Chapter 4: Parametric Contin. and Dynamic Prog.

Math 368

May 22, 2013

1. Correspondence

Use set-valued correspondences in parametric maximization problems and multiple period dynamic programming problems

Definition

Let $\mathbf{S} \subset \mathbb{R}^\ell$, $\mathbf{X} \subset \mathbb{R}^n$.

A **correspondence** $\mathscr{C}$ from $\mathbf{S}$ to $\mathbf{X}$ is a map to nonempty subsets of $\mathbf{X}$:

$$\mathbf{s} \in \mathbf{S} \mapsto \mathscr{C}(\mathbf{s}) \subset \mathbf{X} \quad \text{with } \mathscr{C}(\mathbf{s}) \neq \emptyset$$

$\mathscr{P}(\mathbf{X})$ be collection of all nonempty subsets of $\mathbf{X}$.

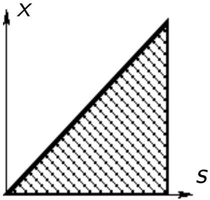
$\mathscr{C}$ is function with values in $\mathscr{P}(\mathbf{X})$.

$\mathscr{C} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$.

Graph of Correspondences

When give examples, draw graph of correspondence

$$\mathscr{C}_1(s) = [0, s] \qquad \text{for } 0 \le s \le 1.$$



Definition

Graph of a correspondence $\mathscr{C} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ is

$$\mathrm{Gr}(\mathscr{C}) = \{\, (\mathbf{s}, \mathbf{x}) \in \mathbf{S} \times \mathbf{X} : \mathbf{s} \in \mathbf{S}, \;\; \mathbf{x} \in \mathscr{C}(\mathbf{s}) \,\} \subset \mathbf{S} \times \mathbf{X}.$$

Definition

$\mathscr{C} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ is a **closed-graph correspondence** p.t.

$\mathrm{Gr}(\mathscr{C})$ is a closed subset of $\mathbf{S} \times \mathbf{X}$.

A correspondence $\mathscr{C} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ is **closed-valued** p.t.

$\mathscr{C}(\mathbf{s})$ is a closed subset of $\mathbf{X}$ for each fixed $\mathbf{s} \in \mathbf{S}$.

compact-valued if each $\mathscr{C}(\mathbf{s})$ is compact.

A correspondence $\mathscr{C}$ is **bounded** p.t. $\exists\ K > 0$ such that

$\mathscr{C}(\mathbf{s}) \subset \overline{\mathbf{B}}(\mathbf{0}, K)$ for all $\mathbf{s} \in \mathbf{S}$.

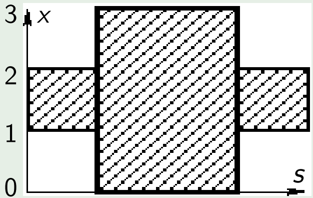
$\mathscr{C}$ is **locally bounded** p.t. for each $\mathbf{s}_0 \in \mathbf{S}$, $\exists\ K > 0$ & $r > 0$ s.t.

$\mathscr{C}(\mathbf{s}) \subset \overline{\mathbf{B}}(\mathbf{0}, K)$ for all $\mathbf{s} \in \mathbf{B}(\theta_0, r) \cap \mathbf{S}$.

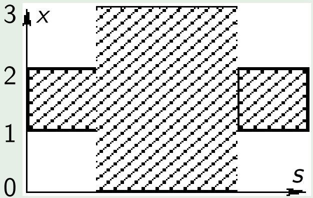
Example

$\mathbf{S} = [0, 2]$, $\mathbf{X} = \mathbb{R}$. Graph in $\mathbf{S} \times \mathbf{X} = [0, 2] \times \mathbb{R}$.

$$\mathscr{C}_2(s) = \begin{cases} [1, 2] & \text{for } 0 \le s < 0.5, \quad 1.5 < s \le 2 \\ [0, 3] & \text{for } 0.5 \le s \le 1.5 \end{cases}$$



$\mathscr{C}_2$



$\mathscr{C}_3$

$$\mathscr{C}_3(s) = \begin{cases} [1, 2] & \text{for } 0 \le s \le 0.5, \quad 1.5 \le s \le 2 \\ [0, 3] & \text{for } 0.5 < s < 1.5. \end{cases}$$

$\mathscr{C}_2$ & $\mathscr{C}_3$ are compact-valued.

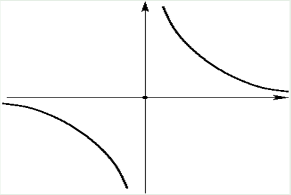
$\mathscr{C}_2$ is closed-graph, but not $\mathscr{C}_3$.

$\mathscr{C}_2$ & $\mathscr{C}_3$ are bounded.

Unbounded Correspondence

Example

$$\mathscr{C}_4(s) = \begin{cases} \left\{ \dfrac{1}{s} \right\} & s \neq 0 \\ \{0\} & s = 0 \end{cases}$$



$\mathscr{C}_4$

$\mathscr{C}_4$ is a closed-graph correspondence, compact valued

$\mathscr{C}_4$ is not bounded, nor locally bounded at $s = 0$.

Parametric Maximization

Maximizing $f(\mathbf{s}, \mathbf{x})$ subject to $\mathbf{x} \in \mathscr{F}(\mathbf{s}) \subset \mathbb{R}^n$

$\mathbf{s} \in \mathbf{S}$: parameter space

both f and domain $\mathscr{F}$ can depend on $\mathbf{s}$.

For $\mathbf{s} \in \mathbf{S}$,

$$f^*(\mathbf{s}) = \max\{f(\mathbf{s}, \mathbf{x}) : \mathbf{x} \in \mathscr{F}(\mathbf{s})\} \in \mathbb{R}$$
$$\mathscr{F}^*(\mathbf{s}) = \{\, \mathbf{x} \in \mathscr{F}(\mathbf{s}) : f(\mathbf{s}, \mathbf{x}) = f^*(\mathbf{s}) \,\} \quad \subset \mathscr{F}(\mathbf{s}) \subset \mathbb{R}^n.$$

For each $\mathbf{s}$, $\mathscr{F}(\mathbf{s})$ and $\mathscr{F}^*(\mathbf{s})$ are sets,

so $\mathscr{F}$ and $\mathscr{F}^*$ are examples of correspondences.

$f^*(\mathbf{s})$ is a number, so f^* is a function

Question: How do $f^*(\mathbf{s})$ and $\mathscr{F}^*(\mathbf{s})$ vary with $\mathbf{s}$?

Example 1

Example

Let $f_1(s,x) = (s - \frac{1}{3})\,x$ for $s \in [0,1] = \mathbf{S}$ and $x \in [0,1] = \mathscr{F}_1$

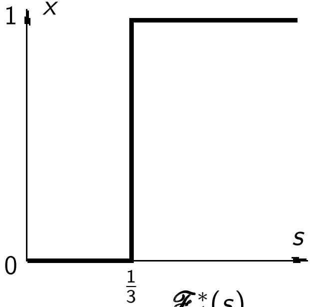
$$\frac{\partial f_1}{\partial x}(s,x) = (s - \tfrac{1}{3}) \begin{cases} < 0 & \text{for } s < \frac{1}{3} \\ \equiv 0 & \text{for } s = \frac{1}{3} \\ > 0 & \text{for } s > \frac{1}{3}. \end{cases}$$

$$\mathscr{F}_1^*(s) = \begin{cases} \{0\} & \text{for } s < \frac{1}{3} \\ [0,1] & \text{for } s = \frac{1}{3} \\ \{1\} & \text{for } s > \frac{1}{3}. \end{cases} \quad \& \quad f_1^*(s) = \begin{cases} 0 & \text{for } s \leq \frac{1}{3} \\ (s - \frac{1}{3}) & \text{for } s > \frac{1}{3}. \end{cases}$$

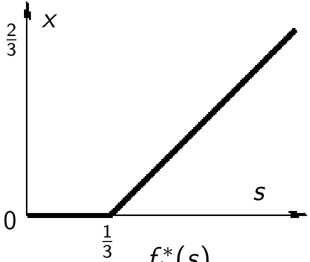
$f_1^*(s)$ is continuous, $\mathscr{F}_1^*(s)$ changes dramatically at $s = \frac{1}{3}$,

In game theory, $\mathscr{F}_1^*(s)$ called **best response correspondence**

$$\mathscr{F}_1^*(s) = \begin{cases} \{0\} & \text{for } s < \frac{1}{3} \\ [0,1] & \text{for } s = \frac{1}{3} \\ \{1\} & \text{for } s > \frac{1}{3}. \end{cases} \quad \text{and} \quad f_1^*(s) = \begin{cases} 0 & \text{for } s \le \frac{1}{3} \\ (s - \frac{1}{3}) & \text{for } s > \frac{1}{3}. \end{cases}$$



$\mathscr{F}_1^*(s)$

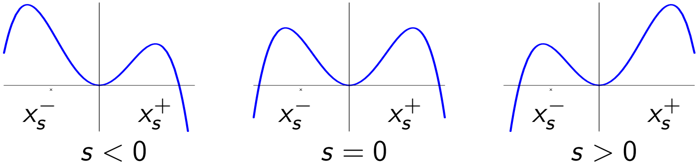


$f_1^*(s)$

$\mathscr{F}_1^*(s)$ is compact valued, bounded, closed graph. f_1^* is continuous

Example 2

Let $f_2(s,x) = -\frac{1}{4}x^4 + \frac{1}{3}\,s\,x^3 + \frac{1}{2}x^2$ for $x \in \mathbb{R} = \mathscr{F}_2$.



$$f_{2x}(s,x) = -x^3 + s\,x^2 + x,$$

Critical points: $0,\ x_s^\pm = \frac{1}{2}\left[s \pm \sqrt{s^2+4}\right].$

For $s=0$: $x_0^\pm = \pm 1$, $f_2(0,\pm 1) = \frac{1}{4} > 0 = f(0,0)$.

$$f_{2xx}(s,x) = -3\,x^2 + 2\,s\,x + 1.$$

$f_{2xx}(s,0) = 1 > 0$: local minimum at $x = 0$.

$f_{2xx}(s,x_s^\pm) = -(x_s^\pm)^2 + 2\left[-(x_s^\pm)^2 + s x_s^\pm + 1\right] - 1 = -(x_s^\pm)^2 - 1 < 0,$

local maximum at x_s^+, x_s^-.

Example 2, continued

Do not calculate $f(s, x_s^{\pm})$ but

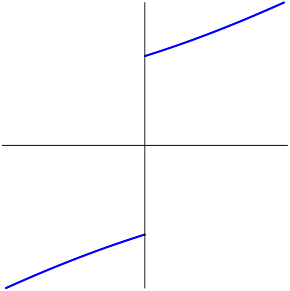
$$\frac{d}{ds} f_2(s, x_s^{\pm}) = f_{2x}(s, x_s^{\pm}) \frac{dx_s^{\pm}}{ds} + \tfrac{1}{3} \left(x_s^{\pm}\right)^3 = \tfrac{1}{3} \left(x_s^{\pm}\right)^3$$

For $s > 0$, $\quad f_2(s, x_s^-) < f(0, \pm 1) < f_2(s, x_s^+)$.

