar{\mathbf{x}}$ on level set for constraints $i \in \mathbf{E} \cup \mathbf{E}'$.

By Lagrange multipliers,

$$\nabla f(\bar{\mathbf{x}}) = \mathbf{c} = \sum_{i\in\mathbf{E}} \bar{y}_i\, \mathbf{R}_j^T - \sum_{i\in\mathbf{E}'} \bar{z}_i\, \mathbf{e}_i$$

By setting $\bar{y}_i = 0$ for $i \notin \mathbf{E}$ and $1 \le i \le m$ and

$\bar{z}_i = 0$ for $i \notin \mathbf{E}'$ and $1 \le i \le n$,

$$\mathbf{c} = \sum_{1\le i\le m} \bar{y}_i\, \mathbf{R}_i^T - \sum_{1\le i\le m} \bar{z}_i\, \mathbf{e}_i = \mathbf{A}^\top \bar{\mathbf{y}} - \bar{\mathbf{z}}. \quad (*)$$

Proof of Necessary Conditions, contin.

Since $\bar{y}_i = 0$ for $b_i - \sum_j a_{ij}\bar{x}_j \neq 0$

$$0 = \bar{y}_i \left(b_i - \sum_j a_{ij}\bar{x}_j\right) \qquad \text{for } 1 \leq i \leq m.$$

Since $\bar{z}_j = 0$ for $\bar{x}_j \neq 0$,

$$0 = \bar{x}_j\,\bar{z}_j \quad 1 \leq j \leq n.$$

In vector-matrix form using (*),

(1) $0 = \bar{\mathbf{y}} \cdot (\mathbf{b} - \mathbf{A}\bar{\mathbf{x}})$

(2) $0 = \bar{\mathbf{x}} \cdot \bar{\mathbf{z}} = \left(\mathbf{A}^\top \bar{\mathbf{y}} - \mathbf{c}\right) \cdot \bar{\mathbf{x}}$

Still need (4): (i) $\bar{y}_i \geq 0$ for resource constraint,

(ii) $\bar{y}_i \leq 0$ for requirement constraint,

(iii) $\bar{y}_i$ is unrestricted for equality constraint,

(iv) $\bar{z}_j = \sum_i a_{ij}\bar{y}_i - c_j \geq 0$ for $x_j \geq 0$, (v) $\bar{z}_j \leq 0$ for $x_j \leq 0$, and

(vi) $\bar{z}_j = 0$ for x_j unrestricted.

Proof of Necessary Conditions, contin.

$\{\mathbf{R}_i^T\}_{i \in \mathbf{E}} \cup \{\text{-}\mathbf{e}_i\}_{i \in \mathbf{E}'} = \{\mathbf{w}_k\}_{k \in \mathbf{E}''}$ are linearly independent

Complete to a basis of $\mathbb{R}^n$ using vectors perp to these first vectors.

$\mathbf{W} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ and $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ s.t. $\mathbf{W}^\top \mathbf{V} = \mathbf{I}$.

Column $\mathbf{v}_k$ perp to $\mathbf{w}_i$ except for $i = k$.

$\mathbf{c} = \sum_{1 \le i \le m} \bar{y}_i \, \mathbf{R}_i^T - \sum_{1 \le i \le n} \bar{z}_i \, \mathbf{e}_i = \sum_j p_j \, \mathbf{w}_j$ where $p_j = \bar{y}_i, \bar{z}_i,$ or 0

$\mathbf{c} \cdot \mathbf{v}_k = (\sum_j p_j \, \mathbf{w}_j) \cdot \mathbf{v}_k = p_k$

Take $i \in \mathbf{E}$. Gradient of this constraint is $\mathbf{R}_i^\top = \mathbf{w}_k$ for some $k \in \mathbf{E}''$.

Set $\delta = \text{-}1$ for resource and $\delta = +1$ for requirement constraints

$\delta \, \mathbf{R}_i^\top$ points into $\mathscr{F}_M$ (unless $=$ constraint).

For small $t \ge 0$, $\bar{\mathbf{x}} + t \, \delta \, \mathbf{v}_k \in \mathscr{F}_M$

$$0 \le f(\bar{\mathbf{x}}) - f(\bar{\mathbf{x}} + t \, \delta \, \mathbf{v}_k) = \text{-}t \, \delta \, \mathbf{c} \cdot \mathbf{v}_k = \text{-}t \, \delta \, p_k,$$

$\text{-}\delta \, p_k \ge 0$, or $\bar{y}_i \ge 0$ for resource and ≤ 0 for requirement constraint

Can't move off for equality constraint, so y_i unrestricted

Proof of Necessary Conditions, contin.

Take $i \in \mathbf{E}'$. $-\mathbf{e}_i = \mathbf{w}_k$ for some $k \in \mathbf{E}''$.

Set $\delta = -1$ if $x_i \geq 0$ and $\delta = 1$ if $x_i \leq 0$

$\delta\, \mathbf{w}_k = -\delta\, \mathbf{e}_i$ points into $\mathscr{F}_M$ (unless x_i unrestricted).

By argument as before, $-\delta\, p_k = -\delta\, \bar{z}_i \geq 0$.

Therefore $\bar{z}_i \geq 0$ if $x_i \geq 0$ and $\bar{z}_i \leq 0$ if $x_i \leq 0$

If x_i is unrestricted, then the equation is not tight and $\bar{z}_i = 0$.

