STABILITY OF ANOSOV DIFFEOMORPHISMS

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In these notes we give a proof of Anosov's theorem on structural stability of diffeomorphisms of a compact C^∞ manifold M without boundary. We also show that the Anosov diffeomorphisms form an open (maybe empty according to M) set in D, where D is the set of C^r diffeomorphisms of M with the C^r topology, $r \geq 1$.

The main references are Moser [1], Mather's appendix in [2] and Hirsch-Pugh [4].

Definition 1. Let $< \, , \, >$ be a C^∞ Riemannian metric on M and $| \cdot |$ its induced norm on $T_{\mathbf{x}}M$ for each $\mathbf{x} \in M$. We say that $f \in D$ is Anosov if

1. the tangent bundle of M splits in a Whitney direct sum of continuous sub-bundles $TM = E^s \oplus E^u$, where E^s and E^u are Df-invariant,
2. there exists constants $c, c' > 0$ and $0 < \lambda < 1$ such that

$$|Df^n_{\mathbf{x}}\mathbf{v}| < c\,\lambda^n |\mathbf{v}|$$
$$|Df^{-n}_{\mathbf{x}}\mathbf{w}| < c'\,\lambda^n |\mathbf{w}|$$

   for all $\mathbf{x} \in M$, $\mathbf{v} \in E^s_{\mathbf{x}}$, and $\mathbf{w} \in E^u_{\mathbf{x}}$ and $n > 0$.

M being compact, this definition is independent of the Riemannian metric $< \, , \, >$. Also, E^s and E^u are uniquely determined by the above conditions.

A vector bundle $\pi : E \to M$ of class C^r is said to be normed if there is a C^s $(0 \leq s \leq r)$ real function $F : E \to R$ such that $F|\pi^{-1}(\mathbf{x})$ defines a norm on $\pi^{-1}(\mathbf{x})$ for every $\mathbf{x} \in M$. We usually denote such a norm by $|\mathbf{v}|$.

Let $\pi : E \to M$ be a normed vector bundle over M. We denote by $\Gamma(E)$ the Banach space of continuous sections of E, with norm $\|\sigma\| = \sup_{\mathbf{x} \in M} |\sigma(\mathbf{x})|$, $\sigma \in \Gamma(E)$.

We denote $\Gamma(TM)$ simply by $\Gamma(M)$. If $f \in D$, then f induces a continuous operator $f_* : \Gamma(M) \to \Gamma(M)$, defined by $f_*\sigma = Df\sigma \circ f^{-1}$, $\sigma \in \Gamma(M)$. That is $f_*\sigma(\mathbf{x}) = Df_{f^{-1}(\mathbf{x})}\sigma(f^{-1}(\mathbf{x}))$. The linearity of f_* is clear and its continuity follows from the fact that M is compact. In fact, f_* is an isomorphism, where $(f_*)^{-1} = (f^{-1})_*$.

In order to prove that the Anosov diffeomorphisms form an open set, we need the following lemmas.

Lemma 1. *$f \in D$ is Anosov if and only if f_* is hyperbolic. Also, if f is Anosov then there is a C^∞ structure of normed vector bundle on TM for which we can take $c = c' = 1$ in Definition 1.*

Proof. If f is Anosov, then $\Gamma(M)$ splits in a direct sum of closed subspaces

$$\Gamma(M) = \Gamma(E^s) \oplus \Gamma(E^u)$$

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where

$$\sigma \in \Gamma(E^s) \quad \Leftrightarrow \quad \sigma(\mathbf{x}) \in E^s, \ \forall\ \mathbf{x} \in M$$
$$\sigma \in \Gamma(E^u) \quad \Leftrightarrow \quad \sigma(\mathbf{x}) \in E^u, \ \forall\ \mathbf{x} \in M.$$

Since E^s and E^u are Df-invariant, $\Gamma(E^s)$ and $\Gamma(E^u)$ are f_*-invariant. Let $f_s = f|\Gamma(E^s)$ and $f_u = f|\Gamma(E^u)$. Then $f_* = f_s \oplus f_u$ and f_s, f_u are (continuous) isomorphisms of $\Gamma(E^s)$, $\Gamma(E^u)$. This implies that

$$\text{Spectrum}(f_*) = \text{Spectrum}(f_s) \cup \text{Spectrum}(f_u).$$

But f being Anosov,

$$\|f_s^n\| \leq c\,\lambda^n$$
$$\|f_u^{-n}\| \leq c'\,\lambda^n.$$

Therefore the spectral radius of f_s and f_u^{-1} are not bigger than $\lambda < 1$. Thus f_* is hyperbolic.

Let us now assume that $f_* : \Gamma(M) \to \Gamma(M)$ is hyperbolic for $f \in D$ and $\Gamma(M)$ with the norm induced as before by a Riemannian metric on M. As in [3], $\Gamma(M)$ can be decomposed in a direct sum of f_*-invariant subspaces $\Gamma(M) = \Gamma^s \oplus \Gamma^u$, so that the spectral radius of $f_s = f_*|\Gamma^s$ and of $f_u^{-1} = f_*^{-1}|\Gamma^u$ are smaller than 1.