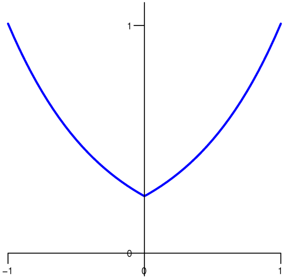
For $s < 0$, $\quad f_2(s, x_s^-) > f(0, \pm 1) > f_2(s, x_s^+)$.

Thus,

$$\mathscr{F}_2^*(s) = \begin{cases} \{x_s^-\} & \text{for } s < 0 \\ \{\text{-}1, 1\} & \text{for } s = 0 \\ \{x_s^+\} & \text{for } s > 0. \end{cases}$$



$\mathscr{F}_2^*(s)$



$f_2^*(s)$

Calculated numerically

$\mathscr{F}_2^*(s)$ jumps at $s = 0$, not continuous

is compact valued, closed graph, locally bounded.

$f_2^*(s)$ is continuous by Parametric Maximization Theorem.

Upper-Hemicontinuous

Continuity of correspondences defined using small region around set.

Definition

For a set $\mathbf{A} \subset \mathbb{R}^n$, **$\epsilon$-neighborhood of A** is

$$\mathbf{B}(\mathbf{A}, \epsilon) = \{\, \mathbf{x} \in \mathbb{R}^n : \text{ there is a } \mathbf{y} \in \mathbf{A} \text{ with } \|\mathbf{x} - \mathbf{y}\| < \epsilon \,\}.$$

Definition

A compact-valued correspondence $\mathscr{C} : \mathbf{S} \subset \mathbb{R}^\ell \to \mathscr{P}(\mathbf{X})$ is

upper-hemicontinuous (uhc) at $\mathbf{s}_0 \in \mathbf{S}$ p.t.

for any $\epsilon > 0$ *exists* $\delta > 0$ such that

if $\mathbf{s} \in \mathbf{B}(\mathbf{s}_0, \delta) \cap \mathbf{S}$ then $\mathscr{C}(\mathbf{s}) \subset \mathbf{B}(\mathscr{C}(\mathbf{s}_0), \epsilon)$.

Image set cannot get a lot bigger for small changes from $\mathbf{s}_0$ to nearby $\mathbf{s}$

$\mathscr{C}$ is **upper-hemicontinuous on S** p.t. it is uhc at each $\mathbf{s}_0 \in \mathbf{S}$.

Continuous Correspondence

Definition

A compact-valued correspondence $\mathscr{C} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ is

continuous at $\mathbf{s}_0 \in \mathbf{S}$ p.t.

for any $\epsilon > 0$, $\exists\ \delta > 0$ s.t. if $\mathbf{s} \in \mathbf{B}(\mathbf{s}_0, \delta) \cap \mathbf{S}$ then

$\mathscr{C}(\mathbf{s}_0) \subset \mathbf{B}(\mathscr{C}(\mathbf{s}), \epsilon)$ &

$\mathscr{C}(\mathbf{s}) \subset \mathbf{B}(\mathscr{C}(\mathbf{s}_0), \epsilon)$.

Image sets $\mathscr{C}(\mathbf{s}_0)$ & $\mathscr{C}(\mathbf{s})$ within small neighborhoods of each other

not a lot smaller nor bigger for small changes from $\mathbf{s}_0$ to nearby $\mathbf{s}$

$\mathscr{C}$ is **continuous on S** p.t. it is continuous at each $\mathbf{s}_0 \in \mathbf{S}$.

Examples of Correspondences

$\mathscr{C}_1$ is continuous correspondence

$\mathscr{C}_2$, $\mathscr{F}_1^*$, and $\mathscr{F}_2^*$ are upper-hemicontinuous not continuous.

$\mathscr{C}_3$ is not uhc at 0.5 or 1.5: $[1,2]$ jumps to $[0,3]$

with changes of $s = 0.5$ to $0.5 + \delta$ or $s = 1.5$ to $1.5 - \delta$

$\mathscr{C}_4$ is neither upper-hemicontinuous nor continuous at $s = 0$:

As s changes from 0 to δ, $\mathscr{C}_4(0) = \{0\}$ changes to $\mathscr{C}_4(\delta) = \left\{\frac{1}{\delta}\right\}$,

$\left\{\frac{1}{\delta}\right\} \not\subseteq \mathbf{B}(\{0\}, \epsilon) = (-\epsilon, \epsilon)$ &

$\{0\} \not\subseteq \mathbf{B}\left(\left\{\frac{1}{\delta}\right\}, \epsilon\right) = \left(-\epsilon + \frac{1}{\delta}, \epsilon + \frac{1}{\delta}\right)$.

Condition to be Upper-Hemicontinuous

Proposition (1)

Let $\mathscr{C} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ *be a compact-valued, locally bded correspondence.*

$\mathscr{C}$ *is upper-hemicontinuous iff* $\mathscr{C}$ *is a closed-graph correspondence.*

See online class book for proof.

Remark

$\mathscr{C}_4(s) = \left\{\frac{1}{s}\right\}$ above shows why correspondence

must be locally bounded in this proposition.

Parametric Maximization Theorem

Theorem (2 Parametric Maximization Theorem)

Assume $f : \mathbf{S} \times \mathbf{X} \to \mathbb{R}$ *is a continuous function and*

$\mathscr{F} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ *is a compact-valued continuous correspondence.*

Then, $f^*(\mathbf{s}) = \max\{\, f(\mathbf{s}, \mathbf{x}) : \mathbf{x} \in \mathscr{F}(\mathbf{s})\,\}$ *is continuous, and*

$\mathscr{F}^*(\mathbf{s}) = \{\, \mathbf{x} \in \mathscr{F}(\mathbf{s}) : f(\mathbf{s}, \mathbf{x}) = f^*(\mathbf{s})\,\}$

is a compact-valued upper-hemicontinuous correspondence.

If $\mathscr{F}^*(\mathbf{s})$ *is single point for each* $\mathbf{s}$*,*

then continuous correspondence or function.

If $f(\mathbf{s}, \mathbf{x})$ is strictly concave fn of $\mathbf{x}$ for each $\mathbf{s}$,

then each $\mathscr{F}^*(\mathbf{s})$ is a single point and so $\mathscr{F}^*$ is continuous.

Domains for Dynamic Programs often have $\mathscr{F}(s) = [0, s]$,

which satisfies theorem.

Example 4

Let $\mathbf{S} = \mathbf{X} = \mathbb{R}_+$ and $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$h(s, x) = x^{\frac{1}{2}} + (s - x)^{\frac{1}{2}} \qquad \text{for } x \in [0, s] = \mathscr{F}(s).$$

h is continuous and $\mathscr{F}(s)$ continuous correspondence.

Parametric Maximization Theorem applies.

Critical point satisfies

$$0 = \frac{\partial h}{\partial x} = \frac{1}{2}\, x^{-\frac{1}{2}} - \frac{1}{2}\,(s - x)^{-\frac{1}{2}},$$

$$x^{-\frac{1}{2}} = (s - x)^{-\frac{1}{2}},$$

$$s - x = x,$$

$$s = 2x,$$

$$\bar{x} = \tfrac{1}{2} s \in [0, s].$$

Example 4, continued

$$\frac{\partial^2 h}{\partial x^2} = -\frac{1}{4}\,x^{-\frac{3}{2}} - \frac{1}{4}\,(s-x)^{-\frac{3}{2}} < 0 \text{ for all } x > 0, \text{ so}$$

$h(s,x)$ is a concave function of x on $[0,s]$

$\bar{x} = \frac{1}{2}s$ is the unique maximizer on $[0,s]$.

$\mathscr{F}^*(s) = \left\{\frac{1}{2}s\right\}$ is a continuous correspondence

$h^*(s) = \left(\frac{1}{2}s\right)^{\frac{1}{2}} + \left(\frac{1}{2}s\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} s^{\frac{1}{2}}$ is continuous as it must be

Consumer Theory

Theorem (3)

Commodity bundles are points in $\mathbb{R}^n_+$.

Parameters are prices $p_i > 0$ *for* $1 \leq i \leq n$ *and income* $I > 0$.

Parameter space $\mathbf{S} = \{\, (\mathbf{p}, I) \in \mathbb{R}^{n+1}_{++} \,\}$.

Budget correspondence $\mathscr{B} : \mathbf{S} \to \mathscr{P}(\mathbb{R}^n_+)$ *is*

$$\mathscr{B}(\mathbf{p}, I) = \{\, \mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq I \,\}.$$

$\mathscr{B}$ *is a continuous, compact-valued correspondence.*

Proof.

Obviously, $\mathscr{B}$ is compact-valued.

Intuitively, it is continuous.

An explicit proof is given in online class book. □

Corollary (4)

$u : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous **utility function**.

Then **indirect utility function** $v : \mathbf{S} \to \mathbb{R}$,

$$v(\mathbf{p}, I) = \max\{u(\mathbf{x}) : \mathbf{x} \in \mathscr{B}(\mathbf{p}, I)\},$$

is continuous.

Also, **demand correspondence** $\mathbf{d} : \mathbf{S} \to \mathscr{P}(\mathbb{R}^n_+)$,

$$\mathbf{d}(\mathbf{p}, I) = \mathscr{B}^* = \{\, \mathbf{x} \in \mathscr{B}(\mathbf{p}, I) : u(\mathbf{x}) = v(\mathbf{p}, I)\,\},$$

is a compact-valued uhc correspondence.

2. Finite Horizon Dyn Prog: Consumption-Savings

T is number of time periods fixed at start of problem.

t period is integer for $0 \leq t \leq T$.

$w_t \in \mathbb{R}_+$ is **wealth** at period t, called **state**. w_0 **initial state**

c_t consumption at period t, chosen with $0 \leq c_t \leq w_t$, **action**

$\mathscr{F}(w_t) = [0, w_t]$ is **feasible action correspondence**, $c_t \in [0, w_t]$.

$w_{t+1} = f(w_t, c_t) = k(w_t - c_t)$ is **transition function**,

where $k \geq 1$ for fixed production rate.

$u(c_t) = \sqrt{c_t}$ is utility of c_t valued at period t. For $0 < \delta \leq 1$,

$r_t(c_t) = \delta^t u(c_t) = \delta^t \sqrt{c_t}$ **reward function** valued at $t = 0$.

Problem: Given k, δ, $u(c)$, and w_0, maximize $\sum_{t=0}^{T} \delta^t u(c_t)$.

subject to $0 \leq c_t \leq w_t$ and $w_{t+1} = k(w_t - c_t)$.

Finite Horizon Dyn Prog

Can solve as KKT problem as function of w_0:

maximize $\sum_{t=0}^{T} \delta^t \, u(c_t)$

subject to $0 \leq c_t \leq w_t$ and $w_{t+1} = k(w_t - c_t)$,

Easier to break up and solve simpler problem at each time period,

starting at $t = T$

called **backward induction**

Treat as a **Dynamic Programming Problem**

Strategy and Value Function

Markovian strategy profile is rule σ_t at each period t for a choice of c_t

as function of only w_t, $c_t = \sigma_t(w_t)$, and not of $w_{t'}$ for $t' \neq t$.