This proves $\bar{\mathbf{y}} \in \mathscr{F}_m$ and satisfies complementary slackness (1) and (2)

QED

Optimality and Complementary Slackness

Corollary

Assume that $\bar{\mathbf{x}} \in \mathscr{F}_M$ *is a feasible solution for primal MLP and* $\bar{\mathbf{y}} \in \mathscr{F}_m$ *is a feasible solution of dual mLP. Then the following are equivalent.*

a. $\bar{\mathbf{x}}$ *is an optimal solution of MLP and* $\bar{\mathbf{y}}$ *is an optimal solution of mLP.*

b. $\mathbf{c} \cdot \bar{\mathbf{x}} = \mathbf{b} \cdot \bar{\mathbf{y}}$.

c. $\mathbf{0} = \bar{\mathbf{x}} \cdot (\mathbf{c} - \mathbf{A}^\top \bar{\mathbf{y}})$ *and* $\mathbf{0} = (\mathbf{b} - \mathbf{A}\bar{\mathbf{x}}) \cdot \bar{\mathbf{y}}$.

Proof.

$(\mathbf{b} \Leftrightarrow \mathbf{c})$ Restatement of Weak Duality Theorem.

$(\mathbf{a} \Rightarrow \mathbf{c})$ By proposition, $\exists\ \bar{\mathbf{y}}'$ that satisfies complementary slackness

By Weak Duality Theorem, $\mathbf{c} \cdot \bar{\mathbf{x}} = \mathbf{b} \cdot \bar{\mathbf{y}}'$.

So, $\mathbf{c} \cdot \bar{\mathbf{x}} = \mathbf{b} \cdot \bar{\mathbf{y}}' \geq \mathbf{b} \cdot \bar{\mathbf{y}} \geq \mathbf{c} \cdot \bar{\mathbf{x}}$.

By Weak Duality Theorem, $\bar{\mathbf{y}}$ satisfies complementary slackness. □

Proof of Corollary, contin.

Proof.

($\mathbf{b} \Rightarrow \mathbf{a}$) If $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ satisfy $\mathbf{c} \cdot \bar{\mathbf{x}} = \mathbf{b} \cdot \bar{\mathbf{y}}$, then

for any $\mathbf{x} \in \mathscr{F}_M$ & $\mathbf{y} \in \mathscr{F}_m$,

$$\mathbf{c} \cdot \mathbf{x} \leq \mathbf{b} \cdot \bar{\mathbf{y}} = \mathbf{c} \cdot \bar{\mathbf{x}} \leq \mathbf{b} \cdot \mathbf{y}$$

$\bar{\mathbf{x}}$ & $\bar{\mathbf{y}}$ must be optimal solutions. □

Duality Theorem

Theorem

Consider two dual problems MLP and mLP.

Then, MLP has an optimal sol'n iff dual mLP has an optimal sol'n.

Proof.

If MLP has an optimal sol'n $\bar{\mathbf{x}}$,

then mLP has feasible sol'n $\bar{\mathbf{y}}$ that satisfies complementary slackness.

By Corollary, $\bar{\mathbf{y}}$ is optimal sol'n of mLP.

Converse is similar □

Duality and Tableau

Theorem

If either MLP or mLP is solved for an optimal sol'n by simplex method, then sol'n of its dual LP is displayed in bottom row of final optimal tableau in the columns associated with slack and artificial variables. (not surplus)

Proof:

Start with MLP. To solve by tableau, need $\mathbf{x} \geq 0$.

Group equations into resource, requirement, and equality constraints.

so tableau for MLP

$$\left[\begin{array}{c|cc|cc|c} \mathbf{A}_1 & \mathbf{I}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_1 \\ \mathbf{A}_2 & \mathbf{0} & -\mathbf{I}_2 & \mathbf{I}_2 & \mathbf{0} & \mathbf{b}_2 \\ \mathbf{A}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{b}_3 \\ \hline -\mathbf{c}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \end{array}\right]$$

Row operations to final tableau realized by by matrix multiplication

$$\left[\begin{array}{ccc|c} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 & \mathbf{0} \\ \hline \bar{\mathbf{y}}_1^\top & \bar{\mathbf{y}}_2^\top & \bar{\mathbf{y}}_2^\top & 1 \end{array}\right] \left[\begin{array}{c|cc|cc|c} \mathbf{A}_1 & \mathbf{I}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_1 \\ \mathbf{A}_2 & \mathbf{0} & -\mathbf{I}_2 & \mathbf{I}_2 & \mathbf{0} & \mathbf{b}_2 \\ \mathbf{A}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{b}_3 \\ \hline -\mathbf{c}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|cc|cc|c} \mathbf{M}_1\mathbf{A}_1 + \mathbf{M}_2\mathbf{A}_2 + \mathbf{M}_3\mathbf{A}_3 & \mathbf{M}_1 & -\mathbf{M}_2 & \mathbf{M}_2 & \mathbf{M}_3 & \mathbf{M}\mathbf{b} \\ \hline \bar{\mathbf{y}}_1^\top\mathbf{A}_1 + \bar{\mathbf{y}}_2^\top\mathbf{A}_2 + \bar{\mathbf{y}}_3^\top\mathbf{A}_3 - \mathbf{c}^\top & \bar{\mathbf{y}}_1^\top & -\bar{\mathbf{y}}_2^\top & \bar{\mathbf{y}}_2^\top & \bar{\mathbf{y}}_3^\top & \bar{\mathbf{y}}^\top\mathbf{b} \end{array}\right]$$