For each $\mathbf{x} \in M$, define

$$E^s_{\mathbf{x}} = \{\sigma(\mathbf{x}) \,|\, \sigma \in \Gamma^s\}$$
$$E^u_{\mathbf{x}} = \{\sigma(\mathbf{x}) \,|\, \sigma \in \Gamma^u\}.$$

It is not hard to see that $E^s = \bigcup_{\mathbf{x}\in M} E^s_{\mathbf{x}}$ and $E^u = \bigcup_{\mathbf{x}\in M} E^u_{\mathbf{x}}$ are continuous subbundles of TM, Df-invariant and $TM = E^s \oplus E^u$. To see that this sum is direct, we let $\mathbf{v} \in E^s_{\mathbf{x}} \cap E^u_{\mathbf{x}}$ for some $\mathbf{x} \in M$. Since the spectral radius of f_s and f_u^{-1} are smaller than 1, there is an integer n_0 so that $\|f_s^{n_0}\| < k$, $\|f_u^{-n_0}\| < k$ with $0 < k < 1$. Define $\sigma^s \in \Gamma^s$ and $\sigma^u \in \Gamma^u$ such that $\sigma^s(f^{-n_0}(\mathbf{x})) = Df_{\mathbf{x}}^{-n_0}\mathbf{v}$, $\|\sigma^s\| = |Df_{\mathbf{x}}^{-n_0}\mathbf{v}|$, $\sigma^u(\mathbf{x}) = \mathbf{v}$, and $\|\sigma^u\| = |\mathbf{v}|$. From this we get $\|f_s^{n_0}(\sigma^s)\| \leq k\,|Df_{\mathbf{x}}^{-n_0}\mathbf{v}|$ and $\|f_u^{-n_0}(\sigma^u)\| \leq k\,|\mathbf{v}|$. This means that $|\mathbf{v}| \leq k\,|Df_{\mathbf{x}}^{-n_0}\mathbf{v}|$ and $|Df_{\mathbf{x}}^{-n_0}\mathbf{v}| \leq k\,|\mathbf{v}|$, which implies that $\mathbf{v} = \mathbf{0}$ since $0 < k < 1$. Thus $TM = E^s \oplus E^u$.

Now set $\lambda = k^{1/n_0} < 1$,

$$c = \sup_{0 \leq i < n_0} \{\|f_s\|^i\,\lambda^{-i}\,\} \qquad \text{and}$$
$$c' = \sup_{0 \leq i < n_0} \{\|f_u^{-1}\|^i\,\lambda^i\,\}.$$

From $\|f_s^{n_0}\| < k$ and $\|f_u^{-n_0}\| < k$ we get

$$|Df_{\mathbf{x}}^n\mathbf{v}| \leq c\,\lambda^n\,|\mathbf{v}|$$
$$|Df_{\mathbf{x}}^{-n}\mathbf{w}| \leq c'\,\lambda^n\,|\mathbf{w}|$$

for each $\mathbf{x} \in M$, $\mathbf{v} \in E^s_{\mathbf{x}}$ and $\mathbf{w} \in E^u_{\mathbf{x}}$. This shows that f is Anosov, finishing the proof of the first part of the lemma.

Finally, we prove that if f is Anosov then there is a C^∞ norm on TM so that we can take $c = c' = 1$ in the above inequalities. Following [3], let ρ be such that

$\lambda < \rho < 1$ and define

$$|\mathbf{v}|_s = \sum_{n=0}^{\infty} \rho^{-n}\, |Df^n_{\mathbf{x}}\mathbf{v}|$$
$$|\mathbf{w}|_u = \sum_{n=0}^{\infty} \rho^{-n}\, |Df^{-n}_{\mathbf{x}}\mathbf{w}|$$

for $\mathbf{v} \in E^s_{\mathbf{x}}$ and $\mathbf{w} \in E^u_{\mathbf{x}}$. For any $\alpha \in T_{\mathbf{x}}M$, α can be written as $\alpha = \mathbf{v} + \mathbf{w}$, with $\mathbf{v} \in E^s_{\mathbf{x}}$ and $\mathbf{w} \in E^u_{\mathbf{x}}$. Define $|\alpha|_1 = |\mathbf{v}|_s + |\mathbf{w}|_u$. Then $|\ |_1$ is a norm equivalent to the original one and

$$|Df_{\mathbf{x}}\mathbf{v}|_1 \le \rho\, |\mathbf{v}|_1$$
$$|Df^{-1}_{\mathbf{x}}\mathbf{w}|_1 \le \rho\, |\mathbf{w}|_1$$

for $\mathbf{v} \in E^s_{\mathbf{x}}$ and $\mathbf{w} \in E^u_{\mathbf{x}}$. Of course, we can only say that $|\ |_1$ is a C^0 norm. But now we approximate $|\ |_1$ by a C^∞ norm so that the above inequalities still hold. □

Let E be a Banach space and E_1, E_2 closed subspaces so that $E = E_1 \oplus E_2$. Given $0 < \tau < 1$, we denote by $\mathcal{L}_\tau$ the hyperbolic isomorphisms L of E leaving E_1, E_2 invariant such that $\|L|E_1\| < \tau$ and $\|L^{-1}|E_2\| < \tau$.

The following lemma, due to Hirsch and Pugh, was proved in [4]. The proof we present here was suggested by Palis.