Use backward induction to find maximal Markovian strategy profile

(T) For $t = T$: maximize $r_T(c) = \delta^T c^{\frac{1}{2}}$ for $0 \leq c = c_T \leq w_T$.

r_T is strictly increasing, so optimal choice is $\bar{c} = w_T$.

Denote this choice by $\bar{c}_T = \sigma^*_T(w_T) = w_T$.

Optimal strategy at period T.

Value function at T-period is maximum payoff for period T

given w_T,

$$V_T(w_T) = r_T(\sigma^*_T(w_T)) = \delta^T w_T^{\frac{1}{2}}.$$

Consumption-Savings Problem, continued

(T-1) Let $w = w_{T\text{-}1}$ at period $t = T\text{-}1$.

$r_{T-1}(c) = \delta^{T-1}\, c^{\frac{1}{2}}$ is one period payoff for action c

$w_T = f_{T\text{-}1}(w_{T\text{-}1}, c) = k(w_{T\text{-}1} - c)$ carry forward to period T

$V_T(k(w-c)) = \delta^{T}\, k^{\frac{1}{2}}\, (w_{T\text{-}1} - c)^{\frac{1}{2}}$ is maximal payoff at period T

For $0 \le c \le w_{T\text{-}1}$, seek to maximize

$$
\begin{aligned}
h(w_{T\text{-}1}, c) &= r_{T-1}(c) + V_T(k(w-c)) \\
&= \delta^{T-1}\, c^{\frac{1}{2}} + \delta^{T}\, k^{\frac{1}{2}}\, (w_{T\text{-}1} - c)^{\frac{1}{2}}
\end{aligned}
$$

$$
0 = \frac{\partial h}{\partial c} = \delta^{T-1}\, \tfrac{1}{2} c^{-\frac{1}{2}} + \delta^{T}\, k^{\frac{1}{2}}\, \tfrac{1}{2} \left(w_{T\text{-}1} - c\right)^{-\frac{1}{2}} (\text{-}1),
$$

$$
c^{-\frac{1}{2}} = \delta\, k^{\frac{1}{2}} \left(w_{T\text{-}1} - c\right)^{-\frac{1}{2}}, \qquad w_{T\text{-}1} - c = \delta^2\, k\, c,
$$

$$
w_{T\text{-}1} = (1 + \delta^2\, k)\, c, \qquad \bar{c} = \frac{w_{T\text{-}1}}{1 + \delta^2\, k}.
$$

Since $\dfrac{\partial^2 h}{\partial c^2} < 0$, this critical point is a maximum.

Consumption-Savings Problem, continued

Optimal strategy is $\bar{c} = \sigma^*_{T\text{-}1}(w_{T\text{-}1}) = \dfrac{w_{T\text{-}1}}{1+\delta^2\,k} \le w_{T\text{-}1}$.

Value fn at period $T\text{-}1$ is maximal payoff for periods $t \ge T\text{-}1$,

$$
\begin{aligned}
V_{T\text{-}1}(w_{T\text{-}1}) = h^*(w_{T\text{-}1}) &= \delta^{T-1}\,\bar{c}^{\frac{1}{2}} + \delta^{T}\,[k(w-\bar{c})]^{\frac{1}{2}} \\
&= \delta^{T-1}\,\bar{c}^{\frac{1}{2}} + \delta^{T}\left[\delta^{2}k^{2}\bar{c}\right]^{\frac{1}{2}} \\
&= \delta^{T-1}\,(1+\delta^2\,k)\,\bar{c}^{\frac{1}{2}} \\
&= \delta^{T-1}\,(1+\delta^2\,k)\,\left[\frac{w_{T\text{-}1}}{1+\delta^2\,k}\right]^{\frac{1}{2}} \\
&= \delta^{T-1}\,(1+\delta^2\,k)^{\frac{1}{2}}\,w_{T\text{-}1}^{\frac{1}{2}}.
\end{aligned}
$$

By backward induction $V_j(w_j) = \delta^j \left(1 + \delta^2 k + \cdots + \delta^{2T-2j} k^{T-j}\right)^{\frac{1}{2}} w_j^{\frac{1}{2}}$.

Valid for $j = T$ and T-1. Assume true $t+1$.

(t) Given $w = w_t$: For $0 \le c \le w$, maximize

$$h(w, c) = r_t(c) + V_{t+1}(k(w-c))$$

$$= \delta^t c^{\frac{1}{2}} + \delta^{t+1} \left(1 + \cdots + \delta^{2T-2t-2} k^{T-t-1}\right)^{\frac{1}{2}} k^{\frac{1}{2}} (w-c)^{\frac{1}{2}}$$

$$0 = \frac{\partial h}{\partial c} = \delta^t \frac{1}{2} c^{-\frac{1}{2}} + \delta^{t+1} \left(1 + \cdots + \delta^{2T-2t-2} k^{T-t-1}\right)^{\frac{1}{2}} k^{\frac{1}{2}} \frac{1}{2} (w-c)^{-\frac{1}{2}} (-1)$$

$$(w-c)^{\frac{1}{2}} = \delta \left(1 + \cdots + \delta^{2T-2t-2} k^{T-t-1}\right)^{\frac{1}{2}} k^{\frac{1}{2}} c^{\frac{1}{2}}$$

$$w-c = \delta^2 k \left(1 + \cdots + \delta^{2T-2t-2} k^{T-t-1}\right) c$$

$$= \left(\delta^2 k + \cdots + \delta^{2T-2t} k^{T-t}\right) c$$

$$w = \left(1 + \delta^2 k + \cdots + \delta^{2T-2t} k^{T-t}\right) c$$

$$\bar{c} = \frac{w}{1 + \cdots + \delta^{2T-2t} k^{T-t}} \le w.$$

Consumption-Savings Problem, continued

Since $\dfrac{\partial^2 h}{\partial c^2} < 0$, this critical point is a maximum.

Optimal strategy: $\bar{c} = \sigma_t^*(w_t) = \dfrac{w_t}{1 + \cdots + \delta^{2T-2t} k^{T-t}} \le w_t$.

$$[k\,(w_t - \bar{c})]^{\frac{1}{2}} = \delta\, k \left[1 + \cdots + \delta^{2T-2t-2} k^{T-t-1}\right]^{\frac{1}{2}} \bar{c}^{\frac{1}{2}}$$

$$\begin{aligned}
h_t^*(w_t) &= \delta^t \bar{c}^{\frac{1}{2}} + V_{t+1}\left(k(w_t - \bar{c})\right) \\
&= \delta^t \bar{c}^{\frac{1}{2}} + \delta^{t+1} \left(1 + \cdots + \delta^{2T-2t-2} k^{T-t-1}\right)^{\frac{1}{2}} [k\,(w_t - \bar{c})]^{\frac{1}{2}} \\
&= \delta^t \bar{c}^{\frac{1}{2}} + \delta^{t+2} k \left(1 + \cdots + \delta^{2T-2t-2} k^{T-t-1}\right) \bar{c}^{\frac{1}{2}} \\
&= \delta^t \left(1 + \delta^2 k + \cdots + \delta^{2T-2t} k^{T-t}\right) \bar{c}^{\frac{1}{2}} \\
&= \frac{\delta^t \left(1 + \cdots + \delta^{2T-2t} k^{T-t}\right) w_t^{\frac{1}{2}}}{\left(1 + \cdots + \delta^{2T-2t} k^{T-t}\right)^{\frac{1}{2}}} \\
&= \delta^t \left(1 + \cdots + \delta^{2T-2t} k^{T-t}\right)^{\frac{1}{2}} w_t^{\frac{1}{2}} = V_t(w_t).
\end{aligned}$$

Consumption-Savings Problem, continued

Shown if the maximum from $t+1$ to T has form of $V_{t+1}(w)$ given

then maximum from t to T has form of $V_t(w)$ given.

Therefore, valid for all $t = T,\ T-1,\ \ldots,\ 0.$

Maximal payoff all periods $t = 0$ to T

$$V_0(w_0) = \left(1 + \delta^2 k + \cdots + \delta^{2T} k^T\right)^{\frac{1}{2}} w_0^{\frac{1}{2}}.$$

Optimal strategy profile is $\boldsymbol{\sigma^*} = (\sigma_0^*, \ldots, \sigma_T^*)$ where

$$\bar{c}_t = \sigma_t^*(w_t) = \frac{w_t}{1 + \cdots + \delta^{2T-2t} k^{T-t}}$$

End of Example

Example with Production

Include some production in feasibility correspondence for $0 \le t \le T$.

w_t be wealth or capital; labor force is held fixed so

production is w_t^β with $0 < \beta < 1$.

$0 \le c \le w_t^\beta$ consumption

$f(w_t, c_t) = w_t^\beta - c_t = w_{t+1}$.

$r_t(w, c) = \delta^t \ln(c)$, where $0 < \delta \le 1$.

(T) Maximizer $\delta^T \ln(c)$ for $0 \le c \le w^\beta$,

$$\sigma_T^*(w) = c^* = w^\beta$$

$$V_T(w) = \delta^T \ln(w^\beta) = \delta^T \beta \ln(w)$$

Example with Production, contin.

$$\textbf{(T}-\textbf{1)}\quad h(w,c) = \delta^{T\text{-}1}\ln(c) + V_T(w^\beta - c)$$
$$= \delta^{T\text{-}1}\ln(c) + \delta^T\,\beta\,\ln(w^\beta - c)$$

$$0 = \frac{\partial h}{\partial c} = \delta^{T\text{-}1}\frac{1}{c} - \delta^T\,\beta\,\frac{1}{w^\beta - c}$$

$$w^\beta - c = \delta\,\beta\,c \qquad\qquad w^\beta = (1 + \delta\,\beta)\,c$$

$$\bar{c} = \sigma^*_{T-1}(w) = \frac{w^\beta}{1 + \delta\,\beta} \le w^\beta$$

$$V_{T\text{-}1}(w) = \delta^{T\text{-}1}\ln(\bar{c}) + V_t(w^\beta - \bar{c})$$
$$= \delta^{T\text{-}1}\ln(\bar{c}) + \delta^T\beta\ln(\delta\,\beta\,\bar{c})$$
$$= \delta^{T\text{-}1}\,[1 + \delta\beta]\,[\beta\ln(w) - \ln(1 + \delta\,\beta)] + \delta^T\beta\ln(\delta\,\beta)$$
$$= \delta^{T\text{-}1}\beta\,[1 + \delta\beta]\,\ln(w) + v_{T\text{-}1}.$$

Example with Production, contin.