Obj fn row is not added to the other rows so last column $= (\mathbf{0}, 1)^\top$,

In final tableau, entries in objective function row ≥ 0,

except for artificial variable columns, so

$$\mathbf{A}^\top\bar{\mathbf{y}} - \mathbf{c} = \left(\bar{\mathbf{y}}^\top\mathbf{A} - \mathbf{c}^\top\right)^\top \geq \mathbf{0}, \;\; \bar{\mathbf{y}}_1 \geq \mathbf{0}, \;\; \bar{\mathbf{y}}_2 \leq \mathbf{0}, \;\; \text{so} \qquad \bar{\mathbf{y}} \in \mathscr{F}_m$$

$\mathbf{c} \cdot \mathbf{x}_{\max} = \bar{\mathbf{y}}^\top\mathbf{b} = \mathbf{b} \cdot \bar{\mathbf{y}}$ so $\bar{\mathbf{y}}$ is minimizer by Optimality Corollary.

Proof continued

Note: $\left(\mathbf{A}^\top \bar{\mathbf{y}}\right)_i = \mathbf{L}_i \cdot \bar{\mathbf{y}}$ $\quad$ $\mathbf{L}_i$ i^{th}-column of $\mathbf{A}$

If $x_i \leq 0$, set $\xi_i = -x_i \geq 0$.

Column in tableau, -1 original column, and new obj fn coef $-c_i$.

Now have $0 \leq (-\mathbf{L}_i) \cdot \bar{\mathbf{y}} - (-c_i)$,

$\mathbf{L}_i \cdot \bar{\mathbf{y}} \leq c_i$ $\quad$ resource constraint of dual.

If x_i arbitrary, set $x_i = \xi_i - \eta_i$.

Then get both $\mathbf{L}_i \cdot \bar{\mathbf{y}} \geq c_i$ & $\mathbf{L}_i \cdot \bar{\mathbf{y}} \leq c_i$,

$\mathbf{L}_i \cdot \bar{\mathbf{y}} = c_i$, $\quad$ equality constraint of dual.

QED

Sensitivity Analysis

Sensitive analysis concerns the extent to which more of a resource would increase the maximum value of a MLP.

Example. In short run,

	in stock	product 1 per item	product 2 per item
profit		\$40	\$10
units of paint	1020	15	10
fasteners	400	10	2
hours of labor	420	3	5

$$\left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 15 & 10 & 1 & 0 & 0 & 1020 \\ \mathbf{10} & 2 & 0 & 1 & 0 & 400 \\ 3 & 5 & 0 & 0 & 1 & 420 \\ \hline \mathbf{-40} & -10 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 0 & 7 & 1 & -1.5 & 0 & 420 \\ \mathbf{1} & .2 & 0 & .1 & 0 & 40 \\ 0 & 4.4 & 0 & -.3 & 1 & 300 \\ \hline 0 & -2 & 0 & 4 & 0 & 1600 \end{array}\right]$$

Sensitivity Analysis, continued

$$\left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 0 & 7 & 1 & -1.5 & 0 & 420 \\ 1 & .2 & 0 & .1 & 0 & 40 \\ 0 & 4.4 & 0 & -.3 & 1 & 300 \\ \hline 0 & -2 & 0 & 4 & 0 & 1600 \end{array}\right] \sim \left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 0 & 1 & \frac{1}{7} & -\frac{3}{14} & 0 & 60 \\ 1 & 0 & -\frac{1}{35} & \frac{1}{7} & 0 & 28 \\ 0 & 0 & -\frac{22}{35} & \frac{9}{14} & 1 & 36 \\ \hline 0 & 0 & \frac{2}{7} & \frac{25}{7} & 0 & 1720 \end{array}\right]$$

Optimal solution is

$$x_1 = 28, \quad x_2 = 60, \quad s_1 = 0, \quad s_2 = 0, \quad s_3 = 36,$$

with optimal profit of 1720.

Sensitivity Analysis, continued

$$\left[\begin{array}{cc|ccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline 0 & 1 & \frac{1}{7} & -\frac{3}{14} & 0 & 60 \\ 1 & 0 & -\frac{1}{35} & \frac{1}{7} & 0 & 28 \\ 0 & 0 & -\frac{22}{35} & \frac{9}{14} & 1 & 36 \\ \hline 0 & 0 & \frac{2}{7} & \frac{25}{7} & 0 & 1720 \end{array}\right]$$

Values of an increase of constrained quantities are

$y_1 = \frac{2}{7}$, $y_2 = \frac{25}{7}$,

$y_3 = 0$ for quantity that is not tight.

Increase is largest for second constraint, limitation on fasteners.

Next consider range that b_2 can be increased

while keeping same basic variables with $s_1 = s_2 = 0$.

Let δ_2 be change in 2nd resource (fasteners):

$$10x_1 + 2x_2 + s_2 = 400 + \delta_2 \qquad \text{starting form of constraint.}$$

s_2 and δ_2 play similar roles (and have similar units),

so the new final tableau adds

δ_2 times s_2-column to right side of equalities.