Lemma 2. *Given τ, $0 < \tau < 1$, there exists $\epsilon > 0$ such that if the isomorphism $T : E \to E$ with respect to the splitting $E = E_1 \oplus E_2$ has the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $L = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathcal{L}_\tau$ and $\|B\| < \epsilon$, $\|C\| < \epsilon$, then T is hyperbolic.*

Proof. First we notice that there exists $\epsilon > 0$ (which depends only on τ) such that if $\|B\| < \epsilon$, $\|C\| < \epsilon$, then T is locally conjugate to L. (See [3].) In fact, we get a global uniformly continuous conjugacy h between T and L, i.e., $\tilde{T}\,h = h\,L$, where $\tilde{T} = T$ near the origin. It is easy to see that the local images of E_1 and E_2, $h(E_1)$ and $h(E_2)$, generate closed linear subspaces $\tilde{E}_1$ and $\tilde{E}_2$, invariant by T and $\tilde{E}_1 \cap \tilde{E}_2 = \mathbf{0}$. Also, $\|T^n|\tilde{E}_1\| < 1$ and $\|T^{-n}|\tilde{E}_2\| < 1$ for some integer n, which imply that the spectral radii of $T|\tilde{E}_1$ and of $T^{-1}|\tilde{E}_2$ are less than one. Notice that $\tilde{E}_1$ and $\tilde{E}_2$ are characterized by the fact that $T^n\mathbf{v} \to \mathbf{0}$ and $T^{-n}\mathbf{w} \to \mathbf{0}$ as $n \to \infty$ for any $\mathbf{v} \in \tilde{E}_1$ and $\mathbf{w} \in \tilde{E}_2$.

Finally, we show that $E = \tilde{E}_1 \oplus \tilde{E}_2$. To see this, it is enough to show that $h(\mathbf{v} + \mathbf{w}) - h(\mathbf{v}) \in \tilde{E}_2$ for small $\mathbf{v} \in \tilde{E}_1$ and $\mathbf{w} \in \tilde{E}_2$. In fact,

$$\|L^{-n}(\mathbf{v} + \mathbf{w}) - L^{-n}\mathbf{v}\| = \|L^{-n}\mathbf{w}\| < \lambda^n\, \|\mathbf{w}\|.$$

Therefore,

$$h(L^{-n}(\mathbf{v} + \mathbf{w})) - h(L^{-n}\mathbf{v}) = T^{-n}(h(\mathbf{v} + \mathbf{w}) - h(\mathbf{v}))$$

converges to the origin as $n \to \infty$ for h uniformly continuous. Thus $E = \tilde{E}_1 \oplus \tilde{E}_2$ and since the spectral radii of $T|\tilde{E}_1$ and of $T^{-1}|\tilde{E}_2$ are less than one, T is hyperbolic. □

We can now prove the following theorem.

Theorem 1. *The Anosov diffeomorphisms form an open set in Diff(M).*

Proof. Let f be an Anosov diffeomorphism. Then f_* is a hyperbolic isomorphism of $\Gamma(M)$. Thus $\Gamma(M) = \Gamma_s \oplus \Gamma_u$, where Γ_s and Γ_u are given by Lemma 1. Γ_s and Γ_u are f_*-invariant and $\|f_*|\Gamma_s\| < \tau$, $\|f_*^{-1}|\Gamma_u\| < \tau$ for some τ such that $0 < \tau < 1$. It is immediate that given $\epsilon > 0$, there is a neighborhood $N(f) \subset \mathrm{Diff}(M)$ with the property that for any $g \in N(f)$, $g_* = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to the splitting $\Gamma(M) = \Gamma_s \oplus \Gamma_u$, where $\|A\| < \tau$, $\|D^{-1}\| < \tau$, $\|B\| < \epsilon$, and $\|C\| < \epsilon$. Thus taking ϵ as in Lemma 2, g_* is hyperbolic, and by Lemma 1, g is Anosov. □

Remark 1. Notice that the map $\rho : \mathrm{Diff}(M) \to \mathrm{Isom}(\Gamma(M))$, defined by $\rho(g) = g_*$, is not continuous. What we used in the proof above was the continuity of the norm of the operators corresponding to the decomposition of g_* with respect to the splitting $\Gamma(M) = \Gamma_s \oplus \Gamma_u$.

As before, we denote by D the space of diffeomorphisms on M with the C^1 topology. We denote by H the space of homeomorphisms on M with the C^0 topology.

Theorem 2. *(Anosov) If f is an Anosov diffeomorphism then f is structurally stable. In particular, there exists a neighborhood V of f in D, a neighborhood U of the identity $id : M \to M$ in H, and a continuous function $h : V \to U$ such that if $g \in V$ then $h = h(g)$ is the unique solution in U of the functional equation*

$$h \circ g = f \circ h.$$

Before proving the theorem we need several definitions, constructions, and lemmas.

Definition 2. Let K_1 and K_2 be compact metric spaces, U an open subset of a Banach space F_1, and V an open subset of a Banach space F_2. Suppose that we have $f : K_1 \to K_2$ and $\bar{f} : K_1 \times U \to K_2 \times V$ continuous such that the following diagram is commutative:

$$\begin{array}{ccccc} K_1 \times U & \xrightarrow{\bar{f}} & K_2 \times V & \xrightarrow{p_2} & V \\ \pi\downarrow & & \downarrow p_1 & & \\ K_1 & \xrightarrow{f} & K_2 & & \end{array}$$

where π, p_1, and p_2 are projections. We say that $\bar{f}$ is *vertically of class C^r* $(r \geq 0)$ if $p_2 \circ f$ has r partial derivatives with respect to the variable in U and the partials are continuous mappings

$$D_2^k\,(p_2 \circ \bar{f}) : K_1 \times U \to L_s^k(F_1, F_2)$$

for $k = 0, \ldots, r$. Here $L_s^k(F_1, F_2)$ are symmetric k-multilinear mappings from F_1 to F_2. In particular, for each fixed $\mathbf{x} \in K_1$, $p_2 \circ \bar{f}(\mathbf{x}, \cdot) : U \to V$ is of class C^r.

