Preface to the First Edition

In recent years, Dynamical Systems has had many applications in science and engineering, some of which have gone under the related headings of chaos theory or nonlinear analysis. Behind these applications there lies a rich mathematical subject which we treat in this book. This subject centers on the orbits of iteration of a (nonlinear) function or of the solutions of (nonlinear) ordinary differential equations. In particular, we are interested in the properties which persist under nonlinear change of coordinates. As such, we are interested in the geometric or topological aspects of the orbits or solutions more than an explicit formula for an orbit (which may not be available in any case). However, as becomes clear in the treatment in this book, there are many properties of a particular solution or the whole system which can be measured by some quantity. Also, although the subject has a geometric or topological flavor, analytic analysis plays an important role (e.g., the local analysis near a fixed point and the stable manifold theory).

There have been several books and monographs on the subject of Dynamical Systems. There are several distinctive aspects which together make this book unique.

First of all, this book treats the subject from a mathematical perspective with the proofs of most of the results included: the only proofs which are omitted either (i) are left to the reader, (ii) are too technically difficult to include in an introductory book, even at the graduate level, or (iii) concern a topic which is only included as a bridge between the material covered in the book and commonly encountered concepts in Dynamical Systems. (Much of the material concerning measures, Liapunov exponents, and fractal dimension is of this latter category.) Although it has a mathematical perspective, readers who are more interested in applied or computational aspects of the subject should find the explicit statements of the results helpful even if they do not concern themselves with the details of the proofs. In particular, the inclusion of explicit formulas for the various bifurcations should be very useful.

Second, this book is meant to be a graduate textbook and not just a reference book or monograph on the subject. This aspect of the book is reflected in the way the background materials are carefully reviewed as we use them. (The particular prerequisites from undergraduate mathematics are discussed below.) The ideas are introduced through examples and at a level which is accessible to a beginning graduate student. Many exercises are included to help the student learn the meaning of the theorems and master the techniques of the proofs and topic under consideration. Since the exercises are not usually just routine applications of theorems but involve similar proofs and or calculations, they are best assigned in groups, weekly or biweekly. For this reason, they are grouped at the end of each chapter rather than in the individual section.

Third, the scope of the book is on the scale of a year long graduate course and is designed to be used in such a graduate level mathematics course in Dynamical Systems. This means that the book is not comprehensive or exhaustive but tries to treat the core concepts thoroughly and treat others enough so the reader will be prepared to read further in Dynamical Systems without a complete mathematical treatment. In fact, this book grew out of a graduate course that I taught at Northwestern University many times between the early 1970s and the present. To the material that I covered in that course, I have added a few other topics: some of which my colleagues treat when they teach the course, others round out the treatment of a topic covered earlier in the book (e.g., Chapter XII*), and others just give greater flexibility to possible courses using this book. Details on which sections form the core of the book are discussed in Section 1.4.

The perspective of the book is centered on multidimensional systems of real variables. Chapters II and III concern functions of one real variable, but this is done mainly because this makes the treatment simpler analytically than that given later in higher dimensions: there are not any (or many) aspects introduced which are unique to one dimension. Some results are proved so they apply in Banach spaces or even complete metric but most of the results are developed in finite dimensions. In particular, no direct connection with partial differential equations or delay equations is given. The fact that the book concerns functions of real rather than complex variables explains why topics such as the Julia set, Mandelbrot set, and Measurable Riemann Mapping Theorem are not treated.

This book treats the dynamics of both iteration of functions and solutions of ordinary differential equations. Many of the concepts are first introduced for iteration of functions where the geometry is simpler, but an attempt has been made to interpret these results for differential equations. A proof of the existence and continuity of solutions with respect to initial conditions is also included to establish the beginnings of this aspect of the subject.

Although there is much overlap in this book and one on ordinary differential equations, the emphasis is different. The dynamical systems approach centers more on properties of the whole system or subsets of the system rather than individual solutions. Even the more local theory in Chapters IV – VII deals with characterizing types of solutions under various hypotheses. Chapters VIII and X deal more directly with more global aspects: Chapter VIII centers on various examples and Chapter X gives the global theory.

Finally, within the various types of Dynamical Systems, this book is most concerned with hyperbolic systems: this focus is most prominent in Chapters VIII, X, XI, and XII. However, an attempt has been made to make this book valuable to people interested in various aspects of Dynamical Systems.