Induction hypothesis with v_j a constant

$$V_j(w) = \delta^j \beta \left[1 + \delta\beta + \dots + \delta^{T-j}\beta^{T-j}\right] \ln(w) + v_j$$

$$\begin{aligned} h(w,c) &= \delta^t \ln(c) + V_{t+1}(w^\beta - c) \\ &= \delta^t \ln(c) + \delta^{t+1} \beta [1 + \dots + \delta^{T-t-1}\beta^{T-t-1}] \ln(w^\beta - c) + v_{t+1} \end{aligned}$$

$$0 = \tfrac{\partial h}{\partial c} = \delta^t \frac{1}{c} - \delta^{t+1} \beta [1 + \dots + \delta^{T-t-1}\beta^{T-t-1}] \frac{1}{w^\beta - c}$$

$$w^\beta - c = [\delta\beta + \dots + \delta^{T-t}\beta^{T-t}]\, c$$

$$w^\beta = [1 + \dots + \delta^{T-t}\beta^{T-t}]\, c$$

$$\bar{c} = \sigma^*_{T-t}(w) = \frac{w^\beta}{1 + \dots + \delta^{T-t}\beta^{T-t}} \le w^\beta$$

Example with Production, contin.

$$V_t(w) = \delta^t \ln(\bar{c}) + V_{t+1}(w^\beta - \bar{c})$$

$$= \delta^t \ln(\bar{c}) + V_{t+1}\left([\delta\,\beta\, + \cdots + \delta^{T-t}\beta^{T-t}]\,\bar{c}\right)$$

$$= \delta^t \ln(\bar{c}) + \delta^t[\delta\beta + \cdots + \delta^{T-t}\beta^{T-t}]\ln(\bar{c})$$

$$+\delta^t[\delta\beta + \cdots + \delta^{T-t}\beta^{T-t}]\ln(\delta\,\beta\, + \cdots + \delta^{T-t}\beta^{T-t}) + v_{t+1}$$

$$= \delta^t[1 + \cdots + \delta^{T-t}\beta^{T-t}]\,\beta\ln(w)$$

$$-\delta^t[1 + \cdots + \delta^{T-t}\beta^{T-t}]\ln(1 + \cdots + \delta^{T-t}\beta^{T-t})$$

$$+\delta^t[\delta\beta + \cdots + \delta^{T-t}\beta^{T-t}]\ln(\delta\,\beta\, + \cdots + \delta^{T-t}\beta^{T-t}) + v_{t+1}$$

$$= \delta^t\beta\,[1 + \cdots + \delta^t\beta^t]\,\ln(w) + v_t$$

End of Example

Supremum

Supremum or **least upper bound** for $f : \mathbf{X} \to \mathbb{R}$ is

M such that $f(\mathbf{x}) \leq M$ for all $\mathbf{x} \in \mathbf{X}$,

and no small number works.

$$\sup\{\, f(\mathbf{x}) : \mathbf{x} \in \mathbf{X} \,\}.$$

If function is bounded above, a finite supremum exists.

Supremum $= \infty$ if $f(\mathbf{x})$ is not bounded above.

Example

$\arctan(x)$ is bounded above on $\mathbb{R}$ but no maximum,

$$\sup\{\, \arctan(x) : x \in \mathbb{R} \,\} = \tfrac{\pi}{2}$$

$\sup\left\{\, \frac{1}{x} : x > 0 \,\right\} = \infty$ $\qquad$ $\sup\left\{\, \frac{1}{x} : x < 0 \,\right\} = 0$

Infimum

Infimum or **greatest lower bound** for $f : \mathbf{X} \to \mathbb{R}$ is

m such that $f(\mathbf{x}) \geq m$ for all $\mathbf{x} \in \mathbf{X}$,

and no larger number works.

$$\inf\{\, f(\mathbf{x}) : \mathbf{x} \in \mathbf{X} \,\}.$$

Infimum $= -\infty$ iff $f(\mathbf{x})$ is not bounded below.

If function is bounded below, a greatest lower bound or infimum exists.

Example

$\arctan(x)$ is bounded below on $\mathbb{R}$ but no minimum,

$$\inf\{\, \arctan(x) : x \in \mathbb{R} \,\} = -\tfrac{\pi}{2},$$

$\inf \left\{ \frac{1}{x} : x > 0 \right\} = 0, \qquad \inf \left\{ \frac{1}{x} : x < 0 \right\} = -\infty$

General Finite-Horizon Dynamic Program

A general FHDP consists of $\{\mathbf{S}, \mathbf{A}, T, (r_t, f_t, \mathscr{F}_t)_{t=0}^{T}\}$.

- T is a positive integer; periods t are taken with $0 \le t \le T$.
- $\mathbf{S}$ is state space, with state at period t given by $s_t \in \mathbf{S}$.
  (In C-S problem, $s_t = w_t \in [0, \infty) = \mathbf{S}$.)
- $\mathbf{A}$ is action space, with action at period t given by a_t.
  (In C-S problem, $c_t \in [0, \infty) = \mathbf{A}$.)
- For each $0 \le t \le T$, following are defined:
  - $r_t : \mathbf{S} \times \mathbf{A} \to \mathbb{R}$ is continuous period-t **reward function**.
    (In C-S problem, $r_t(w, c) = \delta^t \, c^{\frac{1}{2}}$.)
  - $f_t : \mathbf{S} \times \mathbf{A} \to \mathbf{S}$ is continuous period-t **transition function**,
    $s_{t+1} = f_t(s_t, a_t)$ (In C-S problem, $f_t(w, c) = k(w - c)$.)
  - $\mathscr{F}_t : \mathbf{S} \to \mathscr{P}(\mathbf{A})$ is **feasible action correspondences**, and is assumed to be a continuous and compact-valued correspondence on $\mathbf{S}$.
    Only $a_t \in \mathscr{F}_t(s_t)$ are allowed.
    (In C-S problem, $c_t \in [0, w_t] = \mathscr{F}_t(w_t)$.)

Value Function

Total reward for s_0, allowable $\{a_t\}_{t=0}^{T}$, & $s_{t+1} = f_t(s_t, a_t)$ is

$$W\left(s_0, \{a_t\}_{t=0}^{T}\right) = \sum_{t=0}^{T} r_t(s_t, a_t)$$

Value function of continuation FHDP starting at period t,

$$\begin{aligned} V_t(s_t) &= \sup \Big\{ \sum_{j=t}^{T} r_j(s_j, a_j) : a_j \in \mathscr{F}_j(s_j), \\ &\qquad\qquad s_{j+1} = f_j(s_j, a_j) \text{ for } j = t, \dots, T \Big\}. \\ &= \sup \left\{ W\left(s_t, \{a_j\}_{j=t}^{T}\right) : \{a_j\}_{j=t}^{T} \text{ allowable} \right\} \end{aligned}$$

$V(s_0) = V_0(s_0)$ is value function for whole FHDP.

Problem: Find actions that realize supremum so maximum value.

Markovian Strategies

A **Markovian strategy profile** is a collection of (choice) functions

$$\boldsymbol{\sigma} = (\sigma_0, \dots, \sigma_T), \text{ each } \sigma_t : \mathbf{S} \to \mathbf{A}, \;\; a_t = \sigma_t(s_t) \in \mathscr{F}_t(s_t),$$

σ_t function of only s_t

A non-Markovian strategy σ_t is a function of $(s_0, \dots, s_t)$ and not just s_t.

Strategy $\boldsymbol{\sigma}$ and initial state s_0, determines all a_t and s_t: $\quad s_0(s_0, \boldsymbol{\sigma}) = s_0$;

Given $s_t = s_t(s_0, \boldsymbol{\sigma})$ for $0 \le t \le T$,

$$a_t = a_t(s_0, \boldsymbol{\sigma}) = \sigma_t(s_t),$$

$$r_t(s_0, \boldsymbol{\sigma}) = r_t(s_t, a_t), \qquad \text{and}$$

$$s_{t+1} = s_{t+1}(s_0, \boldsymbol{\sigma}) = f_t(s_t, a_t).$$

Total reward for a strategy profile $\boldsymbol{\sigma}$ and initial state s_0 is

$$W(s_0, \boldsymbol{\sigma}) = \sum_{t=0}^{T} r_t(s_0, \boldsymbol{\sigma}).$$

$\boldsymbol{\sigma}^*$ is **optimal strategy profile** p.t. realizes max in defn of value fn,

$$W(s_0, \boldsymbol{\sigma}^*) = V(s_0) \qquad \text{for all } s_0 \in \mathbf{S}.$$

FHDP Bellman Equation & Optimal Strategy

Theorem (4.15)

If a FHDP satisfies assumptions, then following holds:

a. *For* $0 \le t \le T$, $V_t(s_t) < \infty$ *for each* s_t, *is continuous,*

and satisfies **Bellman equation**

$$V_t(s) = \max_{a \in \mathscr{F}_t(s)} r_t(s,a) + V_{t+1}(f_t(s,a)),$$

where $V_{T+1}(f_T(s,a)) \equiv 0.$

b. *There is optimal Markovian strategy profile* $\boldsymbol{\sigma}^*$ *s.t.*

$$W(s_0, \boldsymbol{\sigma}^*) = V(s_0) \text{ for all } s_0.$$

Proof

Use backward induction to get Bellman equation.

For $j = T$, $r(s, a)$ continuous & $\mathscr{F}_T(s)$ is compact, so

$$V_T(s) = \max\{r_T(s, a) : a \in \mathscr{F}_T(s)\} \quad \text{exists for each } s.$$

By Parametric Maximization Theorem,

$$\mathscr{F}^*_T(s) = \{\, a \in \mathscr{F}_T(s) : r_T(s, a) = V_T(s)\,\} \neq \emptyset$$

uhc correspondence and $V_T(s)$ is continuous function of s.

Pick $\sigma^*_T(s) \in \mathscr{F}^*_T(s)$ for all $s \in \mathbf{S}$, $\sigma^*_T : \mathbf{S} \to \mathbf{A}$.

$r_T(s, \sigma^*_T(s)) = W(s, \sigma^*_T) = V_T(s)$, so σ^*_T is an optimal strategy.

Have proved result for $j = T$.

Induction Step

Lemma

For $0 \leq t < T$, suppose that continuation FHDP starting a period $t+1$

has continuous V_{t+1} with $V_{t+1}(s_{t+1}) < \infty$ for each s_{t+1},

also admits a optimal Markovian strategy profile $(\sigma^*_{t+1}, \ldots, \sigma^*_T)$

with $V_{t+1}(s_{t+1}) = W(s_{t+1}, (\sigma^*_{t+1}, \ldots, \sigma^*_T))$.

Then following hold

a. *For each s_t, $V_t(s_t)$ attains a finite max value,*

V_t is continuous, and satisfies

$$V_t(s_t) = \max\{\, r_t(s_t, a_t) + V_{t+1}(f_t(s_t, a_t)) : a_t \in \mathscr{F}_t(s_t) \,\}.$$

b. *There exists $\sigma^*_t(s_t)$, s.t. $(\sigma^*_t, \ldots, \sigma^*_T)$*

is a optimal Markovian strategy profile for starting at period t.

Proof of Lemma

By induction, V_{t+1} is a continuous function,

f_t and r_t are continuous, so

$h_t(s_t, a_t) = r_t(s_t, a_t) + V_{t+1}(f_t(s_t, a_t))$ is continuous

$\mathscr{F}_t$ is is continuous and compact-valued correspondence.

By Param. Maxim. Thm, $\mathscr{F}^*(s_t) \neq \emptyset$ (points that realize max)

max value $h_t^*(s_t)$ is continuous

If select point $\sigma_t^*(s_t) \in \mathscr{F}^*(s_t)$, then

$h_t^*(s_t) = r_t(s_t, \sigma_t^*(s_t)) + V_{t+1}(f_t(s_t, \sigma_t^*(s_t)))$.

optimal strategy.