Definition 3. Let $\pi_1 : E^1 \to M$ and $\pi_2 : E^2 \to N$ be two Riemannian vector bundles of class C^0 over compact metric spaces M and N. Let $\bar{f} : E^1 \to E^2$ be a continuous map that preserves fibers, i.e., there exists a map $f : M \to N$ such that $f \circ \pi_1 = \pi_2 \circ \bar{f}$. We say that $\bar{f}$ is *vertically of class C^r* or $\bar{f}$ is *of class C^r along the fibers*, $0 \leq r \leq \infty$, if the local representatives of $\bar{f}$ in local vector bundle charts are vertically of class C^r (using Definition 2).

Let $f : M \to N$ be a continuous function and $\pi : E \to M$ a Riemannian vector bundle. $f^*(E)$ is the subset of $M \times E$ of pairs $(\mathbf{x}, \mathbf{v})$ such that $f(\mathbf{x}) = \pi(\mathbf{v})$. Let $\pi(f)$ be the projection on the first factor of $M \times E$. $\pi(f) : f^*(E) \to M$ is a vector bundle. There is a Riemannian metric induced on $f^*(E)$ by the inclusion in $M \times E$.

Let $\pi_i : E^i \to M$ $i = 1, 2$ be two Riemannian vector bundles. Let $U \subset E^1$ be an open subset such that $\pi_1|U : U \to M$ is a surjection. Let $\Gamma(U) \subset \Gamma(E^1)$ be the open subset of sections with images in U. We assume U is connected enough so the $\Gamma(U)$ is nonempty.

Let $\bar{f} : U \to E^2$ be a continuous function that preserves fibers covering $f : M \to M$. We denote by

$$\Omega_{\bar{f}} : \Gamma(U) \to \Gamma(f^* E^2)$$

the map induced by composition on the left by $\bar{f}$,

$$\Omega_{\bar{f}} : \gamma \mapsto \bar{f} \circ \gamma.$$

Lemma 3. *If $\bar{f}$ is vertically of class C^r, $0 \leq r \leq \infty$, then $\Omega_{\bar{f}} : \Gamma(U) \to \Gamma(f^* E^2)$ is of class C^r.*

Proof. For $r = 0$, $\Omega_{\bar{f}}$ corresponds to the composition of continuous functions on a compact set. It is a standard result that $\Omega_{\bar{f}}$ is continuous.

Let $\gamma \in \Gamma(U)$. Let $\sigma \in \Gamma(E^1)$ be small enough in norm so that $\gamma + \sigma \in \Gamma(U)$. For each $\mathbf{x} \in M$, we apply Taylor's Theorem to the function $\bar{f}_{\mathbf{x}} : E^1_{\mathbf{x}} \to E^2_{f(\mathbf{x})}$ at the point $\gamma(\mathbf{x})$. We obtain

$$\begin{aligned} \Omega_{\bar{f}}(\gamma + \sigma)(\mathbf{x}) = \Omega_{\bar{f}}(\gamma)(\mathbf{x}) &+ \sum_{k=1}^{r} \frac{1}{k!} D^k \bar{f}_{\mathbf{x}}(\gamma(\mathbf{x}))(\sigma(\mathbf{x}))^k \\ &+ R(\gamma(\mathbf{x}), \sigma(\mathbf{x}))(\sigma(\mathbf{x}))^r. \end{aligned} \tag{1}$$

Here $(\sigma(\mathbf{x}))^k = (\sigma(\mathbf{x}), \ldots, \sigma(\mathbf{x}))$, and $R(\mathbf{x}, \mathbf{y}) \in L^r_x(E^1_{\mathbf{x}}, E^2_{f(\mathbf{x})})$. $L^r_x(E^1_{\mathbf{x}}, E^2_{f(\mathbf{x})})$ are symmetric r-multilinear functions from $E^1_{\mathbf{x}}$ to $E^2_{f(\mathbf{x})}$. Writing formula (1) without evaluation at $\mathbf{x}$ we obtain

$$\Omega_{\bar{f}}(\gamma + \sigma) = \Omega_{\bar{f}}(\gamma) + \sum_{k=1}^{r} \frac{1}{k!} D^k \bar{f}_{\mathbf{x}}(\gamma)(\sigma)^k + R(\gamma, \sigma)(\sigma)^r \tag{2}$$

where we are only taking the derivative of $\bar{f}$ along the fiber and

$$R(\gamma, \sigma) \in L^r_x(\Gamma(E^1), \Gamma(f^* E^2)).$$

We leave it to the reader to check that $R(\ ,\ )$ is continuous and that $R(\gamma, \mathbf{0}) = \mathbf{0}$. By the converse to Taylor's Theorem, [7,2.1], it follows that $\Omega_{\bar{f}}$ is of class C^r and that

$$D^k \Omega_{\bar{f}}(\gamma)(\sigma_1, \ldots, \sigma_k) = D^k \bar{f}_{\mathbf{x}}(\gamma)(\sigma_1, \ldots, \sigma_k)$$

for $\sigma_1, \ldots, \sigma_k \in \Gamma(E^1)$. Then $D^k \Omega_{\bar{f}} : \Gamma(U) \to L^k_x(\Gamma(E^1), \Gamma(f^* E^2))$. □

The following lemma is obvious.