Still need $x_1, x_2, s_3 \geq 0$,

$$0 \leq x_2 = 60 - \tfrac{3\,\delta_2}{14} \qquad \text{or} \qquad \delta_2 \leq 60 \cdot \tfrac{14}{3} = 280,$$

$$0 \leq x_1 = 28 + \tfrac{\delta_2}{7} \qquad \text{or} \qquad \delta_2 \geq -28 \cdot 7 = -196,$$

$$0 \leq s_3 = 36 + \tfrac{9\,\delta_2}{14} \qquad \text{or} \qquad \delta_2 \geq -36 \cdot \tfrac{14}{9} = -56;$$

$$-56 \leq \delta_2 \leq 280.$$

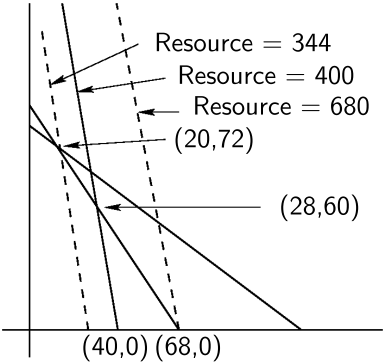
Sensitivity Analysis, continued

Resource can be incr at most 280 units and decr at most 56 units, or

$$-56 \leq \delta_2 \leq 280$$

$$344 = 400 - 56 \leq b_2 \leq 400 + 280 = 680.$$

For this range, x_1, x_2, and s_3 are still basic variables.



Sensitivity for Change of Constraint Constants

For $\delta_2 = 280$,

$$x_1 = 28 + 280 \cdot \tfrac{1}{7} = 68,$$

$$x_2 = 60 - 280 \cdot \tfrac{3}{14} = 0,$$

$$s_3 = 36 + 280 \cdot \tfrac{9}{14} = 216,$$

$$z = 1720 + 280 \cdot \tfrac{25}{7} = 2720 \qquad \text{is optimal value.}$$

For $\delta_2 = -56$,

$$x_1 = 28 - 56 \cdot \tfrac{1}{7} = 20,$$

$$x_2 = 60 + 56 \cdot \tfrac{3}{14} = 72,$$

$$s_3 = 36 - 56 \cdot \tfrac{9}{14} = 0,$$

$$z = 1720 - 56 \cdot \tfrac{25}{7} = 1520 \qquad \text{is optimal value.}$$

General Changes in Tight Constraint

Use optimal (final) tableau that gives maximum

b_i' entry of i^{th}-row of constants in right hand column

$c_j' \geq 0$ entry in j^{th}-column of objective row

a_{ij}' entry in i^{th}-row and j^{th}-column

exclude right side constants and any artificial variable columns.

$\mathbf{C}_j'$ j^{th}-column of $\mathbf{A}'$ (note capital and bold, not c_j)

$\mathbf{R}_i$ i^{th}-row of $\mathbf{A}'$

General Changes in Constraint, contin.

Change in tight r^{th}- resource constraint, $b_r + \delta_r$

Assume that s_r is in k^{th}-column

$$\left[\begin{array}{c|c|c} & s_r & \\ \hline \mathbf{A} & \mathbf{e}^r & \mathbf{b} + \delta_r \mathbf{e}^r \\ \hline -\mathbf{c}^T & 0 & 0 \end{array}\right] \sim \left[\begin{array}{c|c|c} & s_r & \\ \hline \mathbf{A}' & \mathbf{C}'_k & \mathbf{b}' + \delta_r \mathbf{C}'_k \\ \hline \mathbf{c}'^T & c'_k & M + \delta_r c'_k \end{array}\right]$$

z_i basic variable with pivot in i^{th}-row, need $0 \leq z_i = b'_i + \delta_r a'_{ik}$.

For $a'_{ik} < 0$, need $-\delta_r a'_{ik} \leq b'_i$, so

$$\delta_r \leq \min_i \left\{ \frac{b'_i}{-a'_{ik}} : a'_{ik} < 0 \right\}, \quad k^{\text{th}}\text{-column for } s_r$$

For $a'_{ik} > 0$, need $-b'_i \leq \delta_r a'_{ik}$, so

$$-\min_i \left\{ \frac{b'_i}{a'_{ik}} : a'_{ik} > 0 \right\} \leq \delta_r, \quad k^{\text{th}}\text{-column for } s_r.$$

Change in optimal value for δ_r in allowable range: $\delta_r c'_k$

Change in Slack Constraint Constant

Let s_r be for a pivot column in optimal tableau for a slack r^{th}-resource.

To keep same basic variables, need changed amount $b'_r + \delta_r \geq 0$

$\delta_r \geq -b'_r$.

b_r can be increased by an arbitrary amount.

For δ_r is this range, optimal value is unchanged.