Need to show, $V_t(s_t) = h_t^*(s_t)$, i.e., satisfied Bellman equation.

Proof of Lemma, contin.

For any s_t & allowable sequences $s_t^0 = s_t$,

$$a_i^0 \in \mathscr{F}_i(s_i) \ \ \& \ \ s_{i+1}^0 = f_i(s_i^0, a_i^0) \ \text{ for } \ i \geq t$$

$$\begin{aligned}
\textstyle\sum_{i=t}^{T} r_i(s_i^0, a_i^0) &= r_t(s_t^0, a_t^0) + \textstyle\sum_{i=t+1}^{T} r_i(s_i^0, a_i^0) \\
&\leq r_t(s_t^0, a_t^0) \\
&\qquad + \max \left\{ \textstyle\sum_{i=t+1}^{T} r_i(s_i, a_i) : a_i \in \mathscr{F}_i(s_i), \ \ s_{i+1} = f_i(s_i, a_i) \right\} \\
&= r_t(s_t, a_t^0) + V_{t+1}(f_i(s_t, a_t^0)) = h_t(s_t, a_t^0) \\
&\leq \max \{ h_t(s_t, a_t) : a_t \in \mathscr{F}_t(s_t) \} \\
&= h_t^*(s_t).
\end{aligned}$$

Taking supremum over allowable sequences

$$V_t(s_t) \leq h_t^*(s_t).$$

Proof of Lemma, contin.

$$
\begin{aligned}
h_t^*(s_t) &= h_t(s_t, \sigma_t^*(s_t)) \\
&= r_t(s_t, \sigma_t^*(s_t)) + V_{t+1}(f_t(s_t, \sigma_t^*(s_t)) \\
&= r_t(s_t, \sigma_t^*(s_t)) + \textstyle\sum_{i=t+1}^{T} r_i(s_i^*, \sigma_i^*(s_i^*)) \le V_t(s_t),
\end{aligned}
$$

where $s_t^* = s_t$ and $s_{i+1}^* = f_i(s_i^*, \sigma_t^*(s_i^*))$.

$V_t(s_t) \le h_t^*(s_t) \le V_t(s_t)$ so $V_t(s_t) = h_t^*(s_t) < \infty$.

$V_t(s_t) = h_t^*(s_t) = \max\{\, r_t(s_t, a_t) + V_{t+1}(f_t(s_t, a_t)) : a_t \in \mathscr{F}_t(s_t)\,\}$,

V_t satisfies Bellman equation & is continuous.

$$
\begin{aligned}
V_t(s_t) &= r_t(s_t, \sigma_t^*(s_t)) + V_{t+1}(f_t(s_t, \sigma_t^*(s_t))) \\
&= r_t(s_t, \sigma_t^*(s_t)) + W(f_t(s_t, \sigma_t^*(s_t)), (\sigma_{t+1}^*, \dots, \sigma_T^*)) \\
&= W(s_t, (\sigma_t^*, \dots, \sigma_T^*)),
\end{aligned}
$$

so optimal Markovian strategy profile as claimed.

QED

Solution of FHDP by Bellman Equation

Bellman equation determines solution method for a FHDP.

(T) Maximize $r_T(s_T, a_T)$ with parameter s_T determines

strategy $a_T = \boldsymbol{\sigma}^*_T(s_T)$

value function $V_T(s_T) = r^*(s_T) = r_T(s_T, \sigma^*_T(s_T))$.

By backward induction, once strategies σ^*_j and value functions V_j

have been determined for $T \geq j \geq t+1$, then

$a_t = \sigma^*_t(s_t)$ maximizes

$$h_t(s, a) = r_t(s, a) + V_{t+1}(f_t(s, a)).$$

Period t value function

$$V_t(s_t) = h^*_t(s_t) = r_t(s_t, \sigma^*_t(s_t)) + V_{t+1}(f_t(s_t, \sigma^*_t(s_t))).$$

By induction, get back to $V_0(s_0) = V(s_0)$.

Infinite Horizon Dynamic Program

Definition

A **stationary dynamic program** (SDP) is

$\{\mathbf{S},\ \mathbf{A},\ \mathscr{F},\ f,\ r,\ \delta\}$ with infinite horizon:

- $\mathbf{S} \subset \mathbb{R}^n$ is **state space**.
- $\mathbf{A} \subset \mathbb{R}^k$ is **action space**.
- $\mathscr{F} : \mathbf{S} \to \mathscr{P}(\mathbf{A})$, **feasible action correspondence**, is compact-valued, nonempty, continuous correspondence.

  $\mathscr{F}(s) \subset \mathbf{A}$ is set of allowable actions given $s \in \mathbf{S}$.
- $f : \mathbf{S} \times \mathbf{A} \to \mathbf{S}$, continuous **transition function**: $s_{t+1} = f(s_t, a_t)$ from s_t and a_t to s_{t+1} for $t \geq 0$.
- $r : \mathbf{S} \times \mathbf{A} \to \mathbb{R}$ continuous, one-period **reward function** that specifies reward $r(s, a)$ for an action a taken at state s.
- $0 < \delta < 1$ **discount factor**.

  $r(s_t, a_t)$ discounted back to period-0 is $\delta^t r(s_t, a_t)$.

Value Function

Same r, f, $\mathscr{F}$ for all t.

Definition

For allowable actions $a_t \in \mathscr{F}(s_t)$ and states $s_{t+1} = f(s_t, a_t)$

$$W(s_0, \{a_t\}) = \sum_{t=0}^{\infty} \delta^t r(s_t, a_t) \text{ **total reward**.}$$

Discount factor allows infinite sum possibly to be finite.

Definition

Value function $V : \mathbf{S} \to \mathbb{R}$ is supremum of total reward,

$$V(s_0) = \sup \{ W(s_0, \{a_t\}) : \{a_t\} \text{ allowable sequence} \}.$$

Problem: Maximize total reward $W(s_0, \{a_t\}) = \sum_{t=0}^{\infty} \delta^t r(s_t, a_t)$,

for allowable actions $a_t \in \mathscr{F}(s_t)$ and states $s_{t+1} = f(s_t, a_t)$.

So realize $V(s_0)$ as a maximum with some sequence of actions.

Optimal Stationary Strategy

Definition

A **stationary strategy** σ is a choice $\sigma(s) \in \mathscr{F}(s) \subset \mathbf{A}$ for each $s \in \mathbf{S}$

that is same for all periods.

Given a stationary strategy σ and s_0, by induction,

$a_t = a_t(s_0, \sigma) = \sigma(s_t)$,

$s_{t+1} = s_{t+1}(s_0, \sigma) = f(s_t, a_t)$, and

$W(s_0, \sigma) = \sum_{t=0}^{\infty} \delta^t r(s_t, a_t)$ total reward.

An **optimal stationary strategy** σ^* is a stationary strategy s.t.

$W(s_0, \sigma^*) = V(s_0)$ for all $s_0 \in \mathbf{S}$.

SDP Bellman Equation

Theorem (9 SDP Bellman Equation)

Value function $V(s)$ *satisfies* **Bellman Equation**

$$V(s) = \sup_{a \in \mathscr{F}(s)} r(s, a) + \delta\, V(\, f(s, a)\,).$$

Remark

Note that for an infinite horizon SDP,

same function V is on both sides of Bellman equation.

Necessary to solve equation for a function

and not just value for a given value of s.

Proof of Bellman Equation:

For any choice of actions, reward satisfies

$$\begin{aligned}\sum_{t=0}^{\infty} \delta^{t} r(s_t, a_t) &= r(s_0, a_0) + \delta \sum_{t=1}^{\infty} \delta^{t-1} r(s_t, a_t) \\ &= r(s_0, a_0) + \delta \sum_{j=0}^{\infty} \delta^{j} r(s_{j+1}, a_{j+1}). \\ &\leq r(s_0, a_0) + \delta \sup_{a_t,\ t \geq 1} \sum_{j=0}^{\infty} \delta^{j} r(s_{j+1}, a_{j+1}) \\ &\leq r(s_0, a_0) + \delta V(f(s_0, a_0)).\end{aligned}$$

Taking supremum over all allowable a_0,

$$V(s_0) \leq \sup_{a_0 \in \phi(s_0)} \{ r(s_0, a_0) + \delta V(f(s_0, a_0)) \}.$$

Also, can get a sequence of allowable actions s.t.

$W(s_0, \{a_t\})$ is within ϵ of right hand side,

$V(s_0) + \epsilon$ is $\geq$ right hand side, so equality.

Valid even if $= \infty$. **QED**

Outline of Theorems

For properties of value function assume either (i) $r(s, a)$ is bounded or

(ii) 1 sector economy with assumptions E1 − E2 on f, r,

Finite value function: $V(s) < \infty$ for each $s \in \mathbf{S}$,

so $V(s)$ well defined function: (i) Thm 10, (ii) Thm 13

Continuity: $\exists$ unique bounded fn satisfying Bellman equation.

By an iterative process like for FHDP, get sequence of $V_j(s)$ that

converge uniformly to $V(s)$ on bounded intervals $[0, \bar{s}]$,

so $V(s)$ is continuous. (i) Thm 11, (ii) Thm 13

Optimal Strategy: Exists by Param Max: Thms 12, 14(b)

First give examples finding value function using Bellman Equation

Then give precise theorems and proofs to show why works.

Example of Optimal Growth for One-Sector Economy

Optimal Growth of one-sector economy: More general case later

Determine Value Function and Optimal Strategy for

$$0 < \delta < 1,$$
$$\mathbf{S} = \mathbb{R}_+,$$
$$\mathbf{A} = \mathbb{R}_+,$$
$$\mathscr{F}(s) = [0, s],$$
$$f(s,a) = k(s-a), \qquad \text{with } \ k \geq 1 \ \text{ and } \ k\delta^2 < 1,$$
$$r(s,a) = u(a) = a^{\frac{1}{2}}.$$

$r(s,a) = a^{\frac{1}{2}}$ is not bounded on $\mathbb{R}_+$, but $s_t \leq k^t s_0$, $a_t \leq k^t s_0$,

$$\text{so } \ \delta^t r(s_t, a_t) \leq \delta^t u(k^t s_0) = \delta^t (k^t s_0)^{\frac{1}{2}} = \left(\delta k^{\frac{1}{2}}\right)^t s_0^{\frac{1}{2}},$$

$$V(s_0) \leq \sum_{t=0}^{\infty} \left(\delta k^{\frac{1}{2}}\right)^t s_0^{\frac{1}{2}} = \frac{1}{1 - \delta\, k^{\frac{1}{2}}}\, s_0^{\frac{1}{2}} < \infty.$$

Iterative Solution to get V

Solution Method 1: Form sequence of continuous fns $V_j(s)$

that converge to value function $V(s)$

Assume $V_j(s)$ is continuous

$$h_{j+1}(s,a) = r(s,a) + \delta\, V_j(f(s,a)) = a^{\frac{1}{2}} + \delta\, V_j(k(s-a)).$$

$$V_{j+1}(s) = h^*_{j+1}(s) = \max\{\, h_{j+1}(s,a) : a \in \mathscr{F}(s)\,\}.$$

continuous by Param Max Thm. $\mathscr{F}^*(s) \neq \emptyset$.