Lemma 4. *Let $\pi : E \to N$ be a Riemannian vector bundle of class C^0. Let M and N be compact metric spaces. Let $f : M \to N$ be a continuous function. Let $A_f : \Gamma(E) \to \Gamma(f^* E)$ be defined by $\gamma \mapsto \gamma \circ f$. Then for fixed f, A_f is a continuous linear function in γ and hence C^∞.*

Let C be the space of continuous functions from M to M. We give M a C^∞ Riemannian metric. The topology of C is given by the metric $\bar{d}$:

$$\bar{d}(f,g) = \sup\{d(f(\mathbf{x}), g(\mathbf{x})) \ : \ \mathbf{x} \in M\}$$

where d is the distance between points of M induced by the Riemannian structure on M. In Theorem 2 we have

$$H = \{h \in C \ : \ h \text{ is a homeomorphism } \}.$$

We take this opportunity to give the construction that makes C into a Banach manifold. To prove Theorem 2 we only use the local coordinate chart at the identity given by the following lemma.

Lemma 5. *C admits the structure of a C^∞ manifold modeled on a Banach space.*

Proof. Let $\mathcal{U}$ be an open cover of M by convex neighborhoods. (Convex with respect to the Riemannian structure.) Let $\delta > 0$ be a Lebesgue number associated to the open cover, i.e., given a ball B of radius less than or equal to δ there exists a $U \in \mathcal{U}$ such that $B \subset U$.

Let $f \in C$. Let $\Gamma(f)$ denote the Banach space of continuous sections of $f^*(TM)$, $\Gamma(f^*(TM))$. Let $U(f) = U_\delta(f)$ be the open ball in $\Gamma(f)$ of radius δ centered at the zero section. Let $B(f) = B_\delta(f)$ be the open ball in C centered at f or radius δ. We parameterize $B(f)$ by $U(f)$ as follows. Let $\phi_f : U(f) \to B(f)$ be given by

$$(\phi_f(\sigma))(\mathbf{x}) = \exp_{f(\mathbf{x})}(\sigma(\mathbf{x}))$$

for $\sigma \in U(f)$. We have that

$$\begin{aligned}
\bar{d}(\phi_f(\sigma_1), \phi_f(\sigma_2)) &= \sup_{\mathbf{x} \in M} d(\exp_{f(\mathbf{x})} \sigma_1(\mathbf{x}), \exp_{f(\mathbf{x})} \sigma_2(\mathbf{x})\,) \\
&\leq \sup_{\mathbf{x} \in M} \{|\sigma_1(\mathbf{x}) - \sigma_2(\mathbf{x})|\} \\
&\leq \|\sigma_1 - \sigma_2\|.
\end{aligned}$$

Therefore ϕ_f is continuous. On the other hand, ϕ_f has an inverse $\phi_f^{-1} : B(f) \to U(f)$ defined by

$$\phi_f^{-1}(g)(\mathbf{x}) = (\mathbf{x}, \, (\exp_{f(\mathbf{x})})^{-1}(g(\mathbf{x}))).$$

Because the neighborhoods in $\mathcal{U}$ are convex, the expression $(\exp_{f(\mathbf{x})})^{-1}(g(\mathbf{x}))$ is well defined. By the uniform continuity of the exponential on M, it follow there is a constant e such that

$$\begin{aligned}
\|\phi_f^{-1}(g_1) - \phi_f^{-1}(g_2)\| &= \sup_{\mathbf{x} \in M} \{|(\exp_{f(\mathbf{x})})^{-1}(g_1(\mathbf{x})) - (\exp_{f(\mathbf{x})})^{-1}(g_2(\mathbf{x}))|\} \\
&\leq e \sup_{\mathbf{x} \in M} \{d(g_1(\mathbf{x}), g_2(\mathbf{x}))\} \\
&\leq e \, \bar{d}(g_1, g_2).
\end{aligned}$$

Thus ϕ_f^{-1} is continuous.

We have defined an atlas for C, whose local charts are modeled on the Banach spaces $f^*(TM)$ where $f \in C$. To complete the proof, it suffices to show the changes of coordinates are C^∞.

Let $\phi_f : U(f) \to B(f)$ and $\phi_g : U(g) \to B(g)$ be two charts. We need to prove that

$$\phi_g^{-1}\phi_f : U(f) \to U(g)$$

is a diffeomorphism of class C^∞ on it domain of definition.