The specific prerequisites include undergraduate analysis (including the Implicit Function Theorem), linear algebra (including the Jordan canonical form), and point set topology (including Cantor sets). For the analysis, one of the following books should be sufficient background: Apostol (1974), Marsden (1974), or Rudin (1964). For the linear algebra, one of the following books should be sufficient background: Hoffman and Kunze (1961) or Hartley and Hawkes (1970). For the point set topology, one of the following books should be sufficient background: Croom (1989), Hocking and Young (1961), or Munkres (1975). What is needed from these other subjects is an ability to use these tools; knowing a proof of the Implicit Function Theorem does not particularly help someone know how to use it. For this reason, we carefully discuss the way these tools are used just before we use them. (See the sections on the Calculus Prerequisites, Cantor Sets, Real Jordan Canonical Form, Differentiation in Higher Dimensions, Implicit Function Theorem, Inverse Function Theorem, Contraction Mapping Theorem, and Definition of a Manifold.) After using these tools in Dynamical Systems, the reader should gain a much better understanding of the importance of these "undergraduate" subjects. The terminology and ideas from differential topology or differential geometry

* Chapter numbers have been revised to agree with those in the current edition.

are also used, including that of a tangent vector, the tangent bundle, and a manifold. However, most surfaces or manifolds are either Euclidean space, tori, or graphs of functions so these ideas should not be too intimidating. Although someone pursuing Dynamical Systems further should learn manifold theory, I have tried to make this book accessible to someone without prior background in this subject. Thus, the prerequisites for this book are really undergraduate analysis, linear algebra, and point set topology and not advanced graduate work. However, the reader should be warned that most beginning graduate students do not find the material at all trivial. The main complicating aspect seems to be the use of a large variety of methods and approaches. The unifying feature is not the methods used but the type of questions which we are trying to answer. By having patience and reviewing the mathematics from other subjects as they are used, the reader should find the material accessible and rich in content, both mathematical and for applications.

The main topic of the book is the dynamics induced by iteration of a (nonlinear) function or by the solutions of (nonlinear) ordinary differential equations. In the usual undergraduate mathematics courses, some properties of solutions of differential equations are considered but more attention is paid to the specific form of the solution. In connection with functions, they are graphed and their minima and maxima are found, but the iterates of a function are not often considered. To iterate a function we repeatedly have the same function act on a point and its images. Thus, for a function f with initial condition x_0, we consider $x_1 = f(x_0)$, and then $x_n = f(x_{n-1})$ for $n \geq 1$. We are interested in finding the qualitative features and long time limiting behavior of a typical orbit, for either an ordinary differential equation or the iterates of a function. Certainly, fixed points or periodic points are important, but sometimes the orbit moves densely through a complicated set such as a Cantor set. We want to understand and bring a structure to this seemingly random behavior. It is often expressed by saying, "we want to bring order out of chaos." One way of finding this structure is via the tool of *symbolic dynamics*. If there is a real valued function f and a sequence of intervals J_i such that the image of J_i by f covers J_{i+1}, $f(J_i) \supset J_{i+1}$, then it is possible to show that there is a point x whose orbit passes through this sequence of intervals, $f^i(x) \in J_i$. Labels for the intervals then can be used as symbols, hence the name of symbolic dynamics for this approach.

Another important concept is that of structural stability. Some types of systems (iterated functions or ordinary differential equations) have dynamics which are equivalent (topologically conjugate) to that of any of its perturbations. Such a system is called *structurally stable*. Finally, the term *chaos* is given a special meaning and interpretation. There is no one set definition of a chaotic system, but we discuss various ideas and measurements related to chaotic dynamics. One of the ironies is that some chaotic systems are also structurally stable.

Chapter I gives a more detailed introduction into the main ideas that are treated in the book by means of examples of functions and differential equations. Suffice it to say here that these three ideas, symbolic dynamics, structural stability, and chaos, form the central part of the approach to Dynamical Systems presented in this book.

In the year-long graduate course at Northwestern, we cover the the material in Chapter II, Sharkovskii's Theorem and Subshifts of Finite Type from Chapter III, Chapter IV except the Perron-Frobenius Theorem, Chapter V except some of the material on periodic orbits for planar differential equations (and sometime the proof of the Stable Manifold Theorem is omitted), a selection of examples from Chapter VIII, and most of Chapter X. In a given year, other selected topics are usually added from among the following: Chapter VII on bifurcations, the material on topological entropy in Chapter IX, and the Kupka-Smale Theorem. A course which did not emphasize the global hyperbolic theory as much could be obtained by skipping Chapter X and treating additional topics, e.g., Chapter VII or more on the measurements of chaos. Section 1.4 discusses the content of the different chapters and possible selections of sections or topics for a course using this book.