Start $V_0(s) \equiv 0$.

$$h_1(s,a) = a^{\frac{1}{2}} + \delta\, V_0\,(k(s-k)) = a^{\frac{1}{2}}.$$

h_1 is an increasing fn of a, is maximized on $[0,s]$ for $\bar{a} = s$,

$V_1(s) = h_1(s,\bar{a}) = s^{\frac{1}{2}}$, max 1 period, $t = 0$

Iterative Solution Method, contin.

$$h_2(s,a) = a^{\frac{1}{2}} + \delta\, V_1\,(k(s-a)) = a^{\frac{1}{2}} + \delta\, k^{\frac{1}{2}}(s-a)^{\frac{1}{2}}.$$

$$0 = \frac{\partial h_2}{\partial a} = \tfrac{1}{2} a^{-\frac{1}{2}} - \tfrac{1}{2}\delta\, k^{\frac{1}{2}}(s-a)^{-\frac{1}{2}},$$

$$(s-a)^{\frac{1}{2}} = \delta\, k^{\frac{1}{2}} a^{\frac{1}{2}}, \qquad s - a = \delta^2\, k\, a,$$

$$s = (1+\delta^2\, k)a, \qquad \bar{a} = \frac{s}{1+\delta^2\, k}.$$

Maximizer since $\dfrac{\partial^2 h_2}{\partial a^2} < 0$ everywhere on $[0, s]$

$$\begin{aligned} V_2(s) &= h_2(s,\bar{a}) = (\bar{a})^{\frac{1}{2}} + \delta\, k^{\frac{1}{2}}(s-\bar{a})^{\frac{1}{2}} \\ &= (\bar{a})^{\frac{1}{2}} + \delta\, k^{\frac{1}{2}} \delta\, k^{\frac{1}{2}}\,(\bar{a})^{\frac{1}{2}} \\ &= (1+\delta^2\, k)\, s^{\frac{1}{2}} (1+\delta^2\, k)^{-\frac{1}{2}} \\ &= (1+\delta^2\, k)^{\frac{1}{2}}\, s^{\frac{1}{2}}, \quad \text{max 2 periods, } t = 0, 1. \end{aligned}$$

Iterative Solution Method, contin.

Induction hypothesis $V_j(s) = \left(1 + \delta^2\, k + \cdots + \delta^{2j-2}\, k^{j-1}\right)^{\frac{1}{2}}\ s^{\frac{1}{2}}$

$$\sigma_j^*(s) = \left(1 + \delta^2\, k + \cdots + \delta^{2j-2}\, k^{j-1}\right)^{-1} s.$$

Assume true for $j = t$,

$$h_{t+1}(s,a) = r(s,a) + \delta\, V_t(f(s,a))$$

$$= a^{\frac{1}{2}} + \delta \left(1 + \cdots + \delta^{2t-2}\, k^{t-1}\right)^{\frac{1}{2}} \left(k(s-a)\right)^{\frac{1}{2}}.$$

$$0 = \frac{\partial h_{t+1}}{\partial a} = \tfrac{1}{2} a^{-\frac{1}{2}} - \tfrac{1}{2}\delta\, k^{\frac{1}{2}} \left(1 + \cdots + \delta^{2t-2}\, k^{t-1}\right)^{\frac{1}{2}} (s-a)^{-\frac{1}{2}}.$$

$$s - a = \delta^2 k \left(1 + \cdots + \delta^{2t-2}\, k^{t-1}\right) a$$

$$s = \left(1 + \cdots + \delta^{2t}\, k^{t}\right) a,$$

$\bar{a} = \left(1 + \cdots + \delta^{2t}\, k^{t}\right)^{-1} s.$ Verify induction for form of $\sigma_{t+1}^*(s)$.

Iterative Solution Method, contin.

$$\bar{a} = \left(1 + \cdots + \delta^{2t}\, k^t\right)^{-1} s.$$

$$V_{t+1}(s) = h^*_{t+1}(s) = h_{t+1}(s, \bar{a}).$$

$$= \bar{a}^{\frac{1}{2}} + \delta \, \left(1 + \cdots + \delta^{2t-2}\, k^{t-1}\right)^{\frac{1}{2}} \, (k(s - \bar{a}))^{\frac{1}{2}}.$$

$$= \bar{a}^{\frac{1}{2}} + \delta \left(1 + \cdots + \delta^{2t-2}\, k^{t-1}\right)^{\frac{1}{2}} \left(\delta^2 k^2 \left(1 + \cdots + \delta^{2t-2}\, k^{t-1}\right) \bar{a}\right)^{\frac{1}{2}}$$

$$= \bar{a}^{\frac{1}{2}} + \delta^2 k \left(1 + \cdots + \delta^{2t-2}\, k^{t-1}\right) \bar{a}^{\frac{1}{2}}$$

$$= \left(1 + \cdots + \delta^{2t}\, k^t\right) \bar{a}^{\frac{1}{2}}$$

$$= \left(1 + \cdots + \delta^{2t}\, k^t\right) \left(1 + \cdots + \delta^{2t}\, k^t\right)^{-\frac{1}{2}} s^{\frac{1}{2}}.$$

$$= \left(1 + \cdots + \delta^{2t}\, k^t\right)^{\frac{1}{2}} s^{\frac{1}{2}}.$$

Verifies induction for form of $V_{t+1}(s)$.

Iterative Solution Method, contin.

$$V_\infty(s) = \lim_{t\to\infty} V_t(s) = \lim_{t\to\infty} \left(1 + \delta^2 k + \cdots + \delta^{2t-2} k^{t-1}\right)^{\frac{1}{2}} s^{\frac{1}{2}}$$

$$= \frac{s^{\frac{1}{2}}}{(1-\delta^2 k)^{\frac{1}{2}}}.$$

$$V_\infty(s) = \lim_{t\to\infty} V_{t+1}(s) = \lim_{t\to\infty} \max\{\, a^{\frac{1}{2}} + \delta\, V_t(k(s-a)) : 0 \le a \le s \,\}$$

$$= \max\{\, a^{\frac{1}{2}} + \delta\, V_\infty(k(s-a)) : 0 \le a \le s \,\}$$

$V_\infty(s)$ satisfies Bellman equation

Since $V(s)$ is unique locally bounded solution of Bellman eq,

$$V(s) = V_\infty(s) = \frac{s^{\frac{1}{2}}}{(1-\delta^2 k)^{\frac{1}{2}}} \quad \text{value function.}$$

$$\sigma^*(s) = \lim_{t\to\infty} \sigma_t(s) = \lim_{t\to\infty} \left(1 + \cdots + \delta^{2t-2} k^{t-1}\right)^{-1} s$$

$$= (1 - k\,\delta^2)\, s \qquad \text{optimal strategy}$$

Steps Solving SDP by Iteration, Method 1

1. Start with $V_0(s) \equiv 0$ for all s.
2. Using Param Max Thm, by induction continuous

   $$V_{j+1}(s) = \max\{\, r(s,a) + \delta\, V_j(f(s,a)) : a \in \mathscr{F}(s) \,\}.$$

   $V_1(s)$ max over 1 period, $t = 0$

   $V_2(s)$ max over 2 periods, $t = 0, 1$

   $V_j(s)$ max over j periods, $t = 0, \dots, j-1$
3. $V_j(s) \to V(s)$ for each s. max over all periods.

   Converges uniformly on compact intervals so continuous.
4. $\sigma_j(s) \to \sigma^*(s)$ **optimal strategy**

   $$\sigma^*(s) \in \mathscr{F}^*(s) = \{\, a \in \mathscr{F}(s) : r(s,a) + \delta\, V(f(s,a)) \text{ is maximal} \,\}$$

One-Sector Economy, Solution Method 2

Solution Method 2: Find V & σ^* by **guessing** form of V. Guess

$$V(s) = M\, s^{\frac{1}{2}}, \quad M \text{ unspecified parameter}, \quad \text{related to } r(s,a).$$

Use Bellman equation to determine parameter M of guess.

$$h(s,a) = r(s,a) + \delta V(f(s,a)) = a^{\frac{1}{2}} + \delta\, M\, k^{\frac{1}{2}} (s-a)^{\frac{1}{2}}.$$

$$0 = \frac{\partial}{\partial a} h(s,a) = \tfrac{1}{2} a^{-\frac{1}{2}} - \delta\, M\, k^{\frac{1}{2}}\, \tfrac{1}{2} (s-a)^{-\frac{1}{2}},$$

$$a^{-\frac{1}{2}} = \delta\, M\, k^{\frac{1}{2}} (s-a)^{-\frac{1}{2}}, \qquad (s-a)^{\frac{1}{2}} = \delta\, M\, k^{\frac{1}{2}} a^{\frac{1}{2}},$$

$$s - a = a\, \delta^2 M^2 k, \qquad s = a\,(1 + \delta^2 M^2 k), \qquad \text{and}$$

$$\bar{a} = \frac{s}{1 + \delta^2 M^2 k} \le s \qquad \text{critical point}$$

$\dfrac{\partial^2}{\partial a^2} h(s,a) < 0$ on $[0, s]$, so $\bar{a}$ is a maximizer and an optimal strategy.

One-Sector Economy, Solution Method 2, continued

$\bar{a} = \sigma^*(s)$ and $V(s)$ must satisfy Bellman equation:

$$V(s) = r(s, \sigma^*(s)) + V(s, f(s, \sigma^*(s))),$$

$$M\, s^{\frac{1}{2}} = \bar{a}^{\frac{1}{2}} + \delta\, M\, k^{\frac{1}{2}} (s - \bar{a})^{\frac{1}{2}}$$

$$= \bar{a}^{\frac{1}{2}} + \delta\, M\, k^{\frac{1}{2}} \delta\, M\, k^{\frac{1}{2}} \bar{a}^{\frac{1}{2}} = \left(1 + \delta^2 M^2 k\right) \bar{a}^{\frac{1}{2}}$$

$$= \left(1 + \delta^2 M^2 k\right) \left[\frac{s}{1 + \delta^2 M^2 k}\right]^{\frac{1}{2}} \quad = \left(1 + \delta^2 M^2 k\right)^{\frac{1}{2}} s^{\frac{1}{2}}.$$

$$M^2 = 1 + \delta^2 M^2 k,$$

$$M^2 (1 - \delta^2 k) = 1$$

$$M^2 = \frac{1}{1 - \delta^2 k}, \qquad \text{and}$$

$$\bar{M} = \left[\frac{1}{1 - \delta^2 k}\right]^{\frac{1}{2}} \qquad \text{Need } \delta^2 k < 1.$$

Therefore, value function is

$$V(s) = \left[\frac{s}{1-\delta^2 k}\right]^{\frac{1}{2}}.$$

The optimal strategy is

$$\begin{aligned}\sigma^*(s) = \bar{a} &= \frac{s}{1+\delta^2 \bar{M}^2 k} \\ &= \frac{s}{\bar{M}^2} \\ &= (1-\delta^2 k)\, s \le s.\end{aligned}$$

End of Example

Steps Solving SDP using Method 2

1. Guess form of value function with unspecified parameters

   Use $r(s,a)$ or 1st few V_j from Method 1, to make guess.