Let $V(f) = \{\mathbf{v} \in f^*(TM) \ : \ |\mathbf{v}| < \delta\}$ and $V(g) = \{\mathbf{v} \in g^*(TM) \ : \ |\mathbf{v}| < \delta\}$. Then $U(f) = \Gamma(V(f))$, $U(g) = \Gamma(V(g))$. Define the homeomorphism $G : V(f) \to V(g)$ by

$$G(\mathbf{x}, \mathbf{v}) = (\mathbf{x}, (\exp_{g(\mathbf{x})})^{-1} \circ \exp_{f(\mathbf{x})} \mathbf{v}).$$

G is well defined by the convexity of the neighborhoods. We have that $\phi_g^{-1}\phi_f(\mathbf{v}) = G \circ \mathbf{v} = \Omega_G(\mathbf{v})$. G preserves fibers. G is vertically of class C^∞ since along a fixed fiber

$$G(\mathbf{x}, \cdot) = (\mathbf{x}, (\exp_{g(\mathbf{x})})^{-1} \circ \exp_{f(\mathbf{x})} \cdot).$$

By Lemma 3, $\phi_g^{-1}\phi_f$ is of class C^∞. In the same way, $(\Omega_G)^{-1} = \Omega_{G^{-1}} = \phi_f^{-1}\phi_g$ is of class C^∞. □

Remark 2. The tangent space of C at f, T_fC, can be identified with $\Gamma(f^*TM)$. In particular, $T_{\mathrm{id}}C = \Gamma(TM) = \Gamma(M)$.

Remark 3. Let $\Lambda \subset M$ be a compact subset. Let $B(\Lambda, M)$ be the topological space of bounded functions from Λ to M. Then we can give $B(\Lambda, M)$ the structure of a manifold of class C^∞ modeled on bounded sections of $TM|\Lambda$.

Proof of Theorem 2: We want to look at the map $D \times D \times C \to C$ given by $(g_1, g_2, h) \mapsto g_1 \circ h \circ g_2^{-1}$. If $g_1 \circ h \circ g_2^{-1} = h$ then $g_1 \circ h = h \circ g_2$. Thus fixed points of the map give a semiconjugacy between g_1 and g_2. (To be a conjugacy, we need h to be a homeomorphism.) Also $g \circ \mathrm{id} \circ g^{-1} = \mathrm{id}$. We want to prove the stability of this fixed point.

We take local coordinates in C near id, $\phi : U \subset \Gamma(M) \to C$ with $\phi(\sigma)(\mathbf{x}) = \exp_{\mathbf{x}} \sigma(\mathbf{x})$. For neighborhoods V of f in D and U of $\mathbf{0}$ in $\Gamma(M)$,

$$A : V \times V \times U \to \Gamma(M)$$

is well defined by

$$A(g_1, g_2, h) = \phi^{-1}(g_1 \circ \phi(h) \circ g_2^{-1}), \qquad \text{or}$$
$$A(g_1, g_2, h)(\mathbf{x}) = \exp_{\mathbf{x}}^{-1}(g_1 \circ \exp_{g_2^{-1}(\mathbf{x})} \circ (h \circ g_2^{-1}(\mathbf{x}))).$$

For $g_1, g_2 \in V$, define $G(g_2, g_2) : TM \to TM$ by

$$G(g_1, g_2)(\mathbf{v}_{\mathbf{x}}) = \exp_{g_2(\mathbf{x})}^{-1}(g_2 \circ \exp_{\mathbf{x}} \mathbf{v}_{\mathbf{x}}).$$

Then

$$\begin{aligned}(\Omega_{G(g_1,g_2)} A'_{g_2^{-1}} h)(\mathbf{x}) &= G(g_1, g_2) \circ h \circ g_2^{-1}(\mathbf{x}) \\ &= \exp_{\mathbf{x}}^{-1}(g_1 \circ \exp_{g_2^{-1}(\mathbf{x})}(h \circ g_2^{-1}(\mathbf{x}))) \\ &= A(g_1, g_2, h)(\mathbf{x}).\end{aligned}$$

Here $A'_{g_2^{-1}}$ is the map given by Lemma 4.

Lemma 6. *A has a partial derivative with respect to the third variable. When* $g_1 = g_2 = g$ *we have*

$$D_3A(g, g, 0)k = Dg(g^{-1})k \circ g^{-1} = g_*k.$$

$D_3A(g_1, g_2, h)$ is continuous in the first and third variables, uniformly in the second variable, i.e., given (g_1, h) and $\epsilon > 0$ there exists neighborhoods V' of g_1 and U' of h such that for $f_{11}, f_{12} \in V'$, $f_2 \in V$, and $h_1, h_2 \in U'$

$$\|D_3A(f_{11}, f_2, h_1) - D_3A(f_{12}, f_2, h_2)\| < \epsilon.$$

In particular, given $\epsilon > 0$, there exist neighborhoods V' of f and U' of 0 in $\Gamma(M)$ such that the Lipschitz constant

$$L(A(f_{11}, f_2, \cdot)|U' - D_3A(f_{11}, f_2, 0)|U') < \epsilon$$

for $g_1, g_2 \in V'$.

Proof. By Lemmas 3 and 4, the partial derivative of A with respect to the third variable exists. Since $D(\exp_{\mathbf{x}})(0_{\mathbf{x}}) = \mathrm{id} : T_{\mathbf{x}}M \to T_{\mathbf{x}}M$, it follows that

$$D_3A(g, g, 0)k = Dg(g^{-1})k \circ g^{-1}.$$

Let $G_1 = G(f_{11}, f_2)$ and $G_2 = G(f_{12}, f_2)$. Then

$$\begin{aligned} &\|D_3A(f_{11}, f_2, h_1) - D_3A(f_{12}, f_2, h_2)\| \\ &\qquad = \sup_{k \in \Gamma(M) \text{ with } \|k\|=1} \{\|DG_1(h_1 \circ f_2^{-1})k \circ f_2^{-1} - |DG_2(h_2 \circ f_2^{-1})k \circ f_2^{-1}\|\} \\ &\qquad \le \sup_{\mathbf{x} \in M} \{\|DG_1(h_1 \circ f_2^{-1}(\mathbf{x})) - DG_2(h_2 \circ f_2^{-1}(\mathbf{x}))\|\}. \end{aligned}$$