Sensitivity Analysis, continued

$$\left[\begin{array}{cc|ccc|c} x_1 & x_2 & \mathbf{s_1} & s_2 & s_3 & \\ \hline 0 & 1 & \mathbf{\frac{1}{7}} & -\frac{3}{14} & 0 & 60 \\ 1 & 0 & -\mathbf{\frac{1}{35}} & \frac{1}{7} & 0 & 28 \\ 0 & 0 & -\mathbf{\frac{22}{35}} & \frac{9}{14} & 1 & 36 \\ \hline 0 & 0 & \mathbf{\frac{2}{7}} & \frac{25}{7} & 0 & 1720 \end{array}\right]$$

Allowable δ_1 for first resource,

$$\delta_1 \le \min\left\{ 28 \cdot \tfrac{35}{1},\ 36 \cdot \tfrac{35}{22} \right\} = \min\left\{ 980,\ 57.27 \right\} = 57.27.$$

$$\delta_1 \ge -\min\left\{ 60 \cdot \tfrac{7}{1} \right\} = -420.$$

Change of optimal value $1720 + \frac{2}{7} \cdot \delta_1$

Allowable δ_3

$$\delta_3 \ge -36$$

Changes in Objective Function Coefficients

For a change from c_1 to $c_1 + \Delta_1$, changes in tableaux

x_1	x_2	s_1	s_2	s_3	
15	10	1	0	0	1020
10	2	0	1	0	400
3	5	0	0	1	420
$-40-\Delta_1$	-10	0	0	0	0

$\sim$

x_1	x_2	s_1	s_2	s_3	
0	1	$\frac{1}{7}$	$-\frac{3}{14}$	0	60
1	0	$-\frac{1}{35}$	$\frac{1}{7}$	0	28
0	0	$-\frac{22}{35}$	$\frac{9}{14}$	1	36
$-\Delta_1$	0	$\frac{2}{7}$	$\frac{25}{7}$	0	1720

$\sim$

x_1	x_2	s_1	s_2	s_3	
$*$	$*$	$*$	$*$	$*$	$*$
0	0	$\frac{2}{7} - \frac{1}{35}\Delta_1$	$\frac{25}{7} + \frac{1}{7}\Delta_1$	0	$1720 + 28\,\Delta_1$

Changes in Coefficients, contin.

To keep objective function row ≥ 0, need

$$\tfrac{2}{7} - \tfrac{1}{35}\Delta_1 \geq 0 \;\;\&\;\; \tfrac{25}{7} + \tfrac{1}{7}\Delta_1 \geq 0$$

$$\Delta_1 \leq \tfrac{2}{7}\,\tfrac{35}{1} = 10 \;\;\&\;\; \Delta_1 \geq -\tfrac{25}{7} \cdot \tfrac{7}{1} = -25$$

$$15 = 40 - 25 \leq c_1 + \Delta_1 \leq 40 + 10 = 50.$$

Corresponding optimal value of objective function is $1720 + 28\,\Delta_1$

General Changes in Objective Function Coefficients

Consider change Δ_k in coefficient c_k of basic x_k in obj fn,

where x_k is a basic in optimal solution and its pivot is in r^{th} row.

a'_{rj} denote the entries in r^{th} row and j^{th} column,

excluding right side constants and any artificial variable columns.

$c'_j \geq 0$ denote entries in the objective row of optimal tableau

With change, original entry in the Obj Rn row becomes $-c_k - \Delta_k$.

Entry in optimal tableau changes from 0 to $-\Delta_k$

To keep x_k basic, need to add $\Delta_k \mathbf{R}'_r$ to Obj Fn row.

Entry in x_k-column is now 0 and j^{th}-column is $c'_j + \Delta_k a'_{rj}$

For all j, need $c'_j + \Delta_k a'_{rj} \geq 0$.

Changes in Coefficients, contin.

$c_k + \Delta_k$ for basic variable with r^{th}-pivot row, $a'_{rk} = 1$ pivot.

For $a'_{rj} > 0$ in r^{th}-pivot row, need $\Delta_k\, a'_{rj} \geq -c'_j$ or $\Delta_k \geq -\dfrac{c'_j}{a'_{rj}}$,

$$\Delta_k \geq -\min_j \left\{ \frac{c'_j}{a'_{rj}} : a'_{rj} > 0,\ j \neq k \right\}, \text{ maximal decrease of } c_k.$$

For $a'_{rj} < 0$ in r^{th}-pivot row, need $c'_j \geq -a'_{rj}\,\Delta_k$ or $\dfrac{c'_j}{-a'_{rj}} \geq \Delta_k$,

$$\Delta_k \leq \min_j \left\{ \frac{c'_j}{-a'_{rj}} : a'_{rj} < 0 \right\}, \text{ maximal increase of } c_k.$$

If $c'_j = 0$ for $a'_{rj} < 0$, then need $\Delta_k \leq 0$.

If $c'_j = 0$ for $a'_{rj} > 0$ & $j \neq k$, then need $\Delta_k \geq 0$.

Change in optimal value is $\Delta_k b'_r$

Our example does not have any non-basic variables, x_k.

If x_k were a non-basic variable in optimal solution, $x_k = 0$,

then $c'_k + \Delta_k \geq 0$ insures that x_r is a non-basic variable,

$\Delta_k \geq -c'_k$.

i.e., $-c'_k \leq 0$ is min decrease needed to make x_k a basic variable

and a positive contribution to optimal solution.

Weighted averages in $\mathbb{R}^n$:

For three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$,

$$\frac{\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3}{3}$$

is average of each component, average of these vectors.

$$\frac{\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_3 + \mathbf{a}_3}{6} = \frac{\mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3}{6}$$
$$= \tfrac{1}{6}\mathbf{a}_1 + \tfrac{2}{6}\mathbf{a}_2 + \tfrac{3}{6}\mathbf{a}_3$$

is a weighted average of these vectors with weights $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{3}{6}$.