2. Determine the critical point $\bar{a}$ of $h(s,a) = r(s,a) + \delta\, V(f(s,a))$

   using guess for $V(s)$

   Verify that $\bar{a}$ is maximizer for $a \in \mathscr{F}(s)$.

3. Calculate $h^*(s) = h(s, \bar{a})$ in terms of parameters.

4. Use Bellman equat. $V(s) = h^*(s)$

   to solve for unspecified parameters in guess.

   Gives $V(s)$ in terms of original data of problem.

5. Substitute parameters found into $\bar{a}$

   to get optimal strategy, $\sigma^*(s) = \bar{a}$

Vintner Example Attributed to Weitzman

On each day, a vintner can split his effort between:

$b_t \in [0, 1]$ is effort for baking bread and

$1 - b_t$ is effort for squeezing grapes for wine.

$w_{t+1} = 1 - b_t \in [0, 1]$ is amount of wine in next period.

$r(w_t, b_t) = \sqrt{w_t b_t}$ is reward or utility function for each period.

$0 < \delta < 1$ is discount factor.

Maximized $\sum_{t=0}^{\infty} \delta^t \sqrt{w_t b_t}$.

w_t is state variable,

b_t is action, and

$w_{t+1} = 1 - b_t$ is transition function.

Bellman equation

$$V(w) = \max\{ \sqrt{wb} + \delta V(1 - b) : b \in [0, 1] \}.$$

Method 1: function to be maximized as function of b is

$$h_{j+1}(w, b) = w^{\frac{1}{2}} b^{\frac{1}{2}} + \delta\, V_j(1-b) \qquad b \in [0, 1].$$

$V_0(w) = 0$ for all w,

$h_1(w, b) = w^{\frac{1}{2}} b^{\frac{1}{2}}$ is an increasing function of b

Max at $b = 1$: $V_1(w) = h_1(w, 1) = w^{\frac{1}{2}}$.

$h_2(w, b) = w^{\frac{1}{2}} b^{\frac{1}{2}} + \delta\,(1-b)^{\frac{1}{2}}$.

$$0 = \frac{\partial h_2}{\partial b} = \tfrac{1}{2} w^{\frac{1}{2}} b^{-\frac{1}{2}} - \tfrac{1}{2}\delta\,(1-b)^{-\frac{1}{2}},$$

$$w b^{-1} = \delta^2\,(1-b)^{-1},$$

$$w(1-b) = \delta^2\, b,$$

$$w = (w + \delta^2)\, b,$$

$$\bar{b} = \frac{w}{w + \delta^2}.$$

$$V_2(w) = h_2(w, \bar{b}) = w^{\frac{1}{2}} \bar{b}^{\frac{1}{2}} + \delta (1 - \bar{b})^{\frac{1}{2}}$$
$$= w^{\frac{1}{2}} \left(\frac{w}{w + \delta^2} \right)^{\frac{1}{2}} + \delta \left(\frac{\delta^2}{w + \delta^2} \right)^{\frac{1}{2}} = \frac{w + \delta^2}{\left[w + \delta^2\right]^{\frac{1}{2}}} = \left[w + \delta^2\right]^{\frac{1}{2}}.$$

Similar calculations show

$$V_3(w) = (1 + \delta^2)^{\frac{1}{2}} \left[w + \delta^2\right]^{\frac{1}{2}}$$

$$V_4(w) = (1 + \delta^2)^{\frac{1}{2}} \left[w + \delta^2 + \delta^4\right]^{\frac{1}{2}}$$

$$V_{2j}(w) = (1 + \delta^2 + \cdots + \delta^{2j-2})^{\frac{1}{2}} \left[w + \delta^2 + \cdots + \delta^{2j}\right]^{\frac{1}{2}}$$

$$V_{2j+1}(w) = \left(1 + \cdots + \delta^{2j}\right)^{\frac{1}{2}} \left[w + \delta^2 + \cdots + \delta^{2j}\right]^{\frac{1}{2}}$$

Note that constants appeared both before and under square root sign

Converges to value function

$$V(s) = \left(\frac{1}{1 - \delta^2} \right)^{\frac{1}{2}} \left[w + \frac{\delta^2}{1 - \delta^2} \right]^{\frac{1}{2}}.$$

Vintner Example, Solution Method 2

Method 2: Assume

$$V(w) = A(w + C)^{\frac{1}{2}}, \qquad A \text{ and } C \text{ unspecified parameters.}$$

Let

$$h(w, b) = b^{\frac{1}{2}} w^{\frac{1}{2}} + \delta A(1 - b + C)^{\frac{1}{2}}.$$

Critical point

$$0 = \frac{\partial h}{\partial b} = \tfrac{1}{2} b^{-\frac{1}{2}} w^{\frac{1}{2}} - \tfrac{1}{2} \delta A(C + 1 - b)^{-\frac{1}{2}}$$

$$w(C + 1 - b) = \delta^2 A^2 b$$

$$w(C + 1) = b \left[w + \delta^2 A^2 \right]$$

$$\bar{b} = \frac{w(C + 1)}{w + \delta^2 A^2}.$$

Vintner Example, Solution Method 2, continued

As a preliminary step to calculate maximum value:

$$(C + 1 - \bar{b}) = \left(\frac{\bar{b}}{w}\right) \delta^2 A^2,$$

$$\delta A (C + 1 - \bar{b})^{\frac{1}{2}} = \left(\frac{\bar{b}}{w}\right)^{\frac{1}{2}} \delta^2 A^2$$

$$= \frac{(C+1)^{\frac{1}{2}} \delta^2 A^2}{[w + \delta^2 A^2]^{\frac{1}{2}}}.$$

Max value $h(w, \bar{b}) = (w\,\bar{b})^{\frac{1}{2}} + \delta A (1 - \bar{b} + C)^{\frac{1}{2}}$ is

$$\frac{(C+1)^{\frac{1}{2}} w}{[w + \delta^2 A^2]^{\frac{1}{2}}} + \frac{(C+1)^{\frac{1}{2}} \delta^2 A^2}{[w + \delta^2 A^2]^{\frac{1}{2}}} = \frac{(C+1)^{\frac{1}{2}} \left[w + \delta^2 A^2\right]}{[w + \delta^2 A^2]^{\frac{1}{2}}}$$

$$= (C+1)^{\frac{1}{2}} \left[w + \delta^2 A^2\right]^{\frac{1}{2}}.$$

Vintner Example, Solution Method 2, continued

Bellman equation becomes

$$A(w+C)^{\frac{1}{2}} = (C+1)^{\frac{1}{2}} \left[w + \delta^2 A^2\right]^{\frac{1}{2}}.$$

Equating similar coefficients, we get $A = (C+1)^{\frac{1}{2}}$ and $C = \delta^2 A^2$, so

$$A^2 = \delta^2 A^2 + 1,$$

$$(1-\delta^2)\, A^2 = 1,$$

$$\bar{A}^2 = \frac{1}{1-\delta^2}, \qquad \text{and}$$

$$\bar{C} = \frac{\delta^2}{1-\delta^2}.$$

Value function is

$$V(w) = \left(\frac{1}{1-\delta^2}\right)^{\frac{1}{2}} \left[w + \frac{\delta^2}{1-\delta^2}\right]^{\frac{1}{2}} = \frac{\left[w(1-\delta^2) + \delta^2\right]^{\frac{1}{2}}}{1-\delta^2}.$$

Vintner Example, Solution Method 2, continued

Using

$$\bar{A}^2 = \frac{1}{1-\delta^2}$$

$$\bar{C} + 1 = \frac{\delta^2}{1-\delta^2} + 1 = \frac{\delta^2 + 1 - \delta^2}{1-\delta^2} = \frac{1}{1-\delta^2},$$

optimal strategy is

$$\sigma^*(w) = \bar{b} = \frac{w(\bar{C}+1)}{w + \delta^2 \bar{A}^2} = \frac{w\left[\frac{1}{1-\delta^2}\right]}{w + \frac{\delta^2}{1-\delta^2}}$$

$$= \frac{w}{w(1-\delta^2) + \delta^2}.$$

End of Example

3.2 Optimization for Bounded r

First assume bounded r:

SDB Reward fn r is continuous and bounded on $\mathbf{S} \times \mathbf{A}$:

$$r(s, a) \leq K \text{ for all } (s, a) \in \mathbf{S} \times \mathbf{A}.$$

Show if SDB, then $V(s) < \infty$, V is continuous,

and optimal strategy exists.

Finite Value Function

Theorem (10)

Assume SDP with bounded r (SDB).

Then value function $V(s)$ is a bounded function,

$$\exists\ 0 < K' < \infty \ \text{ s.t. } V(s) \leq K' \text{ for each } s.$$

Proof.

Total reward for any choice of actions is bounded:

$$\left|\sum_{t=0}^{\infty} \delta^t r(s_t, a_t)\right| \leq \sum_{t=0}^{\infty} \delta^t \left|r(s_t, a_t)\right| \leq \sum_{t=0}^{\infty} \delta^t K = \frac{K}{1-\delta} = K'.$$

Taking supremum over all allowable $\{a_t\}$,

$V(s_0) \leq K' < \infty$ for all s_0. V is bounded function. □

Continuous Value Function

Theorem (11)

Assume a SDP satisfies SDB & has bounded $V(s)$.

Then, $\exists$ *unique bounded fn that satisfies Bellman equation.*

Unique sol'n is continuous.

So, $V(s)$ *is continuous.*

Continuity of $V : \mathbf{S} \to \mathbb{R}$ cannot be proved directly from Bellman equat

because do not know *a priori* that right hand side is continuous.

Instead continuity is proved by means of a process that

takes a bounded fn and returns another bounded fn.

Earlier showed process for two examples.

Proof of Continuity

Proof: Assume $G : \mathbf{S} \to \mathbb{R}$ is any bounded function.

$$h_G(s,a) = r(s,a) + \delta\, G(f(s,a)),$$

$$\mathscr{T}(G)(s) = h^*_G(s) = \sup\{\, r(s,a) + \delta\, G(f(s,a)) : a \in \mathscr{F}(s)\,\}.$$

Can shown $\mathscr{T}(G) : \mathbf{S} \to \mathbb{R}$ is a new bounded function,

if G_1 and G_2 are two such functions

then $\mathscr{T}(G_1)$ and $\mathscr{T}(G_2)$ are closer together than

G_1 and G_2, i.e., $\mathscr{T}$ is a contraction on space of bounded fns.

Need to show set of bounded functions is complete,

i.e., a Cauchy sequence of fns getting closer together

must converge to a bounded function.

Follows that there is a unique bounded function that $\mathscr{T}$ takes to itself.

So $V(s)$ is unique function satisfying Bellman eq.

Proof of Continuity, contin.

If $V_0 : \mathbf{S} \to \mathbb{R}$ is any bounded function and inductively $V_{j+1} = \mathscr{T}(V_j)$,

then $V_j(s)$ converges to unique bounded function fixed by $\mathscr{F}$.

If V_0 is continuous, then all $V_j(s)$ are continuous

by Parametric Maximization Theorem.