Using the uniformity in the exponential, and letting $f_{11}, f_{12} \to g$ and $h_1, h_2 \to h$, we get that

$$\|DG_1(h_1 \circ f_2^{-1}(\mathbf{x})) - DG_2(h_2 \circ f_2^{-1}(\mathbf{x}))\| \to 0$$

uniformly in f_2 and $\mathbf{x}$. Remember that $G_i(f_2^{-1}(\mathbf{x}), \mathbf{v}) = \exp_{\mathbf{x}}^{-1}(f_{1i} \circ \exp_{f_2^{-1}(\mathbf{x})} \mathbf{v})$. This proves the desired continuity of D_3A.

The Lipschitz constant follows from the above results using the Mean Value Theorem. See [5, 8.6.2] for example. □

Remark 4. (**Caution**) $D_3A(g_1, g_2, h)$ is not continuous in g_2. To see this consider the case of a map defined in the plane so we can disregard the exponentials. Let $h = 0$ and take g_2' arbitrarily near g_2 in the C^1 topology but with $(g_2)^{-1}(\mathbf{x}_0) \neq (g_2')^{-1}(\mathbf{x}_0)$. For each such g_2' there exists a $k \in \Gamma(M)$ such that $\|k\|_0 = 1$ and $|k \circ (g_2)^{-1}(\mathbf{x}_0) - k \circ (g_2')^{-1}(\mathbf{x}_0)| = 1$. Then

$$\begin{aligned} &\|D_3A(g_1, g_2, 0) - D_3A(g_1, g_2', 0)\| \\ &\qquad \ge |D(g_1)_{(g_2)^{-1}(\mathbf{x}_0)}\, k \circ g_2^{-1}(\mathbf{x}_0) - D(g_1)_{(g_2')^{-1}(\mathbf{x}_0)}\, k \circ (g_2')^{-1}(\mathbf{x}_0)|. \end{aligned}$$

This stays bounded away from zero as g_2' goes to g_2.

However the following lemma gives a partial result in this direction.

Lemma 7. *Let $T : D \times D \times \Gamma(M) \to \Gamma(M)$ be defined by*

$$T(g_1, g_2, h) = D_3A(g_1, g_2, 0)h.$$

Then T is continuous in all variables.

In fact $D_3A(g_1, g_2, 0)h$ is a continuous function of g_1, g_2, and h. The point is that it is not necessary to take the supremum over all h of unit length but just those near h_0. The proof is left to the reader.

The following lemma is what we need to prove the stability of the fixed point of A. It is based on the last paragraph of page 144 in [4]. If $D_3A : D \times D \times \Gamma(M) \to$

$L(\Gamma(M), \Gamma(M))$ were continuous, then we could use a standard fixed point theorem or the Implicit Function Theorem.

Lemma 8. *Let P be a topological space. Let $F_1 \oplus F_2$ be a Banach space with the norm equal to the maximum of the norms on the two factors. Let $T : P \times F_1 \oplus F_2 \to F_1 \oplus F_2$ be a function (not necessarily continuous) such that for each $\mathbf{x} \in P$, $T(\mathbf{x}, \cdot) : F_1 \oplus F_2 \to F_1 \oplus F_2$ is a continuous linear isomorphism. Assume $\|T_1(\mathbf{x}, \cdot, 0)^{-1}\| \le \tau$, $\|T_2(\mathbf{x}, 0, \cdot)\| \le \tau$, $\|T_1(\mathbf{x}, 0, \cdot)\| \le \mu$, and $\|T_2(\mathbf{x}, , \cdot, 0)\| \le \mu$ where $T_i(\mathbf{x}, \cdot, 0) : F_1 \to F_i$. We also have $\epsilon > 0$ such that $\tau + \mu + \epsilon < 1$. Let $U_1 \oplus U_2 \subset F_1 \oplus F_2$ be a ball about the origin of radius R. Assume $f : P \times U_1 \oplus U_2 \to F_1 \oplus F_2$ is a function such that for all $\mathbf{x} \in P$, (i) the Lipschitz constant $L(f(\mathbf{x}, \cdot) - T(\mathbf{x}, \cdot)|U_1 \oplus U_2) < \epsilon$ and (ii) $|f(\mathbf{x}, 0, 0)| \le (1 - \tau - \mu - \epsilon)R$. Then there exists a function $u : P \to U_1 \oplus U_2$ such that $f(\mathbf{x}, u(\mathbf{x})) = u(\mathbf{x})$ and $|u(\mathbf{x})| \le |f(\mathbf{x}, 0, 0)|/(1 - \tau - \mu - \epsilon)$. Further, if f and T are continuous then so is u.*

Proof. Define $g : P \times U_1 \oplus U_2 \to F_1 \oplus F_2$ by

$$g(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2) = (T_1(\mathbf{x}, \cdot, \mathbf{0})^{-1}(\mathbf{y}_1 + T_1(\mathbf{x}, \mathbf{y}_1, \mathbf{0}) - f_1(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)), f_2(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)).$$

Note the fixed points of $g(\mathbf{x}, \cdot)$ are the same as those of $f(\mathbf{x}, \cdot)$.