For vectors $\{\mathbf{a}_i\}_{i=1}^k$ and numbers $\sum_{i=1}^k t_i = 1$ with $t_i \geq 0$

$$\sum_{i=1}^k t_i\, \mathbf{a}_i$$

is a weighted average, and is called a **convex combination** of $\{\mathbf{a}_i\}$.

Convex Sets

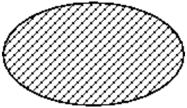
Definition

A set $\mathbf{S} \subset \mathbb{R}^n$ is **convex** provided that

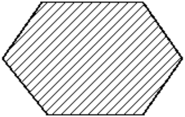
if $\mathbf{x}_0$ and $\mathbf{x}_1$ are any two points in $\mathbf{S}$ then convex combination

$\mathbf{x}_t = (1-t)\mathbf{x}_0 + t\mathbf{x}_1$ is also in $\mathbf{S}$ for all $0 \leq t \leq 1$,

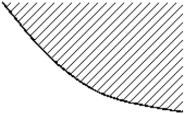
i.e., line segment from $\mathbf{x}_0$ to $\mathbf{x}_1$ in $\mathbf{S}$.



convex

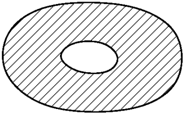


convex



convex

not convex



not convex

Convex Sets, contin.

Each constraint, $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ or $\geq b_i$,

or $x_i \geq 0$, defines a closed half-space in $\mathbb{R}^n$.

$a_{i1}x_1 + \cdots + a_{in}x_n = b_i$ is a hyperplane $(n - 1$ dimensional).

Definition

Any intersection of a finite number of closed half-spaces

and possibly some hyperplanes is called a **polyhedron**.

Convex Sets, contin.

Theorem

a. *Intersection of convex sets,* $\bigcap_j \mathbf{S}_j$*, is convex.*

b. *A polyhedron is convex.* *So feasible set of any LP is convex.*

Proof.

(a) If $\mathbf{x}_0, \mathbf{x}_1 \in \mathbf{S}_j$ & $0 \le t \le 1$,

then $(1-t)\,\mathbf{x}_0 + t\,\mathbf{x}_1 \in \mathbf{S}_j \;\forall\; j$, and

$(1-t)\,\mathbf{x}_0 + t\,\mathbf{x}_1 \in \bigcap_j \mathbf{S}_j$.

(b) Each closed half space & hyperplane is convex,

so intersection is convex. □

Convex Sets, contin.

Theorem

If $\mathbf{S}$ *is a convex set, and* $\mathbf{p}_i \in \mathbf{S}$ *for* $1 \le i \le k$*,*

then any convex combination $\sum_{i=1}^{k} t_i \, \mathbf{p}_i \in \mathbf{S}$*.*

Proof.

Proof is by induction.

For $k = 2$, it follows from the def'n of convex set.

Assume true for $k - 1 \ge 2$.

If $t_k = 1$ & $t_j = 0$ for $1 \le j < k$, then clear.

If $t_k < 1$, then $\sum_{i=1}^{k-1} t_i = 1 - t_k > 0$, & $\sum_{i=1}^{k-1} \frac{t_i}{1-t_k} = 1$.

$\sum_{i=1}^{k-1} \frac{t_i}{1-t_k} \, \mathbf{p}_i \in \mathbf{S}$ by induction hypothesis

So, $\sum_{i=1}^{k} t_i \, \mathbf{p}_i = (1 - t_k) \sum_{i=1}^{k-1} \frac{t_i}{1-t_k} \, \mathbf{p}_i + t_k \, \mathbf{p}_k \in \mathbf{S}$. □

Extreme Points and Vertices

Definition

A point $\mathbf{p}$ in a nonempty convex set $\mathbf{S}$ is called an **extreme point** p.t.

if $\mathbf{p} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1$ with $\mathbf{x}_0$, $\mathbf{x}_1$ in $\mathbf{S}$ and $0 < t < 1$,

then $\mathbf{p} = \mathbf{x}_0 = \mathbf{x}_1$.

An extreme point in a polyhedron is called a **vertex**.

An extreme point of $\mathbf{S}$ must be a boundary point of $\mathbf{S}$.

Disk $\mathbf{D} = \{\, \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1 \,\}$ is convex.

Each point on its boundary is an extreme point.

For next few theorems, consider feasible set

$$\mathscr{F} = \{\, \mathbf{x} \in \mathbb{R}^{n+m}_{+} : \mathbf{A}\mathbf{x} = \mathbf{b} \,\}$$

with slack and surplus variables included in $\mathbf{x}$'s and constraints.