In terms of distance on function space,

distance between functions $V_j(s)$ and $V(s)$ goes to zero,

$V_j(s)$ converges uniformly to $V(S)$, so $V(s)$ is continuous.

More details in online class book.

QED

Value Function as Limit of Finite Horizon Problem

Remark

Take $V_0(s) \equiv 0$ & $V_{j+1} = \mathscr{F}(V_j)$

$V_1(s)$ max over 1 period $t = 0$

$V_2(s)$ max over 2 periods $t = 0, 1$

$V_j(s)$ max over j periods $t = 0, \dots, j-1$

Theorem proves $\lim_{j\to\infty} V_j(x)$ is max for all periods $t \geq 0$,

so value function $V(s)$.

Optimal Strategy

Theorem (12 Optimal Strategy)

Assume a SDP has continuous, finite valued value function $V(s)$ *s.t.*

$$\lim_{t\to\infty} \delta^t V(s_t) = 0$$

for any allowable sequences of $\{a_t\}$ *with* $s_{t+1} = f(x_t, a_t)$.

Then, an optimal stationary strategy exists:

any choice function

$$\sigma^*(s) \in \mathscr{F}^*(s) = \arg\max\{\, r(s,a) + \delta\, V \circ f(s,a) : a \in \mathscr{F}(s)\,\}.$$

is an optimal strategy.

Remark

Theorem is valid for bounded $r(s,a)$ (SDB) so bounded $V(s)$.

Proof of Optimal Strategy Thm, contin.

$h(s, a) = r(s, a) + \delta\, V \circ f(s, a)$ is continuous. By Parm Max Them

$$\begin{aligned} h^*(s_0) &= \max_{a_0 \in [0, s_0]} r(s_0, a_0) + \delta\, V(f(s_0, a_0)) \\ &= \max_{a_0 \in [0, s_0]} \left[r(s_0, a_0) + \delta\ \max_{a_t,\, t \geq 1} \delta^{t-1} r(s_t, a_t) \right] \\ &= \max_{a_t,\, t \geq 0} \delta^t r(s_t, a_t) = V(s_0). \end{aligned}$$

Select $\sigma^*(s) \in \mathscr{F}^*(s) \neq \emptyset$ Show $V(s) = W(s, \sigma^*)$.

$$V(s_t) = r(s_t, \sigma^*(s_t)) + \delta\, V \circ f((s_t, \sigma^*(s_t)) = r(s_t, a_t) + \delta\, V(s_{t+1}),$$

$$\begin{aligned} V(s_0) &= r(s_0, a_0) + \delta\, V(s_1) \\ &= r(s_0, a_0) + \delta\ (r(s_1, a_1) + \delta\, V(s_2)) \\ &= r(s_0, a_0) + \delta\, r(s_1, a_1) + \delta^2 V(s_2) \\ &= r(s_0, a_0) + \delta\, r(s_1, a_1) + \delta^2 r(s_2, a_2) + \delta^3 V(s_3) \\ &= r(s_0, a_0) + \delta\, r(s_1, a_1) + \cdots + \delta^{T-1} r(s_{T-1}, a_{T-1}) + \delta^T V(s_T). \\ &\to W(s_0, \sigma^*) \text{ as } T \to \infty. \end{aligned}$$

Optimal strategy **QED**

$s \in \mathbb{R}_+$ supply of good (state), $c \in [0, s] = \mathscr{F}(s)$ consumption (action),

$r(s, c) = u(c)$ utility, $s_{t+1} = f(s_t - c_t)$ production to next period,

$0 < \delta < 1$ discount. Assumptions on, u & f:

E1. $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, with $u(0) = 0$.

No longer assume $r(s, c) = u(c)$ is bounded.

E2. **a.** $f : \mathbb{R}_+ \to \mathbb{R}$ continuous and increasing.

b. $f(0) = 0$: no free production.

c. Either

(i) $\exists\ \bar{x} > 0$ s. t. $f(x) \le x$ for $x \ge \bar{x}$ or

(ii) $\exists\ \lambda < 1$, s.t.

$$\delta\, u(f(x)) \le \lambda\, u(x) \text{ for } x \ge 0.$$

General One-Sector Economy, continued

No longer assume $r(s, c) = u(c)$ is bounded.

Example considered earlier: $u(c) = c^{\frac{1}{2}}$, $f(x) = k\,x$.

$u(c)$ is unbounded but

u & f satisfy E1 & E2 with c(ii) using $\lambda = \delta\,k^{\frac{1}{2}} < 1$:

$$\delta\,u(f(x)) = \delta\,k^{\frac{1}{2}}x^{\frac{1}{2}} = \lambda\,u(x) \quad \text{for } x \geq 0.$$

$V(s)$ for 1-SecE satisfies Bellman equation by Theorem 9.

Theorem (13)

If a $\mathrm{1\text{-}SecE}$ *satisfies E1- E2, then following are true.*

a. $V(s) < \infty$ *for each* $s \in \mathbb{R}_+$

b. $V(s)$ *is unique bounded solution of Bellman equation and is continuous.*

Proof using E2c(i)

Take any $\bar{s} \geq \bar{x}$. Restrict to $[0, \bar{s}]$. Take $s_0 \in [0, \bar{s}]$.

If $s_t \in [0, \bar{s}]$ & $s_{t+1} = f(s_t - c_t)$, then

$$0 = f(0) \leq f(s_t - c_t) = s_{t+1} \leq f(s_t) \leq f(\bar{s}) \leq \bar{s}.$$

all $s_t \in [0, \bar{s}]$,

r is bounded on $[0, \bar{s}]$.

(a) By Theorem 10, $V(s_0)$ is bounded and finite valued on $[0, \bar{s}]$.

(b) By Theorem 11, $\exists$ unique bounded solution of Bellman equation

$V(s)$ is continuous on $[0, \bar{s}]$.

$\bar{s} \geq \bar{x}$ arbitrary, so $V(s)$ is locally bounded, finite valued, continuous on all $\mathbb{R}_+$.

Proof using E2c(ii)

(a) Take $s_0 \geq 0$. For allowable sequence,

$$\delta\, u(c_t) \leq \delta\, u(s_t) = \delta\, u(f(s_{t-1} - c_{t-1})) \leq \delta\, u(f(s_{t-1})) \leq \lambda\, u(s_{t-1}).$$

$$\delta^t u(c_t) \leq \delta^t u(s_t) \leq \lambda^t u(s_0).$$

$$V(s_0) = \sup\{\textstyle\sum_t \delta^t u(c_t)\} \leq \sum_t \lambda^t u(s_0) < \infty.$$

(b) For $V^*(s) = A\,u(s)$ with $A = \frac{1}{1-\lambda}$,

$$\begin{aligned} u(a) + \delta\, A\, u(f(s-c)) &\leq u(s) + \delta\, A\, u(f(s)) \\ &\leq u(s) + A\,\lambda\, u(s) \\ &= A\, u(s). \end{aligned}$$

$$\begin{aligned} \mathscr{T}(V^*)(s) &= \sup\{\, u(c) + \delta\, A\, u(f(s,c)) : a \in [0,s]\,\} \\ &\leq A\, u(s) = V^*(s). \end{aligned}$$

Proof using E2(iii)(b), contin.

Let $V_0^*(s) = V^*(s)$ and $V_{j+1}^* = \mathscr{T}(V_j^*)$ for $j \geq 0$.

Since $V_1^*(s) = \mathscr{T}(V_0^*)(s) \leq V_0^*(s)$ for all s,

$V_{j+1}^*(s) \leq V_j^*(s)$ for all s by induction.

For each $s \geq 0$,

$V_j^*(s) \geq 0$ is a decreasing sequence,

$\lim_{j\to\infty} V_j^*(s)$ converges to $V_\infty^*(s)$

that satisfies the Bellman equation and so is the value function.

QED

General One-Sector Economy, continued

Theorem (14)

If a 1-SecE *satisfies E1- E2, then following hold.*

a. $V : \mathbb{R}_+ \to \mathbb{R}$ *is increasing.*

b. *There is an optimal strategy* $\sigma^*(s)$.

$$V(s) = u(\sigma^*(s)) + \delta\, V\left(f(s - \sigma^*(s))\right).$$

Proof (a).

Let c_t^* & s_t^* be optimal sequences with $c_t^* = \sigma^*(s_t^*)$, $s_{t+1}^* = f(s_t^* - c_t^*)$.

$s_0' > s_0^*$. Set $c_0' = c_0^* - s_0^* + s_0' > c_0^*$. $c_0' \leq s_0'$,

$$s_0' - c_0' = s_0^* - c_0^* \geq 0, \text{ so } s_1' = f(s_0' - c_0') = s_1^*.$$

Let $c_t' = c_t^*$ & $s_t' = s_t^*$ for $t \geq 1$. Allowable, not necessarily optimal

$$V(s_0') \geq \sum_t \delta^t u(c_t') = V(s_0^*) - u(c_0^*) + u(c_0') > V(s_0^*).$$ □

Proof of Optimal Strategy

Proof (b).

If E2c(i) is satisfied & $\bar{s} \geq \bar{x}$, then

$V(s)$ is bounded on $[0, \bar{s}]$ by proof of Thm 12(a)

Theorem 11 shows optimal strategy exists on $[0, \bar{s}]$, so $\mathbb{R}_+$.

If E2c(ii) is satisfied, then by proof of Theorem 12(a),

$$\delta^T V(s_T) = \sum_{t=T}^{\infty} \delta^t u(c_t) \leq \sum_{t=T}^{\infty} \lambda^t u(s_0) \to 0 \text{ as } T \to \infty,$$

since series $\sum_{t=1}^{\infty} \lambda^t$ converges.

Theorem 11 shows optimal strategy exists on all $\mathbb{R}_+$. □

These theorems show why the examples worked to find optimal strategy and continuous, increasing value function.

General One-Sector Economy, continued

More of the properties of value function and optimal strategy

of earlier example hold generally with following assumptions:

Assumptions on 1-SecE

E3. Utility function u is strictly concave on $\mathbb{R}_+$.

E4. Production function f is concave on $\mathbb{R}_+$.

E5. Utility function u is C^1 on $\mathbb{R}_{++}$ with $u'(0+) = \lim_{c \to 0+} u'(c) = \infty$.

E6. Production fn f is C^1 on $\mathbb{R}_{++}$ with $f'(0+) = \lim_{x \to 0+} f'(x) > 0$.

E3 – E6 are satisfied for $u(c) = c^{\frac{1}{2}}$ & $f(x) = k\,x$.

General One-Sector Economy, continued

Theorem

If a 1-SecE *satisfies E1- E4 with* u *&* f *concave,*

then the following hold.

a. V *is concave.*

b. *Correspondence* $\mathscr{F}^*$ *that gives maximizers of Bellman Equation is single-valued.*

Therefore, optimal strategy σ^* *is uniquely determined and is a continuous function on* $\mathbb{R}_+$.

Proofs this and following are given in online class book.

General One-Sector Economy, continued

Theorem (20, S12.27)

If a 1-SecE *satisfies E1- E6,*

$$\textit{with } u'(0+) = \infty \ \ \& \ \ f'(0+) > 0,$$

then optimal strategy σ^* *is increasing on* $\mathbb{R}_+$.