There are several other books which give introductions into other aspects or approaches to Dynamical Systems. For other graduate level mathematical introductions to Dynamical Systems, see Devaney (1989), Irwin (1980), Nitecki (1970), and Palis and de Melo (1982). For a more comprehensive treatment of Dynamical Systems, see Katok and Hasselblatt (1994). Some books which emphasize the dynamics of iteration of a function of one variable are Alsedà, Llibre, and Misiurewicz (1993), Block and Coppel (1992), and de Melo and Van Strien (1993). Carleson and Gamelin (1993) gives an introduction to the dynamics of functions of a complex variable. Chow and Hale (1982) gives a more thorough treatment of the bifurcation aspects of Dynamical Systems. The article by Boyle (1993) gives a more thorough introduction into symbolic dynamics as a separate subject and not just how it is used to analyze diffeomorphisms or vector fields. Some books which concentrate on Hamiltonian dynamics are Abraham and Marsden (1978), Arnold (1978), and Meyer and Hall (1992). For an introduction to applications of Dynamical Systems, see Guckenheimer and Holmes (1983), Hirsch and Smale (1974), Wiggins (1990, 1988), and Ott (1993). For applications to ecology, see Hirsch (1982, 1985, 1988, 1990), Hofbauer and Sigmund (1988), Hoppensteadt (1982), May (1975), and Waltman (1983). There are many books written on Dynamical Systems by people in fields outside mathematics, including Lichtenberg and Lieberman (1983), Marek and Schreiber (1991), and Rasband (1990).

I have tried very hard to give references to original papers. However, there are many researchers working in Dynamical Systems and I am not always aware of (or remember) contributions by various people to which I should give credit. I apologize for my omissions. I am sure there are many. I hope the references that I have given will help the reader start finding the related work in the literature.

When referring to a theorem in the same chapter, we use the number as it appears in the statement, e.g., Theorem 2.2 which is the second theorem of the second section of the current chapter. If we are referring to Theorem 2.2 from Chapter VI in a chapter other other than Chapter VI, we refer to it as Theorem VI.2.2 to indicate it comes from a different chapter.

There are not any specific references in this book to using a computer to simulate a dynamical system. However, the reader would benefit greatly by seeing the dynamics as it unfolds by such simulation. The reader can either write a program for him or herself or use several of the computer packages available. On an IBM Personal Computer, I have used the program *Phaser* which comes with the book by Koçak (1989). The program *Dynamics* by Yorke (1990) runs on both IBM Personal Computers and Unix/X11 machines. There are several other programs for IBM Personal Computers but I have not used them myself. Also, the program *DSTool* by J. Guckenheimer, M. R. Myers, F. J. Wicklin, and P. A. Worfolk runs on Unix/X11 machines. Many of the programming languages come with a good enough graphics library that it is not difficult to write one's own specialized program. However, for the X-Window environment on a Unix computer, I found the *VOGLE* library (C graphics C functions) a very helpful asset to write my own programs. There are several programs for the Macintosh, including *MacMath* by Hubbard and West (1992), but I have not used them.

Over the years, I have had many useful conversations with colleagues at Northwestern University and from elsewhere, especially people attending the Midwest Dynamical Systems Seminars. Those colleagues in Dynamical Systems at Northwestern University include Keith Burns, John Franks, Don Saari, Robert Williams, and many postdoctoral

instructors and visitors. Those attending the Midwest Dynamical Systems Seminars are too numerous to list, but surely Charles Conley is one who bears mentioning and will long be remembered by many of us. I also owe a great debt to the people who taught me about Dynamical Systems, including Morris Hirsch, Charles Pugh, and Steve Smale. The perspective on Dynamical Systems which I learned from them is still very evident in the selection and treatment of topics in this book.

I would also like to thank the many people who found typographical errors, conceptual errors, or points that needed to be clarified in earlier drafts of this book. I would especially like to thank Keith Burns, Beverly Diamond, Roger Kraft, and Ming-Chia Li. Keith Burns taught out of a preliminary version and made many suggestions for improvements, clarifications, and changed arguments; Beverly Diamond made many suggestions for improvements in grammar and other editing matters; Roger Kraft made both mathematical and typographical corrections; in addition to noting out typographical errors, Ming-Chia Li pointed out aspects which needed clarifying.

This text was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX. The figures were produced using DsTool, Xfig, Maple, and Vogle graphics C Library. I would like to thank Len Evens who supplied me with some macros which were used with $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX to produce the chapter and section titles and numbers, and the index and table of contents.

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Preface to the Second Edition

The second edition of this book has provided the opportunity for correcting many minor typographical errors or mistakes. Needless to say, the basic approach and content of the book have stayed the same. The discussion of the saddle node bifurcation has been rewritten using notation to make it easier to understand. In an attempt to expand the comparison of results for diffeomorphisms and flows, I have added a section on the horseshoe for a flow with a transverse homoclinic point. This section makes explicit the meaning and interpretation of a horseshoe in the case of a flow instead of a diffeomorphism. Another subsection on horseshoes for nontransverse homoclinic points indicates some recent extensions to the understanding of how horseshoes arise. Also added is a section proving the ergodicity of a hyperbolic toral automorphism. This proof is fairly simple but introduces an important technique which is used to prove ergodicity in other situations. Finally, a new chapter on Hamiltonian systems has been added. This chapter treats mainly local properties near fixed points, but should prove of interest to some of the readers.

Future typographical errors and additions will be posted on a website at Northwestern <http://math.nwu.edu/∼clark>. I would appreciate being informed of further typographical errors or suggestions by email.

Clark Robinson
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