Vertices

Theorem

Let $\mathscr{F} = \{\, \mathbf{x} \in \mathbb{R}^{n+m}_{+} : \mathbf{A}\mathbf{x} = \mathbf{b} \,\}$*.*
$\mathbf{x} \in \mathscr{F}$ *is a vertex of* $\mathscr{F}$ *if and only if basic feasible solution,*
i.e., columns of $\mathbf{A}$ *with* $x_j > 0$ *linearly independent set of vectors.*

Proof:
By reindexing columns and variables, can assume that

$$x_1 > 0, \ldots, x_r > 0 \quad x_{r+1} = \cdots = x_{n+m} = 0.$$

(a) Assume $\{\, \mathbf{A}_j \,\}_{j=1}^{r}$ linearly dependent: $\exists\ (\beta_1, \ldots, \beta_r) \neq \mathbf{0}$
$\beta_1 \mathbf{A}_1 + \cdots + \beta_r \mathbf{A}_r = \mathbf{0}$.
For $\boldsymbol{\beta}^{\top} = (\beta_1, \ldots, \beta_r, 0, \ldots, 0)$, $\mathbf{A}\boldsymbol{\beta} = \mathbf{0}$,

For small λ, $\mathbf{w}_1 = \mathbf{x} + \lambda\boldsymbol{\beta} \geq \mathbf{0}$, & $\mathbf{w}_2 = \mathbf{x} - \lambda\boldsymbol{\beta} \geq \mathbf{0}$, $\mathbf{A}\mathbf{w}_i = \mathbf{A}\mathbf{x} = \mathbf{b}$.
so $\mathbf{w}_1, \mathbf{w}_2 \in \mathscr{F}$ & $\mathbf{x} = \frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2$, so not vertex.

(b) Conversely, assume that $\mathbf{x} \in \mathscr{F}$ is not vertex,

$$\mathbf{x} = t\,\mathbf{y} + (1-t)\,\mathbf{z} \qquad \text{for} \;\; 0 < t < 1,$$

with $\mathbf{y} \neq \mathbf{z}$ in $\mathscr{F}$.

For $r < j$,

$$0 = x_j = t\,y_j + (1-t)\,z_j.$$

Since both $y_j \geq 0$ and $z_j \geq 0$, both must be zero for $j > r$.

Because $\mathbf{y} \neq \mathbf{z}$ are both in $\mathscr{F}$,

$$\mathbf{b} = \mathbf{A}\mathbf{y} = y_1\,\mathbf{A}_1 + \cdots + y_r\,\mathbf{A}_r,$$

$$\mathbf{b} = \mathbf{A}\mathbf{z} = z_1\,\mathbf{A}_1 + \cdots + z_r\,\mathbf{A}_r,$$

$$\mathbf{0} = (y_1 - z_1)\,\mathbf{A}_1 + \cdots + (y_r - z_r)\,\mathbf{A}_r,$$

and columns $\{\,\mathbf{A}_j\,\}_{j=1}^{r}$ are linearly dependent. **QED**

Some Vertex is an Optimal Solution

For any convex combination

$$f\left(\sum t_j \mathbf{x}_j\right) = \mathbf{c} \cdot \sum t_j \mathbf{x}_j = \sum t_j \mathbf{c} \cdot \mathbf{x}_j = \sum t_j f(\mathbf{x}_j).$$

Theorem

Assume that $\mathscr{F} = \{\, \mathbf{x} \in \mathbb{R}_+^{n+m} : \mathbf{A}\mathbf{x} = \mathbf{b} \,\} \neq \emptyset$ *for bounded MLP.*

Then following hold.

a. *If* $\mathbf{x}^0 \in \mathscr{F}$, *then there exists a basic feasible* $\mathbf{x}^b \in \mathscr{F}$ *s.t.*

$$f(\mathbf{x}^b) = \mathbf{c} \cdot \mathbf{x}^b \geq \mathbf{c} \cdot \mathbf{x}^0 = f(\mathbf{x}^0).$$

b. *There is at least one optimal basic solution.*

c. *If two or more basic solutions are optimal, then*

any convex combination of them is also an optimal solution.

Proof a.

If $\mathbf{x}^0$ is already a basic feasible sol'n then done.

Otherwise, columns $\mathbf{A}$ for $x_i^0 > 0$ are lin. depen.

Let $\mathbf{A}'$ be matrix with only these columns.

$\exists$ $\mathbf{y}' \neq \mathbf{0}$ such that $\mathbf{A}'\mathbf{y}' = \mathbf{0}$.

Adding 0 in other entries, get $\mathbf{y} \neq \mathbf{0}$ s.t. $\mathbf{A}\mathbf{y} = \mathbf{0}$.

$\mathbf{A}(-\mathbf{y}) = \mathbf{0}$, so can assume that $\mathbf{c} \cdot \mathbf{y} \geq 0$.

$\mathbf{A}\left[\mathbf{x}^0 + t\,\mathbf{y}\right] = \mathbf{A}\,\mathbf{x}^0 = \mathbf{b}$,

If $y_i \neq 0$, then $x_i^0 > 0$, so $\mathbf{x}^0 + t\,\mathbf{y} \geq 0$ for small t is in $\mathscr{F}$.

Proof a, contin.

Case 1. Assume that $\mathbf{c} \cdot \mathbf{y} > 0$ and some component $y_i < 0$.

$x_i^0 > 0$ and $x_i^0 + t\, y_i = 0$ for $t_i = -\frac{x_i^0}{y_i} > 0$.

As t increases from 0 to t_i, objective function increases from

$\mathbf{c} \cdot \mathbf{x}^0$ to $\mathbf{c} \cdot [\mathbf{x}^0 + t_i \mathbf{y}]$.

If more than one $y_i < 0$, then select one with smallest t_i.