First we show $g(\mathbf{x}, \cdot)$ is a contraction with contraction constant $\tau + \mu + \epsilon$. Let $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$ and $\mathbf{y}' = (\mathbf{y}'_1, \mathbf{y}'_2)$.

$$\begin{aligned}
&|g_1(\mathbf{x}, \mathbf{y}) - g_1(\mathbf{x}, \mathbf{y}')| \\
&\qquad \le \tau(|\mathbf{y}_1 - \mathbf{y}'_1| + L(T_1 - f_1)|\mathbf{y} - \mathbf{y}'| + |T_1(\mathbf{x}, \mathbf{0}, \mathbf{y}_2 - \mathbf{y}'_2)|) \\
&\qquad \le \tau(1 + \epsilon + \mu)|\mathbf{y} - \mathbf{y}'| \\
&\qquad \le (\tau + \epsilon + \mu)|\mathbf{y} - \mathbf{y}'| \\
&|g_2(\mathbf{x}, \mathbf{y}) - g_2\mathbf{x}, \mathbf{y}')| \\
&\qquad \le |T_2(\mathbf{x}, \mathbf{y} - \mathbf{y}')| + L(f_2(\mathbf{x}, \cdot) - T_2(\mathbf{x}, \cdot))|\mathbf{y} - \mathbf{y}'| \\
&\qquad \le (\tau + \epsilon + \mu)|\mathbf{y} - \mathbf{y}'|.
\end{aligned}$$

By [6, 10.1.1], [4, 1.1], or [5], g has a fixed point, $u(\mathbf{x})$, for each $\mathbf{x} \in P$ with

$$|u(\mathbf{x})| \le |g(\mathbf{x}, \mathbf{0})|/(1 - \tau - \mu - \epsilon) \le |f(\mathbf{x}, \mathbf{0})|/(1 - \tau - \mu - \epsilon).$$

Now assume f and T are continuous, so g is continuous. For $\mathbf{x}_0 \in P$, by [6,10.1.1], [4, 1.1], or [5],

$$|u(\mathbf{x}_0) - u(\mathbf{x})| \le |g(\mathbf{x}, u(\mathbf{x}_0)) - u(\mathbf{x}_0)|/(1 - \tau - \mu - \epsilon).$$

This shows that u is continuous. ∎

$T(f, f, \cdot) = D_3A(f, f, 0) = f_* : \Gamma(M) \to \Gamma(M)$ is hyperbolic and has a splitting $\Gamma(M) = \Gamma(E^u) \oplus \Gamma(E^s)$. We can approximate E^u and E^s by smooth bundles F^u and F^s. Let $F_1 = \Gamma(F^u)$ and $F_2 = \Gamma(F^s)$. For a small neighborhood V of f we can insure that $T(g_1, g_2, h) = D_3A(g_2, g_2, 0)h$ satisfies Lemma 8. This follows from the continuity of the norms of the coordinate functions and Lemmas 6 and 7. On $F_1 \oplus F_2$ we take the norm $|(\mathbf{y}_1, \mathbf{y}_2) = \max\{|\mathbf{y}_1|, |\mathbf{y}_2|\}$.

Thus there exists neighborhoods V_1 of f in D and U'_1 of 0 in $\Gamma(M)$ such that for $g_1, g_2 \in V_1$ there exists a unique $k = u'(g_1, g_2) \in U'_1$ such that $A(g_1, g_2, k) = k$. Let $U_1 = \phi(U'_1) = \exp(U'_1) \subset C$, $u = \phi \circ u'$, and $h = u(g_1, g_2)$. Then $g_1 \circ h = h \circ g_2$. Also u is a continuous function of g_1 and g_2. Let U_2 be a smaller neighborhood of id in C such that for all $h_1, h_2 \in U_2$, $h_1 \circ h_2 \in U_1$. This exists since composition is continuous. By continuity of u or by continuity of A and the estimate

$$|u'(g_1, g_2) - \mathrm{id}| \leq |A(g_1, g_2, \mathrm{id}) - \mathrm{id}|/(1 - \tau - \mu - \epsilon),$$

there exists a smaller neighborhood V_2 of f in D such that for $g_1, g_2 \in V_2$, $u(g_1, g_2) \in U_2$. If $g \in V_2$ let $h = u(g, f)$ and $h' = u(f, g)$. Then $g \circ h = h \circ f$ and $f \circ h' = h' \circ g$. Thus $h \circ h' \circ g = h \circ f \circ h' = g \circ h \circ h'$. Also $h' \circ h \circ f = f \circ h' \circ h$. $h' \circ h, h \circ h' \in V_1$ so by uniqueness we get $h' \circ h = h \circ h' = \mathrm{id}$. Thus h is a homeomorphism. □

Remark 5. The proof given above applies directly to prove the local stability of basic sets. See [4, Theorem 7.3]

Remark 6. Using the Implicit Function Theorem instead of Lemma 8, we can solve for an h such that $g \circ h = h \circ f$. We do this by always keeping f fixed. h has to be onto by a degree argument. Using the stable manifold theorem of [4] or [5], we can show f is expansive, i.e., there exists an $r > 0$ such that for any two points $\mathbf{x}, \mathbf{y} \in M$ there exists an integer n such that the distance from $f^n(\mathbf{x})$ to $f^n(\mathbf{y})$ is greater than r. From this property, it can be shown that h has to be one to one.

Remark 7. The proof indicated in the Remark 6 does not apply in the general setting of a basic set of Remark 5.

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