Have constructed point in $\mathscr{F}$ with one more zero component of $\mathbf{x}^0$,

fewer components $y_i < 0$,

and a greater value of objective function.

Can continue until either columns are linearly independent or

all $y_i \geq 0$.

Proof a, continued

Case 2. If $\mathbf{c} \cdot \mathbf{y} > 0$ and $\mathbf{y} \geq \mathbf{0}$,

then $\mathbf{x}^0 + t\,\mathbf{y} \in \mathscr{F}$ for all $t > 0$,

$\mathbf{c} \cdot [\mathbf{x}^0 + t\,\mathbf{y}] = \mathbf{c} \cdot \mathbf{x}^0 + t\,\mathbf{c} \cdot \mathbf{y}$ is arbitrarily large.

MLP is unbounded and has no maximum, contradiction.

Case 3. If $\mathbf{c} \cdot \mathbf{y} = 0$: $f(\mathbf{x}^0 + t\,\mathbf{y}) = \mathbf{c} \cdot \mathbf{x}^0$ unchanged

Some $y_i \neq 0$. Considering $\mathbf{y}$ & $-\mathbf{y}$ can assume some $y_i < 0$.

$\exists$ first $t_i > 0$, to make another $x_i^0 + t_i y_i = 0$.

Eventually, get corresponding columns linearly independent, and
a basic solution as claimed in part (**a**).

Proof b,c

(b) Only finitely many basic feasible solutions, $\{\mathbf{p}_j\}_{j=1}^{N}$.

$f(\mathbf{x}) \leq \max_{1 \leq j \leq N} f(\mathbf{p}_j)$ for $\mathbf{x} \in \mathscr{F}$ by part (a)

Maximum can be found among $f(\mathbf{p}_j)$.

(c) Assume $f(\mathbf{p}_{j_i}) = M = \max\{ f(\mathbf{x}) : \mathbf{x} \in \mathscr{F} \}$ for $i = 1, \dots, \ell > 1$.

$$\sum_{i=1}^{\ell} t_{j_i} \mathbf{p}_{j_i} \in \mathscr{F}$$

$$f\left(\sum_{i=1}^{\ell} t_{j_i} \mathbf{p}_{j_i}\right) = \sum_{i=1}^{\ell} t_{j_i} f(\mathbf{p}_{j_i}) = \sum_{i=1}^{\ell} t_{j_i} M = M. \quad \text{optimal}$$

QED

Validation of Simplex Method

If $\exists$ degen. basic feasible sol'ns with fewer than m positive basic var, then simplex method can cycle by row reduction to matrices with same positive basic variables but different sets of pivots: interchange one zero basic variable with zero free variable. Same vertex of feasible set

Need to insure don't repeat same set of basic variables at a vertex (cycle)

Theorem

If a maximum solution exists for a linear programming problem and simplex algorithm does not cycle among degen basic feasible sol'ns, then simplex algorithm locates a maximum solution in finitely many steps.

Sketch of Proof of Theorem

Assume never reach a degenerate basic solution.

Then reach $\mathbf{p}_0$ with all pivoting to $\mathbf{p}_1, \dots, \mathbf{p}_k$ have $f(\mathbf{p}_j) \leq f(\mathbf{p}_0)$

Complete to set of all basic feasible sol'ns (vertices) $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_\ell$

Set of all convex combinations, convex hull, is bounded polyhedron

$\mathbf{H} = \{ \sum_{i=0}^{\ell} t_i \mathbf{p}_i : t_i \geq 0, \ \sum_{i=0}^{\ell} t_i = 1 \} \subset \mathscr{F}$.

Edge of $\mathbf{H}$ from $\mathbf{p}_i$ to $\mathbf{p}_j$ corresponds to pivoting

(as in proof of Theorem 3.4.2(a)) where

one constraint becomes $\neq b_i$ and another become $= b_j$,

Positive cone out from $\mathbf{p}_0$ determined by $\{ \mathbf{p}_i - \mathbf{p}_0 \}_{i=1}^{k}$

$\mathbf{C} = \{ \mathbf{p}_0 + \sum_{i=1}^{k} y_i (\mathbf{p}_i - \mathbf{p}_0) : y_i \geq 0 \} \supset \mathbf{H}$. (geometrically)

Let $\mathbf{q}$ be any vertex of $\mathbf{H}$ (basic solution), $\mathbf{q} \in \mathbf{H} \subset \mathbf{C}$.

$\mathbf{q} - \mathbf{p}_0 = \sum_{i=1}^{k} y_i (\mathbf{p}_i - \mathbf{p}_0)$ with all $y_i \geq 0$

$f(\mathbf{q}) - f(\mathbf{p}) = \sum_{i=1}^{k} y_i [f(\mathbf{p}_i) - f(\mathbf{p}^*)] \leq 0$. **End Proof**

Degenerate Basic Solutions

Homework Prob gives example of a degenerate basic solution.

A basic variable is $= 0$ in addition to free (non-pivot) variables.

When leaving, variable which is becomes > 0 must be

free (non-pivot) variable and not a basic (pivot) variable $= 0$

First pivoting at the degenerate solution

interchanges a basic variable $= 0$ and a free variable,

so new free variable can be made > 0 with next pivoting

when the value of objection function is increased;

all same values of variables, so the same point in $\mathscr{F}$.

Matter of how row reduction relates to movement on feasible set.